RAIRO. ANALYSE NUMÉRIQUE

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# Addendum to the paper "Finite element solution of quasistationary nonlinear magnetic field"

*RAIRO. Analyse numérique*, tome 17, nº 4 (1983), p. 407-415
<http://www.numdam.org/item?id=M2AN_1983__17_4_407_0>

R.A.I.R.O. Analyse numérique/Numerical Analysis
(vol. 17, n° 4, 1983, p. 407 à 415)

# ADDENDUM TO THE PAPER « FINITE ELEMENT SOLUTION OF QUASISTATIONARY NONLINEAR MAGNETIC FIELD » (*)

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Communicated by P. G. CIARLET

Résumé. — *Une version corrigée et étendue du Théorème 2 du papier ci-dessus (ce journal, vol. 16 (1982), pp. 161-191) est établie.*

Abstract. — *A corrected and extended version of Theorem 2 of the above mentioned paper (this journal, vol. 16 (1982), pp. 161-191) is given.*

In the above mentioned paper (this journal, vol. 16 (1982), pp. 161-191; in what follows it is denoted by [Z]) we proved Theorem 2 on a weak as well as on a strong convergence of a fully discrete approximate solution $U^\delta$ to the exact solution $u$ of the problem $P'$. The assertion $\| u_R - U_R^\delta \|_{C([0,T]; H_R)} \to 0$ is not correct because the approximate solution $U^\delta$ is not continuously extended on the interval $[0, T]$. Here, we correct it in two ways and prove it even without any regularity requirement on $u$. Before doing it we remark that (1.1), (1.2), ..., (2.1), (2.2), etc. mean equations from [Z]. The equations in this addendum are denoted by (1), (2), etc. Further, we extend the discrete values $U^i \in V^h$ defined by (3.24) continuously on $[0, T]$. We define a new approximate solution $\mathscr{U}^\delta \in C([0, T]; V)$ by

$$\mathscr{U}^\delta = U^1 \quad \text{in} \quad [0, t_1], \quad \mathscr{U}^\delta = U^{i-1} + \frac{t - t_{i-1}}{\Delta t} \Delta U^i \quad \text{in} \quad (t_{i-1}, t_i], \quad (1)$$

$$2 \leqq i \leqq r, \qquad r = T\,\Delta t^{-1}$$

$$(\Delta U^i = U^i - U^{i-1}).$$

(*) Received in September 1982.
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R.A.I.R.O. Analyse numérique/Numerical Analysis, 0399-0516/1983/407/$ 5.00

THEOREM : *Let the assumptions* (1)-(5) *of* [Z] *be fulfilled, let*

$$f^M \in L^{p'}(0, T ; \overline{V}'_M), \quad M = R, S, \quad 1 < p < \infty, \quad \frac{1}{p} + \frac{1}{p'} = 1 \quad \textit{and} \quad u_0 \in H_R .$$

*Then there exists a unique function*

$$u \in W_R = \{ u \mid u \in L^p(0, T ; V) ; u'_R \in L^{p'}(0, T ; \overline{V}'_R) \}$$

*satisfying* (3.17) *and* (3.18). *Further, the approximate solutions* $U^\delta$ *and* $\mathcal{U}^\delta$ *defined by* (3.24) *and by* (3.25) *and* (1), *respectively, exist, are unique and for* $\delta \equiv (h, \Delta t) \to 0$

$$U^\delta \to u \quad in \quad L^p(0, T ; V) \quad weakly, \tag{2}$$

$$\sup_{0 < t \leqq T} | u_R(t) - U_R^\delta(t) |_R \to 0, \qquad \| u_R - \mathcal{U}_R^\delta \|_{C([0,T]; H_R)} \to 0 . \tag{3}$$

*If, in addition,*

$$a(u, u - v) - a(v, u - v) \geqq \gamma [u - v]^p \quad \forall u, v \in V, \qquad \gamma = \text{const} > 0 \tag{4}$$

*where* [.] *is a seminorm with property*

$$[v] + \lambda \mid v_R \mid_R \geqq \beta \| v \| \quad \forall v \in V, \qquad \lambda, \beta = \text{const} > 0 \tag{5}$$

*then*

$$\| u - U^\delta \|_{L^p(0,T; V)} \to 0, \qquad \| u - \mathcal{U}^\delta \|_{L^p(0,T;V)} \to 0 . \tag{6}$$

*Remark* 1 : Let $H$ be a Hilbert space which is dense and continuously imbedded in a separable reflexive Banach space $V$, let $u_0 \in H$, $f \in L^{p'}(0, T ; V')$ and let $A(u)$ be a nonlinear operator from $V$ in $V'$. We consider the problem

$$u' + A(u) = f, \qquad u(0) = u_0 \tag{7}$$

and we assume that

i) $A(u)$ is hemicontinuous and monotone, $\| A(u) \|_{V'} \leqq C \| u \|^{p-1}$ and $\langle A(u), u \rangle \geqq \alpha [u]^p \quad \forall u \in V$, $\alpha = \text{const} > 0$ ([.] satisfies

$$[v] + \lambda \mid v \mid \geqq \beta \| v \| \qquad \forall v \in V, \qquad \lambda, \beta = \text{const} > 0) .$$

Then

$$U^\delta \to u \quad \text{in} \quad L^p(0, T ; V) \quad \text{weakly},$$

$$\sup_{0 < t \leqq T} | u(t) - U^\delta(t) | \to 0, \qquad \| u - \mathcal{U}^\delta \|_{C([0,T]; H)} \to 0 .$$

If, in addition,

ii) $\langle A(u), u-v \rangle - \langle A(v), u-v \rangle \geqq \gamma[u-v]^p \quad \forall u, v \in V, \quad \gamma = \text{const} > 0$
then

$$\| u - U^\delta \|_{L^p(0,T;V)} \to 0, \qquad \| u - \mathcal{U}^\delta \|_{L^p(0,T;V)} \to 0.$$

These assertions follow from the Theorem ($H_R = H$, $H_S = \emptyset$).

*Remark* 2 : We consider the problem (7) with a linear operator $A(u)$ from $V$ in $V'$. We assume that

i) $\qquad \langle A(u), u \rangle \geqq \alpha[u]^2 \qquad \forall u \in V, \quad \alpha = \text{const} > 0.$

Then

$$\| u - \mathcal{U}^\delta \|_{C([0,T];H)} \to 0, \qquad \| u - \mathcal{U}^\delta \|_{L^2(0,T;V)} \to 0.$$

From i) it follows

$$\langle A(u), u-v \rangle - \langle A(v), u-v \rangle = \langle A(u-v), u-v \rangle \geqq \alpha[u-v]^2 \quad \forall u, v \in V.$$

Hence $A(u)$ is monotone and satisfies ii) of Remark 1 with $p = 2$. From monotonicity and linearity it follows that $A(u)$ is continuous. Evidently, the assumptions i), ii) of Remark 1 are fulfilled.

*Remark* 3 : Theorem 3 of [Z] is true without assuming $u \in C([0, T]; H_0^1(\Omega))$ if we correct (4.8) as follows :

$$\sup_{0 < t \leqq T} \| u(t) - U^\delta(t) \|_{L^2(R)} \to 0, \qquad \| u - \mathcal{U}^\delta \|_{C([0,T];L^2(R))} \to 0,$$

$$\| u - U^\delta \|_{L^2(0,T;H_0^1(\Omega))} \to 0, \qquad \| u - \mathcal{U}^\delta \|_{L^2(0,T;H_0^1(\Omega))} \to 0.$$

*Remark* 4 : We consider a set of, in general, nonuniform partitions of the interval $[0, T]$ : $0 = t_0 < t_1 < \dots < t_r = T$. We denote $\Delta t = \max_{1 \leqq i \leqq r} \Delta t_i$, $\Delta t_i = t_i - t_{i-1}$, and we assume that the set of partitions has the following properties : 1) $\Delta t \to 0$, 2) $\min_{1 \leqq i \leqq r} \Delta t_i \geqq \sigma_0 \Delta t$ for all partitions where $\sigma_0$ is a positive constant which does not depend on the chosen partition (nor on $h$). Let the approximate solution be defined by means of the Euler backward scheme, i.e. by

$$(U_R^i - U_R^{i-1}, z_R)_R + \Delta t_i \, a(U^i, z) = \Delta t_i \langle f^i, z \rangle \quad \forall z \in V^h, \quad U_R^0 = u_0$$

where now $f^i = \dfrac{1}{\Delta t_i} \displaystyle\int_{t_{i-1}}^{t_i} f(\tau)\, d\tau$. Then all results of the paper [Z] as well as of this addendum are true for the extended approximate solutions $U^\delta$ and $\mathcal{U}^\delta$. The changes in the proofs are very small.

*Proof of the Theorem* : We have to prove (3) and (6). The other assertions are proved in [Z].

As $u_R \in \{ \omega \mid \omega \in L^p(0, T; \overline{V}_R); \omega' \in L^{p'}(0, T; \overline{V}'_R) \}$ $u_R$ belongs to $C([0, T]; H_R)$ (see lemma 1 in [Z]). To prove the first part of (3) we use a discrete version of an idea which is used in [4] (see references in [Z]) in the proof of Theorem 1.2, pp. 209-210. We derive an estimate for $\max\limits_{1 \leqq i \leqq j} | U_R^i - v_R^i |_R$ where $v$ is an arbitrary function from $C^1([0, T]; V^h)$. Then we take for $v$ an approximation of $u$.

Let $v^0 = v(0)$ and let $v_{\Delta t}$ denote the function $v_{\Delta t} = v^i \equiv v(t_i)$ in $(t_{i-1}, t_i$, $i = 1, \dots, r$. First, we consider $\{ U^i \}_{i=1}^r$ defined by (3.29). Using (3.23), (3.9) and (3.10) we get

$$\frac{1}{2} | U_R^j - v_R^j |_R^2 - \frac{1}{2} | U_R^0 - v_R^0 |_R^2 \leqq \sum_{i=1}^{j} (\Delta(U_R^i - v_R^i), U_R^i - v_R^i)_R =$$

$$= \sum_{i=1}^{j} (\Delta U_R^i, U_R^i - v_R^i)_R - \sum_{i=1}^{j} (\Delta v_R^i, U_R^i - v_R^i)_R =$$

$$= - \Delta t \sum_{i=1}^{j} a(U^i, U^i - v^i) + \Delta t \sum_{i=1}^{j} \langle f^i, U^i - v^i \rangle$$

$$- \sum_{i=1}^{j} \left( \int_{t_{i-1}}^{t_i} v' \, dt, U_R^\delta - (v_{\Delta t})_R \right)_R = - \int_0^{t_j} a(U^\delta, U^\delta - v_{\Delta t}) \, dt$$

$$+ \int_0^{t_j} \langle f, U^\delta - v_{\Delta t} \rangle \, dt - \int_0^{t_j} (v'_R, U_R^\delta - (v_{\Delta t})_R)_R \, dt =$$

$$= - \int_0^{t_j} [a(U^\delta, U^\delta - u) - a(u, U^\delta - u)] \, dt$$

$$+ \int_0^{t_j} [a(u, u - v_{\Delta t}) - a(U^\delta, u - v_{\Delta t})] \, dt + \int_0^{t_j} \langle u'_R - v'_R, U_R^\delta - (v_{\Delta t})_R \rangle_R \, dt \,.$$

From (3.37) and (3.13) it follows that the second integral is bounded from above by

$$C \| u - v_{\Delta t} \|_{L^p(0,T;V)} \leqq C \{ \| u - v \|_{L^p(0,T,V)} + \| v - v_{\Delta t} \|_{L^p(0,T,V)} \}$$
$$\leqq C \{ \| u - v \|_{W_R} + \Delta t \| v \|_{C^1([0,T],V)} \} \,.$$

Here $\| u \|_{W_R} = \| u \|_{L^p(0,T,V)} + \| u'_R \|_{L^{p'}(0,T,\overline{V}'_R)}$ and $C$ is a generic constant not necessarily the same at any two places. $C$ as well as $C_i$ $(i = 0, \dots, 3)$ introduced later do not depend on $\delta = (h, \Delta t)$ and on the functions $v$, $z$ ($C$ depends on $u$). The third integral is bounded from above by

$$\begin{aligned}\| u'_R - v'_R \|_{L^{p'}(0,T;\bar{V}'_R)} \| U^\delta_R - (v_{\Delta t})_R \|_{L^p(0,T;\bar{V}_R)} &\leqq \\ &\leqq C \| u - v \|_{W_R} \| U^\delta - v + v - v_{\Delta t} \|_{L^p(0,T;V)} \\ &\leqq C \| u - v \|_{W_R} \{ C + \| v \|_{L^p(0,T;V)} + \| v - v_{\Delta t} \|_{L^p(0,T;V)} \} \\ &\leqq C \{ \| u - v \|_{W_R} + \Delta t \| v \|_{C^1([0,T];V)} \}\end{aligned}$$

if we assume that

$$\| u - v \|_{W_R} \leqq 1 . \tag{8}$$

The result of all these estimates can be written in this way :

$$\begin{aligned}| U^j_R - v^j_R |^2_R + \int_0^{t_j} [a(U^\delta, U^\delta - u) - a(u, U^\delta - u)] \, dt &\leqq \\ &\leqq | U^0_R - v^0_R |^2_R + C \{ \| u - v \|_{W_R} + \Delta t \| v \|_{C^1([0,T];V)} \} .\end{aligned}$$

As (see lemma 1 in [Z])

$$| U^0_R - v^0_R |_R = | u(0)_R - v(0)_R |_R \leqq \| u_R - v_R \|_{C([0,T];H_R)} \leqq C \| u - v \|_{W_R}$$

and

$$| u^j_R - U^j_R |_R \leqq \| u_R - v_R \|_{C([0,T];H_R)} + | U^j_R - v^j_R |_R$$

it easily follows that

$$Y^\delta \leqq C_0 \{ \| u - v \|_{W_R} + \Delta t \| v \|_{C^1([0,T];V)} \} \tag{9}$$

where

$$Y^\delta = \max_{1 \leqq i \leqq r} | u^i_R - U^i_R |^2_R + \int_0^T [a(U^\delta, U^\delta - u) - a(u, U^\delta - u)] \, dt .$$

By means of (9) we prove that

$$Y^\delta \to 0 . \tag{10}$$

To this end we remark that $C^1([0, T]; V)$ is dense in $W_R$ (the proof is the same as in case $H_R = H$, $H_S = \varnothing$; see, e.g., [4], p. 144) and that $\bigcup_{j=1}^{\infty} C^1([0, T]; V^{h_j})$ is dense in $C^1([0, T]; V)$ if $h_j \to 0$ (the proof is the same as the proof of lemma 1.5 in [4], p. 209). We also remark that

$$\| z \|_{W_R} \leqq C_1 \| z \|_{C^1([0,T];V)} \qquad \forall z \in C^1([0, T]; V) .$$

If (10) is not true there exists an $\varepsilon_0 > 0$ and a sequence

$$\{ \delta_n \}_{n=1}^{\infty}, \delta_n = (h_n, \Delta t_n) \to 0 ,$$

with the property $Y^{\delta_n} \geqq \varepsilon_0$ for $n \geqq 1$. Let $w \in C^1([0, T]; V)$ be such that

$$\| u - w \|_{W_R} < \frac{1}{4 C_0} \varepsilon_0$$

and $n_0$ be so large that

$$\Delta t_n < \frac{\varepsilon_0}{2 C_0} (\| w \|_{C^1([0,T];V)} + 1)^{-1} , \qquad n \geqq n_0 .$$

As $\bigcup_{n=n_0}^{\infty} C^1([0, T]; V^{h_n})$ is dense in $C^1([0, T]; V)$ there exists $v \in C^1([0, T]; V^{h_{n_1}})$ with $n_1 \geqq n_0$ such that

$$\| w - v \|_{C^1([0,T];V)} < \frac{1}{4 C_0 C_1} \varepsilon_0 .$$

We may assume that $\varepsilon_0 < \min (2 C_0, 4 C_0 C_1)$. Then

$$\| u - v \|_{W_R} \leqq \| u - w \|_{W_R} + \| w - v \|_{W_R} < \frac{1}{4 C_0} \varepsilon_0 +$$

$$+ C_1 \| w - v \|_{C^1([0,T];V)} < \frac{1}{2 C_0} \varepsilon_0 < 1 .$$

Therefore we may use (9) with this $v$ and we get

$$Y^{\delta_{n_1}} < C_0 \left[ \frac{1}{2 C_0} \varepsilon_0 + \Delta t_{n_1} \| v \|_{C^1([0,T];V)} \right] < \varepsilon_0$$

which is in contradiction with $Y^{\delta_n} \geqq \varepsilon_0$ for $n \geqq 1$.

Now we show that (10) is true in case that $\{ U^i \}_{i=1}^{r}$ is defined by (3.59). To this end we remark that (3.61) is true if we replace $U_R^i$ by any $\omega_R^i \in H_R$ such that $\omega_R^0 = \omega_R^{-1}$. Estimating the term $-(\omega_R^j, \omega_R^{j-1})_R$ from below by $-\mid \omega_R^j \mid_R^2 - \frac{1}{4} \mid \omega_R^{j-1} \mid_R^2$ we get

$$\frac{1}{4} \mid \omega_R^j \mid_R^2 - \frac{1}{2} \mid \omega_R^0 \mid_R^2 \leqq \sum_{i=1}^{j} \left( \frac{3}{2} \omega_R^i - 2 \omega_R^{i-1} + \frac{1}{2} \omega_R^{i-2}, \omega_R^i \right)_R =$$

$$= \sum_{i=1}^{j} \left( \frac{3}{2} \Delta \omega_R^i - \frac{1}{2} \Delta \omega_R^{i-1}, \omega_R^i \right)_R . \quad (11)$$

We choose $\omega_R^i = U_R^i - v_R^i$ (as before, $v^i = v(t_i)$, $i = 0, ..., r$; in addition, we define $v^{-1} = v^0$ so that $\Delta v^0 = 0$) and we extend $u'_R$ by zero outside $(0, T)$ keeping the same notation $u'_R$ for this extension. Similarly as before we obtain

$$\frac{1}{4} | U_R^j - v_R^j |_R^2 - \frac{1}{2} | U_R^0 - v_R^0 |_R^2 \leqq \sum_{i=1}^{j} \left( \frac{3}{2} U_R^i - 2 U_R^{i-1} + \frac{1}{2} U_R^{i-2}, U_R^i - v_R^i \right)_R -$$

$$- \sum_{i=1}^{j} \left( \frac{3}{2} \Delta v_R^i - \frac{1}{2} \Delta v_R^{i-1}, U_R^i - v_R^i \right)_R = - \Delta t \sum_{i=1}^{j} a(U^i, U^i - v^i)$$

$$+ \Delta t \sum_{i=1}^{j} \langle f^i, U^i - v^i \rangle - \frac{3}{2} \sum_{i=1}^{j} \left( \int_{t_{i-1}}^{t_i} v'_R(t), U_R^\delta - (v_{\Delta t})_R \right)_R$$

$$+ \frac{1}{2} \sum_{i=2}^{j} \left( \int_{t_{i-1}}^{t_i} v'_R(t - \Delta t), U_R^\delta - (v_{\Delta t})_R \right)_R = - \int_0^{t_j} [a(U^\delta, U^\delta - u)$$

$$- a(u, U^\delta - u)] \, dt + \int_0^{t_j} [a(u, u - v_{\Delta t}) - a(U^\delta, u - v_{\Delta t})] \, dt$$

$$+ \int_0^{t_j} \langle u'_R(t), U_R^\delta - (v_{\Delta t})_R \rangle_R \, dt - \frac{3}{2} \int_0^{t_j} \langle v'_R(t), U_R^\delta - (v_{\Delta t})_R \rangle_R \, dt$$

$$+ \frac{1}{2} \int_{t_1}^{t_j} \langle v'_R(t - \Delta t), U_R^\delta - (v_{\Delta t})_R \rangle_R \, dt .$$

The first two integrals on the right-hand side are the same as before. The last three are equal to

$$\frac{3}{2} \int_0^{t_j} \langle u'_R(t) - v'_R(t), U_R^\delta - (v_{\Delta t})_R \rangle_R \, dt -$$

$$- \frac{1}{2} \int_{t_1}^{t_j} \langle u'_R(t - \Delta t) - v'_R(t - \Delta t), U_R^\delta - (v_{\Delta t})_R \rangle_R \, dt$$

$$- \frac{1}{2} \int_0^{t_j} \langle u'_R(t) - u'_R(t - \Delta t), U_R^\delta - (v_{\Delta t})_R \rangle_R \, dt .$$

The first two terms can be estimated as before. The last is bounded by

$$C\alpha(\Delta t)(1 + \Delta t \, \| v \|_{C^1([0,T];V)}) , \quad \alpha(\Delta t) = \| u'_R(t) - u'_R(t - \Delta t) \|_{L^{p'}(0,T;\bar{V}'_R)} .$$

$Y^\delta$ is bounded from above by

$$Y^\delta \leqq C_2 \{ \| u - v \|_{W_R} + \Delta t \, \| v \|_{C^1([0,T];V)} \} + C_3 \, \alpha(\Delta t) . \tag{12}$$

As $u'_R \in L^{p'}(0, T ; \overline{V'_R})$ it holds $\alpha(\Delta t) \to 0$, hence $Y^\delta \to 0$.

Let $M(\delta) = \max_{1 \leqq i \leqq r} | u_R^i - U_R^i |_R$, $m(\Delta t) = \max | u_R(\tau_1) - u_R(\tau_2) |_R$, $\tau_1, \tau_2 \in [0, T]$, $| \tau_1 - \tau_2 | \leqq \Delta t$. Then

$$M(\delta) \to 0 , \qquad m(\Delta t) \to 0 \quad \text{if} \quad \delta \to 0 .$$

We have $| u_R(t) - U_R^\delta(t) |_R \leqq m(\Delta t) + M(\delta) \to 0$ which proves the first part of (3). Now, let $\| u_R - \mathscr{U}_R^\delta \|_{C([0,T];H_R)} = | u_R(\tau_0) - \mathscr{U}_R^\delta(\tau_0) |_R$. If $\tau_0 \in [0, t_1]$ then

$$| u_R(\tau_0) - \mathscr{U}_R^\delta(\tau_0) |_R \leqq | u_R(\tau_0) - u_R^1 |_R + | u_R^1 - U_R^1 |_R \leqq m(\Delta t) + M(\delta) .$$

If $\tau_0 \in (t_{i-1}, t_i)$, $i \geqq 2$ then

$$| u_R(\tau_0) - \mathscr{U}_R^\delta(\tau_0) |_R \leqq | u_R(\tau_0) - u_R^i |_R + | u_R^i - U_R^i |_R + | U_R^\delta(\tau_0) - \mathscr{U}_R^\delta(\tau_0) |_R \leqq$$
$$\leqq m(\Delta t) + M(\delta) + \left| \frac{t_i - \tau_0}{\Delta t} \Delta U_R^i \right|_R \leqq m(\Delta t) + M(\delta) + | \Delta U_R^i |_R .$$

As $| \Delta U_R^i |_R \leqq | U_R^i - u_R^i |_R + | \Delta u_R^i |_R + | u_R^{i-1} - U_R^{i-1} |_R \leqq m(\Delta t) + 2 M(\delta)$ we see that $\| u_R - \mathscr{U}_R^\delta \|_{C([0,T];H_R)} \leqq 2 m(\Delta t) + 3 M(\delta) \to 0$.

If $a(u, v)$ satisfies (4) it follows from (9) and (10) that $\int_0^T [u - U^\delta]^p \, dt \to 0$. Further,

$$\int_0^T | u_R - U_R^\delta |_R^p \, dt = \sum_{i=1}^r \int_{t_{i-1}}^{t_i} | u_R - U_R^i |_R^p \, dt \leqq$$
$$\leqq \sum_{i=1}^r \int_{t_{i-1}}^{t_i} [| u_R(t) - u_R^i |_R + | u_R^i - U_R^i |_R]^p \, dt$$
$$\leqq T[m(\Delta t) + M(\delta)]^p \to 0 .$$

Consequently, $\| u - U^\delta \|_{L^p(0,T;V)} \to 0$ owing to (5).

Finally, from $\int_0^T \| u(t) - U^\delta(t) \|^p \, dt \to 0$ it follows

$$\int_0^T \| u(t - \Delta t) - U^\delta(t - \Delta t) \|^p \, dt \to 0$$

(here $u$ and $U$ are extended by zero outside $(0, T)$). As

$$\| u(t) - u(t - \Delta t) \|_{L^p(0,T;V)} \to 0$$

we get $\int_0^T \| U^\delta(t) - U^\delta(t - \Delta t) \|^p \, dt \to 0$. On the other hand

$$\int_0^T \| U^\delta(t) - U^\delta(t - \Delta t) \|^p \, dt = \Delta t \left\{ \| U^1 \|^p + \sum_{i=2}^{r} \| \Delta U^i \|^p \right\},$$

thus $\Delta t \sum_{i=2}^{r} \| \Delta U^i \|^p \to 0$. We have

$$\int_0^T \| u - \mathcal{U}^\delta \|^p \, dt \leqq 2^{p-1} \left\{ \int_0^T \| u - U^\delta \|^p \, dt + \int_0^T \| U^\delta - \mathcal{U}^\delta \|^p \, dt \right\}$$

and

$$\int_0^T \| U^\delta - \mathcal{U}^\delta \|^p \, dt = \sum_{i=2}^{r} \int_{t_{i-1}}^{t_i} \left\| \frac{t_i - t}{\Delta t} \Delta U^i \right\|^p dt \leqq \Delta t \sum_{i=2}^{r} \| \Delta U^i \|^p \to 0$$

which proves the last assertion of the Theorem.