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# APPROXIMATION OF BURGERS' EQUATION BY PSEUDO-SPECTRAL METHODS (*) (**)

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Résume — *On applique des méthodes pseudo-spectrales de collocation basées sur des développements polynomiaux de Chebyshev et de Legendre à l'équation de Burgers stationnaire monodimensionnelle L'analyse numérique est construite à partir de théorèmes abstraits concernant l'approximation en dimension finie d'une classe de problèmes non linéaires.*

Abstract — *Pseudo-spectral (collocation) methods for the stationary one dimensional Burgers' equation based on Chebyshev and Legendre polynomial expansions are considered The numerical analysis is developed by means of some abstract theorems concerning finite dimensional approximations of a class of nonlinear problems.*

## INTRODUCTION.

The advection-diffusion equation :

$$- u_{xx} + \lambda(uu_x - f) = 0\,, \quad \lambda \in \mathbb{R}^+ \tag{0.1}$$

known as steady-state Burgers' equation, is commonly used in many applications, since it describes numerous transport phenomena of interest to engineers and scientists. Even, (0.1) is an elliptic regularization of the hyperbolic Burgers' equation relative to nonlinear evolution transport.

In recent years a considerable number of numerical finite difference and finite element methods have been proposed in this field, particularly when $\lambda$ is large and advection is dominating. We refer for instance to the " upwind " finite differences, first considered by Courant, Isaacson and Rees in 1952, and to

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the extension of the upwinding technique to finite elements first used by Zienkiewicz and his school in 1977. Spectral and pseudo-spectral methods for the linearized Burgers' equation were proposed by Gottlieb and Orszag [8], Kreiss and Oliger [9] and by other authors more recently. In [13] Nickell, Gartling and Strang analyze a spectral decomposition coupled with finite element methods to solve numerically (0.1).

In [11] the analysis of an approximation to (0.1) by *spectral* methods based on Legendre and Chebyshev polynomials is given by the authors. Homogeneous boundary conditions on the interval $I = (-1, 1)$ are taken into account. For the same problem, in this paper we analyze *pseudo-spectral* methods using the same orthogonal polynomials. Stability and convergence analysis is more complicate than in [11] since the effects of the errors arising from numerical integrations have to be considered here. However, pseudo-spectral numerical schemes are more convenient for their computational aspects (indeed, the FFT algorithm can be successfully used in general). Furthermore, as Gottlieb and Orszag emphasized in [8], Chebyshev polynomials have high resolution power for thin boundary layers (which may occur when $\lambda$ is large).

Problem (0.1) may be written equivalently in the abstract form :

$$\{ \lambda, u \} \in \mathbb{R}^+ \times V, \quad u + TG(\lambda, u) = 0, \tag{0.2}$$

where $V$ and $W$ are two Banach spaces with $W \subset V'$, $G$ is a differentiable mapping from $\mathbb{R} \times V$ into $W$, $T \in \mathscr{L}(V'; V)$ and $T$ is compact from $W$ into $V$.

In Section 1 we present some general stability and convergence results relative to the approximation of problem (0.2) by discrete problems which may be written as follows

$$\{ \lambda, u_N \} \in \mathbb{R}^+ \times V_N, \quad u_N + T_N G_N(\lambda, u_N) = 0. \tag{0.3}$$

In (0.3) $V_N$ is a finite dimensional subspace of $V$ for any $N \in \mathbb{N}$, while $T_N$ and $G_N$ are some approximations of the operators $T$ and $G$. The formulation (0.3) looks to be particularly adapted to describe approximations of problems like (0.2) by pseudo-spectral methods (in [12], for instance, the authors carry out the analysis of a pseudo-spectral method to approximate the three dimensional, periodic, Navier-Stokes equations). Also, (0.3) is the typical form of finite element approximations to (0.2) which make use of numerical integration. Due to this generality, it is an authors' opinion that Section 1 has an interest in itself, independently of its application to problem (0.1) which is developed in next Sections. Results of Section 1 generalize those by Brezzi, Rappaz and Raviart [4] which are confined to the case $G_N \equiv G$. Relatively to the nonsingular solutions of (0.2) we provide abstract bounds for the error norms $\| u - u_N \|_V$

and $\| u - u_N \|_H$, for any Hilbert space $H$ which contains algebraically and topologically $V$. In addition we state sufficient conditions to have quadratic convergence of a Newton iterative method to solve (0.3).

In Section 2 the Burgers problem (0.1) is written in the form (0.2).

Let $\Lambda$ be any compact subset of $\mathbb{R}^+$ and assume that the mapping

$$\lambda \in \Lambda \to u(\lambda) \in H^{\sigma}_{\omega}(I)$$

is continuous for some $\sigma \geqslant 1$ ($\omega$ is equal to 1 for the Legendre approximation, and $\omega(x) = (1 - x^2)^{-1/2}$ for the Chebyshev approximation). In Section 3 we establish the following error estimate between $u$ and its pseudo-spectral approximation $u_N$ :

$$\forall \lambda \in \Lambda \; \| u_N(\lambda) - u(\lambda) \|_{H^1_\omega(I)} + N^{1-e(\omega)} \| u_N(\lambda) - u(\lambda) \|_{L^2_\omega(I)} = 0(N^{1-\sigma}) \quad (0.4)$$

where $e(\omega) = 0$ for the Chebyshev weight and $e(\omega) = 1/2$ for the Legendre weight.

The estimate (0.4) is established using the abstract results of Section 1.

Throughout this paper $C$ will denote a generic positive constant, independent of the discretization parameter $N$, not necessarily the same in different contexts.

An outline of the paper is a follows :

1. ABSTRACT RESULTS : APPROXIMATIONS OF BRANCHES OF NON SINGULAR SOLUTIONS.
   1.1. Approximation in the energy norm.
   1.2. Error estimates in lower order norms.
   1.3. The Newton method to solve the discrete problem.

2. THE BURGERS' EQUATION : PRELIMINARIES.

3. APPROXIMATION BY PSEUDO-SPECTRAL METHODS : STABILITY AND CONVERGENCE.

## 1. ABSTRACT RESULTS : APPROXIMATIONS OF BRANCHES OF NON SINGULAR SOLUTIONS.

### 1.1. Approximation in the energy norm.

Let $\Lambda$ be a compact interval of the real line, $V$ and $W$ be two Banach spaces, and assume that $W$ is contained into $V'$ (dual space of $V$) with continuous imbedding. Let $T \in \mathscr{L}(V'; V)$ and assume that $T$ is compact from $W$ into $V$; finally, let $G : \Lambda \times V \to W$ be a $C^1$ mapping. We set

$$\forall \{ \lambda, u \} \in \Lambda \times V \quad F(\lambda, u) = u + TG(\lambda, u), \qquad (1.1)$$

and we consider the problem find $(\lambda, u) \in \Lambda \times V$ such that

$$F(\lambda, u) = 0 \tag{1 2}$$

Throughout this section we make the following assumption

$$\left.\begin{array}{l}\text{there exists a branch } \{ (\lambda, u(\lambda)), \lambda \in \Lambda \} \text{ of non singular solutions} \\ \text{of (1 2), in the sense that there exists a constant } \alpha > 0 \text{ such that} \\ \qquad \forall \lambda \in \Lambda , \quad \forall v \in V , \quad \| (\mathrm{Id} + T D_u G[\lambda, u(\lambda)]) v \|_V \geqslant \alpha \| v \|_V \end{array}\right\} \quad (H1)$$

The symbol $D_u G[\lambda_0, u_0]$ (resp $D_\lambda G[\lambda_0, u_0]$) denotes the Frechet derivative, with respect to $u$(resp to $\lambda$) of $G(\lambda, u)$, computed at the point $(\lambda_0, u_0)$ Id is the identity operator

Let $N$ be a parameter which will tend to infinity in the applications In order to approximate the branch $\{ (\lambda, u(\lambda)), \lambda \in \Lambda \}$ we introduce a family $\{ V_N \}_N$ of finite dimensional subspaces of $V$, and a family $\{ T_N \}_N$ of operators belonging to $\mathscr{L}(V', V_N)$ If not otherwise specified, for any $N$ the space $V_N$ is equipped by the norm of $V$

Let us now introduce the mapping $F_N \quad \Lambda \times V \to V$ defined by

$$F_N(\lambda, u) = u + T_N G(\lambda, u), \tag{1 3}$$

and consider the approximate problem find $\{ \lambda, u_N \} \in \Lambda \times V_N$ such that

$$F_N(\lambda, u_N) = 0 \tag{1 4}$$

The following result is due to Brezzi, Rappaz and Raviart (see [4, theorem 6] and replace suitably $h$ by $N$)

THEOREM 1 1 *Let $m \geqslant 1$ be an integer, assume that $G$ is a $C^{m+1}$ mapping from $\Lambda \times V$ into $W$, and that $D^{m+1} G$ is bounded over any bounded subset of $\Lambda \times V$ Let $\Pi_N \quad V \to V_N$ be continuous operator satisfying*

$$\forall v \in V \quad \lim_{N \to \infty} \| \Pi_N v - v \|_V = 0 , \tag{1 5}$$

*moreover assume that*

$$\lim_{N \to \infty} \| T_N - T \|_{\mathscr{L}(W, V)} = 0 \tag{1 6}$$

*Then there exist a neighborhood $\theta$ of the origin in $V$ and, for $N \geqslant N_0$ large enough, a unique $C^{m+1}$ mapping $\lambda \in \Lambda \to u_N(\lambda) \in V_N$, such that*

$$\forall \lambda \in \Lambda , \quad F_N(\lambda, u_N(\lambda)) = 0 , \quad u_N(\lambda) - u(\lambda) \in \theta \tag{1 7}$$

*Furthermore, there exist some positive constants $K_l$ $(0 \leqslant l \leqslant m)$ independent of $\lambda$ and $N$ such that the following estimates hold*

$$\forall \lambda \in \Lambda, \quad \forall l = 0, \quad , m \quad \| u_N^{(l)}(\lambda) - u^{(l)}(\lambda) \|_V \leqslant$$
$$\leqslant K_l \sum_{k=0}^{l} \{ \| u^{(k)}(\lambda) - \Pi_N u^{(k)}(\lambda) \|_V \tag{1 8}$$
$$+ \| (T_N - T) G^{(k)}(\lambda, u(\lambda), \quad , u^{(k)}(\lambda)) \|_V \} \qquad \square$$

In the estimates (1 8) we use the notation $\phi^{(0)} = \phi$ for any function $\phi$ Moreover, denoting by $\mathscr{L}_k(X, Y)$ the space of all continuous $k$-linear mapping of $X^k$ into $Y$, the operators

$$G^{(k)} \quad \Lambda \times V \times \mathscr{L}_1(\Lambda, V) \times \quad \times \mathscr{L}_k(\Lambda, V) \to \mathscr{L}_k(\Lambda, V')$$

are defined by the recurrence formula

$$G^{(k)}(\lambda, \quad , u^{(k)}) = D_\lambda G^{(k-1)}(\lambda, \quad , u^{(k-1)}) + D_u G^{(k-1)}(\lambda, \quad , u^{(k-1)}) u^{(1)} +$$
$$+ \sum_{j=1}^{k-1} D_{u^{(j)}} G^{(k-1)}(\lambda, \quad , u^{(k-1)}) u^{(j+1)}$$

Let us now define a more general class of problems which approximate (1 2) To this end, let $Z$ be a Banach space such that $V_N \subset Z \subset V$, the later imbedding being continuous We assume that there exists a real number $r \geqslant 0$ such that

$$\forall v \in V_N \quad \| v \|_Z \leqslant C N^r \| v \|_V \tag{1 9}$$

For any $N$ let $G_N \quad \mathbb{R} \times V_N \to V'$ be a mapping, which will " approximate " $G$ in the applications, and define $F_N^* \quad \Lambda \times V_N \to V_N$ by

$$F_N^*(\lambda, u_N) = u_N + T_N G_N(\lambda, u_N) \tag{1 10}$$

For the approximate problem find $u_N \in V_N$ such that

$$F_N^*(\lambda, u_N) = 0, \tag{1 11}$$

the following theorem holds

THEOREM 1 2 *Assume that the hypotheses of Theorem 1 1 hold Moreover assume that for any $\lambda \in \Lambda$, $u(\lambda)$ belong to $Z$ Let $G_N \quad \Lambda \times V_N \to V'$ be a $C^{m+1}$*

*mapping, and assume that there exists a positive increasing function* $K : \mathbb{R}^+ \to \mathbb{R}^+$ *such that*

$$\| D^l G_N[\lambda, v] \|_{\mathscr{L}_l([\mathbb{R}\times Z]^{l-1}, \mathbb{R}\times V; W)} \leqslant K(|\lambda| + \| v \|_Z) \, (1 \leqslant l \leqslant m+1) . \quad (*) \tag{1.12}$$

*Furthermore we assume that* :

$$\lim_{N\to\infty} \operatorname*{Sup}_{\lambda\in\Lambda} \| D_u G[\lambda, \Pi_N u(\lambda)] - D_u G_N[\lambda, \Pi_N u(\lambda)] \|_{\mathscr{L}(V_N, V')} = 0 \tag{1.13}$$

$$\lim_{N\to\infty} \operatorname*{Sup}_{\lambda\in\Lambda} N^r \| F_N^*(\lambda, \Pi_N u(\lambda)) \|_V = 0 , \tag{1.14}$$

$$\lim_{N\to\infty} \operatorname*{Sup}_{\lambda\in\Lambda} \| u(\lambda) - \Pi_N u(\lambda) \|_Z = 0 . \tag{1.15}$$

*Then, for* $N \geqslant N_0$ *large enough and for any* $\lambda \in \Lambda$ *there exist a positive constant* $\bar{K}$ *independent of* $N$ *and* $\lambda$ *and a unique* $C^{m+1}$ *mapping* $\lambda \in \Lambda \to u_N(\lambda) \in V_N$, *such that*

$$F_N^*(\lambda, u_N(\lambda)) = 0 , \quad \| u_N(\lambda) - \Pi_N u(\lambda) \|_V \leqslant \bar{K} N^{-r} . \tag{1.16}$$

*Moreover there exists a positive constant* $K_0$ *independent of* $N$ *and* $\lambda$ *such that*

$$\| u_N(\lambda) - u(\lambda) \|_V \leqslant K_0 \{ \| u(\lambda) - \Pi_N u(\lambda) \|_V + \| (T_N - T) G(\lambda, u(\lambda)) \|_V + \\ + \| T_N(G_N - G)(\lambda, \Pi_N u(\lambda)) \|_V \} . \tag{1.17}$$

*If, in addition, we assume that*

$$\lim_{N\to\infty} \operatorname*{Sup}_{\lambda\in\Lambda} \| u^{(l)}(\lambda) - \Pi_N u^{(l)}(\lambda) \|_Z = 0 , \quad (1 \leqslant l \leqslant m) \tag{1.18}$$

$$\lim_{N\to\infty} \operatorname*{Sup}_{\lambda\in\Lambda} \| u^{(m+1)}(\lambda) - \Pi_N u^{(m+1)}(\lambda) \|_V = 0 \tag{1.19}$$

*then for any* $\lambda \in \Lambda$ *there exist some positive constants* $K_l$, $1 \leqslant l \leqslant m$, *independent of* $N$ *and* $\lambda$, *such that*

$$\| u_N^{(l)}(\lambda) - u^{(l)}(\lambda) \|_V \leqslant K_l \sum_{k=0}^{l} N^{(l-k)r} \{ \| u^{(k)}(\lambda) - \Pi_N u^{(k)}(\lambda) \|_V + \\ + \| (T_N - T) G^{(k)}(\lambda, u(\lambda), \dots, u^{(k)}(\lambda)) \|_V \\ + \| T_N(G^{(k)} - G_N^{(k)})(\lambda, \Pi_N u(\lambda), \dots, \Pi_N u^{(k)}(\lambda)) \|_V \} . \tag{1.20}$$

(*) If $A_1, \dots, A_l$, $B$ are $l+1$ Banach spaces, $\mathscr{L}_l(A_1, A_2, \dots, A_l, B)$ denote the set of all continuous mappings from $A_1 \times \dots \times A_l$ into $B$ which are linear in each variable

*Proof* : Under the hypotheses of Theorem 1.1 the implicit function theorem allows to state that $\lambda \to u(\lambda)$ is a $C^1$ mapping. Moreover, thanks to (1.5) the operator $\Pi_N$ is uniformely bounded in $N$, so we get

$$\forall \lambda, \mu \in \Lambda \quad \| \Pi_N(u(\lambda) - u(\mu)) \|_V \leqslant C \, | \lambda - \mu | . \tag{1.21}$$

To complete the proof we need the following two lemmas.

LEMMA 1.1 : *If* (1.12), ..., (1.15) *and the hypotheses of Theorem* 1.1 *hold, then for* $N \geqslant N_0$ *large enough* $D_u F_N^*[\lambda, \Pi_N u(\lambda)]$ *is an isomorphism of* $V_N$ *which satisfies*

$$\| (\mathrm{Id} + T_N D_u G_N[\lambda, \Pi_N u(\lambda)]) v \|_V \geqslant \frac{\alpha}{2} \| v \|_V \quad \forall v \in V_N \tag{1.22}$$

($\alpha$ *is the constant defined by the assumption* ($H$1)).

*Proof* : Since $\Lambda$ is compact, using ($H$1) and the continuity of the operators $D_u G : \Lambda \times V \to \mathscr{L}(V ; W)$ and $T : W \to V$, we get that there exists $\eta_0 > 0$ such that for any $w \in V$ which verifies $\| w - u(\lambda) \|_V \leqslant \eta_0$ it follows

$$\| (\mathrm{Id} + T D_u G[\lambda, w]) v \|_V \geqslant \frac{3 \alpha}{4} \| v \|_V \quad \forall v \in V . \tag{1.23}$$

Moreover, using the continuity of $\lambda \to u(\lambda)$ we have that there exists $M_0 > 0$ such that for any $N \geqslant M_0$ and any $\lambda \in \Lambda$ we have

$$\| \Pi_N u(\lambda) - u(\lambda) \|_V \leqslant \eta_0 . \tag{1.24}$$

We use the inequality

$$\begin{aligned} \| (\mathrm{Id} + T_N D_u G_N[\lambda, \Pi_N u(\lambda)]) v \|_V \geqslant \| (\mathrm{Id} + T D_u G[\lambda, \Pi_N u(\lambda)]) v \|_V - \\ - \| (T - T_N) (D_u G_N[\lambda, \Pi_N u(\lambda)] v) \|_V - \\ - \| T(D_u(G - G_N) [\lambda, \Pi_N u(\lambda)] v) \| . \end{aligned} \tag{1.25}$$

Thanks to (1.23) and (1.24) we get

$$\| (\mathrm{Id} + T D_u G[\lambda, \Pi_N u(\lambda)]) v \|_V \geqslant \frac{3 \alpha}{4} \| v \|_V \quad \forall N \geqslant M_0 . \tag{1.26}$$

On the other hand, using (1.6), (1.12) for $l = 1$, and (1.15) it follows that there exists $M_1 > 0$ such that for any $N \geqslant M_1$ and any $\lambda \in \Lambda$ we have

$$\| (T - T_N) (D_u G_N[\lambda, \Pi_N u(\lambda)] v) \|_V \leqslant \frac{\alpha}{8} \| v \|_V ; \tag{1.27}$$

finally, using (1.13) and the continuity of $T$ we get that there exists $M_2 > 0$ such that for any $N \geqslant M_2$ and any $\lambda \in \Lambda$ we have

$$\| T(D_u(G - G_N)[\lambda, \Pi_N u(\lambda)] v) \|_V \leqslant \frac{\alpha}{8} \| v \|_V . \tag{1.28}$$

Now we obtain (1.22) from (1.25), ... (1 28). Now the proof is complete since $V_N$ is finite dimensional. □

LEMMA 1.2 : *If* (1.12), ..., (1.15) *and the hypotheses of theorem* 1.1 *hold, we get*

$$\sup_{\lambda \in \Lambda} \| D_\lambda F_N^*[\lambda, \Pi_N u(\lambda)] \|_{\mathscr{L}(\mathbb{R} \times V_N, V_N)} \leqslant C \tag{1.29}$$

*($V_N$ is equipped with the norm of $V$). Moreover, there exists an increasing function $K_1 : \mathbb{R}^+ \to \mathbb{R}^+$ such that, if*

$$\{ \mu, v \} \in \Lambda \times V_N \quad and \quad N^r(| \lambda - \mu | + \| \Pi_N u(\lambda) - v \|_V) \leqslant \xi , \tag{1.30}$$

*then*

$$\| DG_N[\lambda, \Pi_N u(\lambda)] - DG_N[\mu, v] \|_{\mathscr{L}(\mathbb{R} \times V, W)} \leqslant K_1(\xi) (| \lambda - \mu | + \\ + \| \Pi_N u(\lambda) - v \|_Z) . \tag{1.31}$$

*Furthermore, if* (1.18) *and* (1.19) *are satisfied, there exists an increasing function $K_2 : \mathbb{R}^+ \to \mathbb{R}^+$ such that, if* (1.30) *holds, we get*

$$\| D^l G_N[\lambda, \Pi_N u(\lambda)] - D^l G_N[\mu, v] \|_{\mathscr{L}_l((\mathbb{R} \times Z)^{l-1}, \mathbb{R} \times V, W)} \leqslant \\ \leqslant K_2(\xi) (| \lambda - \mu | + \| \Pi_N u(\lambda) - v \|_V) \quad (2 \leqslant l \leqslant m) . \tag{1.32}$$

*Proof* : Using (1 15), (1.6) and (1.12) with $l = 1$ we get immediately (1 29) Next from (1.12) it follows that

$$\forall \{ \mu, v \} \in \Lambda \times V_N \; \| D^2 G_N[\mu, v] \|_{\mathscr{L}_2(\mathbb{R} \times Z, \mathbb{R} \times V, W)} \leqslant K(| \mu | + \| v \|_Z) . \tag{1.33}$$

Since

$$\frac{d}{dt} (DG_N[\lambda + t(\mu - \lambda), \Pi_N u(\lambda) + t(v - \Pi_N u(\lambda))]) (t_0) = \\ = D_\lambda DG_N[(1 - t_0) \lambda + t_0 \mu, (1 - t_0) \Pi_N u(\lambda) + t_0 v] \cdot (\mu - \lambda) + \\ + D_u DG_N[(1 - t_0) \lambda + t_0 \mu, (1 - t_0) \Pi_N u(\lambda) + t_0 v] \cdot (v - \Pi_N u(\lambda)) , \\ \forall t_0 \in \mathbb{R} .$$

Applying the mean value theorem to

$$DG_N[\lambda + t(\mu - \lambda), \quad \Pi_N u(\lambda) + t(v - \Pi_N u(\lambda))],$$

and using (1 9), (1 30) and (1 33) we have

$$\| DG_N[\lambda, \Pi_N u(\lambda)] - DG_N[\mu, v] \|_{\mathscr{L}(\mathbb{R} \times V\, W)} \leqslant$$
$$\leqslant K(|\lambda| + \| \Pi_N u(\lambda) \|_Z + \xi) |\lambda - \mu| + \| \Pi_N u(\lambda) - v \|_Z \quad (1\ 34)$$

Therefore (1 31) holds taking $K_1(\xi) = K(|\lambda| + \| \Pi_N u(\lambda) \|_Z + \xi)$

Finally, arguing in a similar way and using (1 12) with $l = 3, \quad , m + 1$ the property (1 32) can be proved □

Let us go back to the proof of Theorem 1 2 Due to (1 21), (1 22), (1 29) and (1 31), we can apply Theorem 1 of [4] to the mapping $F_N^*$ in the following situation

the space $X$ of the theorem is $\mathbb{R}$ provided with the norm $N^r |\lambda|$,
the space $Y$, $Z$ of the theorem are $V_N$ provided with the norm $N^r \| v \|_V$,
finally $y(\lambda)$ becomes $\Pi_N u(\lambda)$

We note that for all mapping $A_N \in \mathscr{L}(\mathbb{R} \times V_N, V_N)$, we have

$$\| A_N \|_{\mathscr{L}(X \times Y\, Z)} = \| A_N \|_{\mathscr{L}(\mathbb{R} \times V_N\, V_N)}$$

Then by the above mentioned result, for $N \geqslant N_0$ large enough there exists a constant $\overline{K} > 0$, independent of $N$, and a unique $C^1$ mapping $\lambda \in \Lambda \to u_N(\lambda) \in V_N$ such that (1 16) holds, together with the inequality

$$\| u_N(\lambda) - \Pi_N u(\lambda) \|_V \leqslant C \| F_N^*(\lambda, \Pi_N u(\lambda)) \|_V, \quad \forall \lambda \in \Lambda \quad (1\ 35)$$

Using (1 1), (1 2) and (1 10), we have

$$\| F_N^*(\lambda, \Pi_N u(\lambda)) \|_V = \| F_N^*(\lambda, \Pi_N u(\lambda)) - F(\lambda, u(\lambda)) \|_V \leqslant$$
$$\leqslant \| \Pi_N u(\lambda) - u(\lambda) \|_V + \| (T_N - T) G(\lambda, u(\lambda)) \|_V +$$
$$+ \| T_N(G_N - G)(\lambda, \Pi_N u(\lambda)) \|_V + \| T_N(G(\lambda, \Pi_N u(\lambda)) - G(\lambda, u(\lambda))) \|_V, \quad (1\ 36)$$

using (1 6), (1 5) and the differentiability of $G$ we obtain

$$\| T_N(G(\lambda, \Pi_N u(\lambda)) - G(\lambda, u(\lambda))) \|_V \leqslant C \| G(\lambda, \Pi_N u(\lambda)) - G(\lambda, u(\lambda)) \|_W \leqslant$$
$$\leqslant C \| \Pi_N u(\lambda) - u(\lambda) \|_V \quad (1\ 37)$$

Now (1 17) is a consequence of (1 35), (1 36) and (1 37)

In order to prove (1 20) we apply the theorem 2 of [4] to $F_N^*$, indeed, from (1 9) we have

$$\forall A_N \in \mathscr{L}_l(\mathbb{R} \times V_N ; V_N) \quad \| A_N \|_{\mathscr{L}_l(X \times Y, Z)} \leqslant C \| A_N \|_{\mathscr{L}_l((\mathbb{R} \times Z)^{l-1}, \mathbb{R} \times V, W)},$$

and

$$\forall v \in V_N \quad \| v \|_{\mathscr{L}_l(X,Y)} \equiv \| v \|_{\mathscr{L}_l(X,Z)} \leqslant C N^{(1-l)r} \| v \|_V .$$

Due to (1 18) and (1 19), using the above mentioned theorem we get the inequality

$$\| u_N^{(l)}(\lambda) - u^{(l)}(\lambda) \|_V \leqslant C \sum_{k=0}^{l} N^{(l-k)r} \| F_N^{*(k)}(\lambda, \Pi_N u(\lambda), \ldots, \Pi_N u^{(k)}(\lambda)) \|_V \tag{1.38}$$

for $l = 1, \ldots, m$ and for any $\lambda \in \Lambda$. Finally we can obtain (1.20) from (1.38), using (1.6), (1.15), (1.18), (1.19), the hypotheses that $G$ is a $C^{m+1}$ mapping, and the identities

$$F^{(k)}(\lambda, u(\lambda), \ldots, u^{(k)}(\lambda)) \equiv 0 \quad \text{for} \quad k = 0, \ldots, m . \qquad \square$$

### 1.2. Error estimates in lower order norms

Let $Y, H, K$ be three Banach spaces, equipped with the norms $\| \cdot \|_Y$, $\| \cdot \|_H$ and $\| \cdot \|_K$ respectively, such that the following imbeddings hold

$$K \hookrightarrow V \hookrightarrow H \hookrightarrow Y .$$

Moreover, assume that $T$ can be extended to a compact operator from $Y$ into $H$, and that $D_u G[\lambda, v]$ can be extended to $D_u G[\lambda, v] \in \mathscr{L}(H ; Y)$.

THEOREM 1.3 : *Let the hypotheses of Theorem 1.2 hold, and let $u_N$ be the solution of (1.11). Assume that for any $\lambda$ in $\Lambda$ :*

*the mapping $v \in K \to D_u G[\lambda, v] \in \mathscr{L}(H ; Y)$ is continuous ;* (1.39)

*the mapping $D_u F[\lambda, u(\lambda)]$ is an isomorphism of $H$ ;* (1.40)

$$\lim_{N \to \infty} \| T - T_N \|_{\mathscr{L}(Y,H)} = 0 \tag{1.41}$$

*$u(\lambda)$ and $u_N(\lambda)$ belong to $K$ and $\| u_N(\lambda) - u(\lambda) \|_K$ tends to zero with $1/N$ .* (1.42)

*Then, for N large enough the following estimate holds*

$$\forall\lambda \in \Lambda \quad \| u(\lambda) - u_N(\lambda) \|_H \leqslant C \{ \| F_N(\lambda, u(\lambda)) \|_H + \\ + \| T_N(G - G_N)(\lambda, u_N(\lambda)) \|_H \} \quad (1.43)$$

*Remark* 1.1 : From (1.43) we derive also that

$$\forall\lambda \in \Lambda \quad \| u(\lambda) - u_N(\lambda) \|_H \leqslant C \{ \| (T_N - T) G(\lambda, u(\lambda)) \|_H + \\ + \| (G - G_N)(\lambda, u_N(\lambda)) \|_Y \} . \qquad \square$$

*Proof* : Since $F_N^*(\lambda, u_N(\lambda)) = 0$, we have

$$\begin{aligned} F_N(\lambda, u(\lambda)) &= F_N(\lambda, u(\lambda)) - F_N^*(\lambda, u_N(\lambda)) \\ &= D_u F[\lambda, u(\lambda)] (u(\lambda) - u_N(\lambda)) + \\ &+ (T_N - T) D_u G[\lambda, u(\lambda)] (u(\lambda) - u_N(\lambda)) + T_N(G(\lambda, u(\lambda)) \\ &- G(\lambda, u_N(\lambda)) - D_u G[\lambda, u(\lambda)] (u(\lambda) - u_N(\lambda))) \\ &+ T_N(G(\lambda, u_N(\lambda)) - G_N(\lambda, u_N(\lambda))) . \end{aligned} \quad (1.44)$$

Let us examine each term of the right hand side.

First, using (1.40) we deduce that there exists a positive constant $\beta$ such that

$$\| D_u F[\lambda, u(\lambda)] (u(\lambda) - u_N(\lambda)) \|_H \geqslant \beta \| u(\lambda) - u_N(\lambda) \|_H . \quad (1.45)$$

Next, thanks to (1.39) and (1.41) we get

$$\| (T_N - T)(D_u G[\lambda, u(\lambda)]) (u(\lambda) - u_N(\lambda)) \|_H \leqslant \varepsilon(N) \| u(\lambda) - u_N(\lambda) \|_H \quad (1.46)$$

where $\varepsilon(N)$ tends to 0 with $1/N$.

On the other hand, setting $u_\theta(\lambda) = \theta u_N(\lambda) + (1 - \theta) u(\lambda)$, for any $\theta \in [0, 1]$, and for any $\lambda \in \Lambda$, we have

$$G(\lambda, u(\lambda)) - G(\lambda, u_N(\lambda)) - D_u G[\lambda, u(\lambda)] (u(\lambda) - u_N(\lambda)) = \\ = \int_0^1 (D_u G[\lambda, u_\theta(\lambda)] - D_u G[\lambda, u_\theta(\lambda)]) \, d\theta (u(\lambda) - u_N(\lambda)) .$$

Then by (1 39) and (1 41) it follows

$$\| T_N \{ G(\lambda, u(\lambda)) - G(\lambda, u_N(\lambda)) - D_u G[\lambda, u(\lambda)] (u(\lambda) - u_N(\lambda)) \} \|_H \leqslant$$
$$\leqslant C\varepsilon (N) \| u(\lambda) - u_N(\lambda) \|_H \quad (1\ 47)$$

where

$$\varepsilon'(N) = \left\| \int_0^1 (D_u G[\lambda, u_\theta(\lambda)] - D_u G[\lambda, u]) \, d\theta \right\|_{\mathscr{L}(H\ Y)}$$

tends to zero with $1/N$ due to (1 39) and to (1 42) Finally (1 43) holds from (1 44), , (1 47) taking $C = (\beta - \varepsilon(N) - \varepsilon'(N))^{-1}$ □

COROLLARY 1 1 *Assume that* (1 39), (1 41), (1 42) *hold, together with the hypotheses of Theorem* 1 2 *and with the following regularity assumption*

$$\text{if } v \in H \text{ verifies } v + TD_u G[\lambda, u(\lambda)] \, v = 0 \text{ then } v \in V \qquad (1\ 48)$$

*Then* (1 43) *holds*

*Proof* To check that (1 40) holds let us note that $D_u F[\lambda, u(\lambda)]$ is a compact operator of $\mathscr{L}(H, H)$ Then by the Fredholm alternative we only need to check that

$$\text{if} \quad D_u F[\lambda, u(\lambda)] \, v = 0 \quad \text{then} \quad v = 0 \qquad (1\ 49)$$

Besides that (1 49) follows easily from (1 48) and hypothesis ($H$ 1) □

### 1.3. The Newton method to solve the approximate problem

In this section $\lambda$ is considered to be fixed

We assume that $G_N \quad \Lambda \times V_N \to V$ is a $C^2$ mapping, and that there exists a positive constant $\delta$ independent of $N$ such that for any $\lambda \in \Lambda$

$$\| T_N D_u^2 G_N[\lambda, u_N(\lambda)] \|_{\mathscr{L}_2(V_N\ V_N)} \leqslant \delta \,, \qquad (1\ 50)$$

There exists a mapping $\varepsilon \quad \mathbb{R}^+ \to \mathbb{R}^+$ such that $\varepsilon(x)/x$ vanishes when $x$ tends to zero, and for any $\lambda \in \Lambda$, for any $v \in V_N$ the following estimates hold

$$\| T_N(D_u G_N[\lambda, v] - D_u G_N[\lambda, u_N(\lambda)] - D_u^2 G_N[\lambda, u_N(\lambda)] (v - u_N(\lambda))) \|_{\mathscr{L}(V\ V)} \leqslant$$
$$\leqslant \varepsilon(\| v - u_N(\lambda) \|_V) \,, \quad (1\ 51)$$

$$\| T_N(G_N(\lambda, v) - G_N(\lambda, u_N(\lambda)) - D_u G_N[\lambda, u_N(\lambda)] (v - u_N(\lambda)) -$$
$$- D_u^2 G_N[\lambda, u_N(\lambda)] (v - u_N(\lambda))^2) \|_V \leqslant \varepsilon(\| v - u_N(\lambda) \|_V^2) \quad (1\ 52)$$

Arguing as in the proof of Lemma 1 1 we can establish the following result

LEMMA 1 3 *Assume that the hypotheses* (1 12), , (1 15), (1 48) *and those of Theorem* 1 1 *hold Let* $\alpha$ *be the constant defined by the property* ($H$1) *There exists a constant* $\eta > 0$ *such that for* $N \geqslant N_2$ *large enough, and for any* $v \in V_N$ *which satisfies* $\| v - \Pi_N u(\lambda) \|_V \leqslant \eta$, *the mapping* $D_u F_N[\lambda, v]$ *is an isomorphism of* $V_N$, *thus*

$$\| D_u F_N[\lambda, v] w \|_V \geqslant \frac{\alpha}{4} \| w \|_V \quad \forall w \in V_N \tag{1 53}$$

*Moreover it follows that*

$$\varepsilon(x)/x \leqslant 1 \quad \forall x \leqslant \eta \qquad \square \tag{1 54}$$

Let $v^0 \in V_N$ be given and consider the following Newton scheme find $v^{n+1} \in V_N$ ($n \geqslant 0$) by solving

$$D_u F_N[\lambda, v^n] v^{n+1} = D_u F_N[\lambda, v^n] v^n - F_N(\lambda, v^n) \tag{1 55}$$

THEOREM 1 4 *Assume that the hypotheses of the previous lemma hold Moreover, assume that if* $N \geqslant N_3$

$$\| u_N(\lambda) - \Pi_N u(\lambda) \|_V < \eta/2 \tag{1 56}$$

*Then, there exists* $\rho > 0$ *such that, if* $v^0$ *satisfies*

$$\| v^0 - u_N(\lambda) \|_V \leqslant \rho\,, \tag{1 57}$$

*the Newton iterates* ($v^n$) *are univocally defined by* (1 55) *and converge quadratically to the solution* $u_N(\lambda)$ *of* (1 11)

*Proof* As $\lambda$ is fixed we shall drop any dependence on it along this proof Denoting by $\rho$ the minimum between $\eta/2$ and $\alpha/(4(\delta + 2))$, from (1 56) and (1 57) it follows that $\| v^0 - \Pi_N u \|_V \leqslant \eta$

Then by Lemma 1 3 the first iterate $v^1$ of (1 55) is univocally defined By induction on $n$ we shall prove that for any $n$ $\| v^{n+1} - u_N \|_V \leqslant \rho$ As a matter of fact, assume that $\| v^n - u_N \|_V \leqslant \rho$ By (1 56) we get that $\| \Pi_N u - v^n \|_V \leqslant \eta$ Taking $v = v^n$ and $w = v^{n+1} - u_N$ in (1 53) gives

$$\| (\mathrm{Id} + T_N D_u G_N[v^n]) (v^{n+1} - u_N) \|_V \geqslant \frac{\alpha}{4} \| v^{n+1} - u_N \|_V \tag{1 58}$$

On the other hand, from (1.55), (1.11), (1.51), (1.52) and (1.54) we get

$$\begin{aligned}\|(\mathrm{Id} + T_N D_u G_N[v^n])(v^{n+1} - u_N)\|_V &= \| T_N(D_u G_N[v^n] v^n) - \\ &\quad - T_N G_N(v^n) - D_u F_N^*[v^n] u_N \|_V \leqslant \\ &\quad \leqslant \| T_N D_u^2 G_N[u_N](v^n - u_N)^2 \|_V + 2 \| v^n - u_N \|_V^2 .\end{aligned}$$

Finally, by (1.50) and (1.58) it follows that

$$\| v^{n+1} - u_N \|_V \leqslant \frac{4(\delta + 2)}{\alpha} \| v^n - u_N \|_V^2 \tag{1.59}$$

and therefore $\| v^{n+1} - u_N \|_V \leqslant \rho$. Then all the Newton iterates are univocally defined by (1.55), and, due to (1.59), they converge quadratically to $u_N$. $\square$

## 2. THE BURGERS' EQUATION : PRELIMINARIES

We denote by $I$ the interval $(-1, 1)$ and by $x$ the current variable of $I$. We consider two weight functions : $\omega(x) \equiv 1$ (Legendre weight), and

$$\omega(x) = (1 - x^2)^{-1/2}$$

(Chebyshev weight). We make use in this paper of the weighted Sobolev spaces $H_\omega^s(I)$. They are defined as follows : for $s = 0$ we set

$$H_\omega^0(I) \equiv L_\omega^2(I) = \{ \phi : I \to \mathbb{R} \mid \phi \text{ is measurable and } (\phi, \phi)_\omega < +\infty \} \tag{2.1}$$

where $(\phi, \psi)_\omega = \int_I \phi(x)\psi(x)\omega(x)\,dx$ denotes the inner product of $L_\omega^2(I)$. For any integer $s > 0$, we set

$$H_\omega^s(I) = \{\phi \in L_\omega^2(I) \mid D^k \phi \in L_\omega^2(I), \quad 0 \leqslant k \leqslant s\}, \tag{2.2}$$

where $D = d/dx$; $H_\omega^s(I)$ is equipped with the following norm

$$\| \phi \|_{s,\omega}^2 = \sum_{k=0}^{s} \int_I (D^k \phi)^2 (x)\,\omega(x)\,dx .$$

For any real, non integral $s$, the space $H_\omega^s(I)$ is defined by the complex interpolation method (see e.g. [3, Ch. 4]). For any integer $s > 0$ we denote by $H_{0,\omega}^s(I)$ the closure of $\mathcal{D}(I)$ into $H_\omega^s(I)$; finally for non integer $s > 0$ we define $H_{0,\omega}^s(I)$ by interpolation. If $\omega \equiv 1$ the spaces $H_\omega^s(I)$ and $H_{0,\omega}^s(I)$ coincide with the classical Sobolev spaces $H^s(I)$ and $H_0^s(I)$ respectively, provided $s \notin \mathbb{N} + \frac{1}{2}$ (see e g. [3, 10]). For $\omega(x) = (1 - x^2)^{-1/2}$ some properties of spaces $H_\omega^s(I)$ have been given in [11], we shall constantly refer to them along this paper The same results for the weight $\omega \equiv 1$ are well known, and we still refer to [1, 10] for the proofs

Let $\varepsilon$ be a positive real number, and $f$ be a given function of $L_\omega^2(I)$.

We consider the following problem : find $u \in H_{0,\omega}^1(I)$ solution of

$$- \varepsilon u_{xx} + u u_x = f \quad \text{in } I . \tag{2.3}$$

Correspondingly to Legendre's and Chebyshev's weight $\omega$ we set

$$V = H_{0,\omega}^1(I) , \quad W = \text{dual space of } H_{0,\omega}^{3/4}(I) , \tag{2.4}$$

and we introduce the bilinear form $c : V \times V \to \mathbb{R}$ defined by

$$c(u, v) = \int_I u_x (v\omega)_x \, dx . \tag{2 5}$$

LEMMA 2 1 : *There exist three positive constants $\alpha$, $\beta$, $\gamma$ such that for any $u \in H_\omega^1(I)$ and $v \in V$ we have*

$$\| v \|_{0,\omega} \leqslant \alpha \| v_x \|_{0,\omega} \quad (\textit{Poincaré's inequality}) \tag{2 6}$$

$$c(v, v) \geqslant \beta \| v \|_{1,\omega}^2 \tag{2.7}$$

$$| c(u, v) | \leqslant \gamma \| u_x \|_{0,\omega} \| v_x \|_{0,\omega} . \quad \square \tag{2.8}$$

If $\omega \equiv 1$ the above results are well known : note that $c(., .)$ is the classical $H_0^1$ inner product. For $\omega(x) = (1 - x^2)^{-1/2}$, (2.6), (2 7) and (2 8) have been proved in [5] Thanks to this lemma, the norm defined by $c(v, v)^{1/2}$, $\forall v \in V$, is equivalent to $\| v \|_{1 \omega}$ (we note that if $\omega$ is not constant $c(., .)$ is not an inner product since it is not symmetric).

Let us define the linear operator $T : V' \to V$ by

$$c(Tg, \phi) = \langle g, \phi \rangle \quad \forall \phi \in V ; \tag{2.9}$$

we recall that (see [11, Remark 1.2 and Theorem 2 4]) for all $s \in [-1, 0[$

$$T \text{ is continuous from } H_{0,\omega}^{-s}(I)' \text{ into } V \cap H_\omega^{2+s}(I) . \tag{2.10}$$

We define also the mapping $G : \mathbb{R} \times V \to W$ by

$$G(\lambda, u) = \lambda(u u_x - f) ; \tag{2 11}$$

we note that $G$ is a $C^{\infty}$ mapping, and for any $k \in \mathbb{N}$, $D^k G$ is bounded over any bounded subset of $\mathbb{R} \times V$. Moreover, since $W$ is obtained by interpolation between $V'$ and $L^2_\omega(I)$, it contains topologically $L^2_\omega(I)$, so we have

$$\| G(\lambda, u) \|_W \leqslant \| G(\lambda, u) \|_{0,\omega} .$$

then, since $V \subset L^{\infty}(I)$ (see [11, Theorem 2.2]) we get

$$\| G(\lambda, u) \|_W \leqslant | \lambda | \; \| uu_x - f \|_{0,\omega} \leqslant C_1 | \lambda | (\| u \|_{L\infty(I)} \| u \|_{1,\omega} + \| f \|_{0,\omega}) \leqslant$$
$$\leqslant C | \lambda | (\| u \|^2_{1,\omega} + \| f \|_{0,\omega}) . \quad (2.12)$$

In addition we get that

$$T \text{ is a compact operator from } W \text{ into } V . \quad (2.13)$$

This property follows easily from (2.10); indeed, $T$ maps continuously $W$ into $H^{5/4}_\omega(I) \cap V$, which in turn is compactly imbedded into $V$ (see [11, Theorem 2.1]) so (2.13) holds.

Let us set $F : \mathbb{R} \times V \to V$,

$$F(\lambda, u) = u + TG(\lambda, u) ; \quad (2.14)$$

the problem (2.3) can be equivalently written as follows : find $u \in V$ such that

$$F(\lambda, u) = 0 , \quad (2.15)$$

where $\lambda = 1/\varepsilon$.

It can be easily seen that problem (2.3) admits a unique solution. Thus for any compact subset $\Lambda$ of $\mathbb{R}^+$ the branch $\{ \{ \lambda, u(\lambda) \}, \lambda \in \Lambda \}$ is non singular, i.e. it satisfies condition $(H1)$. So in this paper $\Lambda$ will denote a generic compact interval of $\mathbb{R}^+$.

## 3. APPROXIMATIONS BY PSEUDO-SPECTRAL METHODS : STABILITY AND CONVERGENCE

Let us denote by $\{ p_n \}_{n=0}^{\infty}$ the family of polynomials which are orthogonal with respect to the $L^2_\omega(I)$ inner product $(.,.)_\omega$. It is well known (see e.g. [14]) that if $\omega \equiv 1$ we have $p_n = \lambda_n L_n$, where $\lambda_n = ((2n+1)/2)^{1/2}$ and $L_n$ is the $n$-th degree Legendre polynomial. If $\omega(x) = (1 - x^2)^{-1/2}$ then $p_n = \tau_n T_n$, with $\tau_0 = (1/\pi)^{1/2}$, $\tau_n = \sqrt{2}\, \tau_0$ if $n \geqslant 1$, and $T_n$ is the $n$-th degree Chebyshev polynomial of the first kind.

We denote by $F^{GL}_{\omega,N} = \{ (x_j, \omega_j) \} \mid 0 \leqslant j \leqslant N \}$ the Gauss-Lobatto integration formula relatively to the weight $\omega$, with nodes

$$-1 = x_0 < x_1 < \cdots < x_N = 1$$

and weights $\omega_j > 0$ (see e.g. [7]). Then we have

$$\forall g \in \mathbb{P}_{2N-1}(I) \quad \int_I g(x)\,\omega(x)\,dx = \sum_{j=0}^{N} g(x_j)\,\omega_j \tag{3.1}$$

where $\mathbb{P}_m(I)$ denotes the space of polynomials of degree $\leqslant m$ over $I$.

We introduce a bilinear form over $C^0(I)$ by setting

$$(\phi, \psi)_{N,\omega} = \sum_{j=0}^{N} \phi(x_j)\,\psi(x_j)\,\omega_j\,, \tag{3.2}$$

and an interpolation operator $P_c : C^0(\bar{I}) \to \mathbb{P}_N(I)$ defined by

$$(P_c\, u)(x_j) = u(x_j)\,, \quad 0 \leqslant j \leqslant N\,. \tag{3.3}$$

It is easy to check that for any $u \in C^0(\bar{I})$, we have

$$P_c\, u = \sum_{k=0}^{N} \tilde{u}_k\, p_k\,, \quad \tilde{u}_N = \frac{(u, p_N)_{N,\omega}}{(p_N, p_N)_{N,\omega}}\,, \quad \tilde{u}_k = (u, p_k)_{N,\omega}\,, \quad k \leqslant N-1\,. \tag{3.4}$$

Using (3.2) and (3.3) we also have that

$$\forall u, \phi \in C^0(\bar{I}) \quad (P_c\, u, \phi)_{N,\omega} = (u, \phi)_{N,\omega} \tag{3.5}$$

$$\forall \phi \in \mathbb{P}_N(I)\,, \quad \forall \psi \in \mathbb{P}_{N-1}(I) \quad (\phi, \psi)_{N,\omega} = (\phi, \psi)_\omega\,. \tag{3.6}$$

Following [6], the triple $(I, F^{GL}_{\omega,N}, P_c)$ is called a *Legendre* (or *Chebyshev*) *spectral interpolation system*, according that $\omega \equiv 1$ (or $\omega(x) = (1 - x^2)^{-1/2}$, respectively).

To approximate (2.3) we introduce the following pseudo-spectral problem : find $u_N \in V_N$ such that

$$\forall \phi \in V_N \; -(u_{N_{xx}}, \phi)_{N,\omega} + \frac{\lambda}{2}([P_c\, u_N^2]_x, \phi)_{N,\omega} = \lambda(f, \phi)_{N,\omega} \tag{3.7}$$

where $V_N = \{\, \phi \in \mathbb{P}_N(I) \mid \phi(-1) = \phi(1) = 0 \,\}$.

*Remark* 3.1 : For any $j = 1, \ldots, N-1$ let $\phi_j$ denote the function of $V_N$ defined by : $\phi_j(x_k) = \delta_{jk}$, $k = 0, \ldots, N$. Then from (3.7) we get

$$\begin{cases} -\, u_{N_{xx}}(x_j) + \dfrac{\lambda}{2}[P_c\, u_N^2]_x\,(x_j) = \lambda f(x_j)\,, & 1 \leqslant j \leqslant N-1 \\ u_N(x_0) = u_N(x_N) = 0\,. & \end{cases} \tag{3.8}$$

Conversely, multiplying the first equation of (3 8) by $\phi(x_j)\,\omega_j$ ($\phi \in V_N$) and adding over $j$ from 0 up to $N$ we get that $u_N$ satisfies (3 7) Hence (3 7) and (3 8) are equivalent

Since $\frac{1}{2}(u_N^2)_x = u_N\, u_{N_x}$, (3 8) should be a standard *collocation method* for (2 3) at the nodes $x_j$, $1 \leqslant j \leqslant N-1$, if $(u_N^2)_x$ was used instead of $[P_c\, u_N^2]_x$ On the other hand, to interpolate before making derivatives is one of the features of pseudo-spectral methods since it is quite easy to implement successfully this process (see e g [8, 9]) Finally, we note that for defining correctly (3 7), $f$ must be continuous, and for that it is enough to require that $f \in H_\omega^s(I)$, for some $s > \frac{1}{2}$ ([11, Theorem 2 2]) For ease of exposition only, we shall assume that $f$ belongs to $H_\omega^1(I)$ $\square$

We want to develop the analysis of the pseudo-spectral problem (3 7) in the abstract framework of Section 1 For that we define the operator $T_N \quad V' \to V_N$ by

$$\forall \phi \in V_N \quad c(T_N\, g, \phi) = \langle g, \phi \rangle\,, \tag{3 9}$$

moreover we define $\Pi_N \quad V \to V_N$ by

$$\forall \phi \in V_N \quad c(\Pi_N\, v, \phi) = c(v, \phi) \tag{3 10}$$

Using (2 9) we get immediately

$$T_N = \Pi_N \circ T \tag{3 11}$$

Let us recall the following result which holds for both Legendre and Chebyshev weights (see [11], Theorems 1 1 and 1 4])

$$\forall u \in H_\omega^\sigma(I) \cap V\,,\ \sigma \geqslant 1\,, \quad \| u - \Pi_N\, u \|_{\mu\,\omega} \leqslant C N^{\mu-\sigma} \| u \|_{\sigma\,\omega} \quad 0 \leqslant \mu \leqslant 1 \tag{3 12}$$

Thanks to (3 12) and using density arguments we can show that

$$\forall v \in V \lim_{N \to \infty} \| v - \Pi_N\, v \|_{1\,\omega} = 0 \tag{3 13}$$

Moreover it can be proved that

$$\lim_{N \to \infty} \| T - T_N \|_{\mathscr{L}(W\, V)} = 0 \tag{3 14}$$

From (2.4), (2.10), (3.11) and (3.12) we have

$$\forall g \in W, \quad \| (T - T_N) g \|_{1,\omega} = \| (\mathrm{Id} - \Pi_N) Tg \|_{1,\omega} \leqslant CN^{-1/4} \| Tg \|_{5/4,\omega} \leqslant$$
$$\leqslant CN^{-1/4} \| g \|_W ,$$

so that $\| T - T_N \|_{\mathscr{L}(W,V)} \leqslant CN^{-1/4}$ and (3.14) follows. Finally for any $\lambda \in \Lambda$ we set

$$\forall v \in V_N, \ \forall \phi \in V \quad \langle G_N(\lambda, v), \phi \rangle = \frac{\lambda}{2} ([P_c v^2]_x, \phi)_\omega - \lambda(f, \phi)_{N,\omega} . \quad (3.15)$$

LEMMA 3.1 : *For any $\lambda \in \mathbb{R}$ and $v \in V_N$ the operator $G_N(\lambda, v)$ belongs to $V'$.*

*Proof* : The linearity is obvious by definition, so let us check the continuity.

(i) We start by proving the following inequality

$$\forall z \in L^2_\omega(I), \ \forall \phi \in V \quad \left| \int_I z(\phi\omega)_x \, dx \right| \leqslant C \| z \|_{0,\omega} \| \phi \|_{1,\omega} . \quad (3.16)$$

If $\omega \equiv 1$ it is a consequence of the Cauchy-Schwarz inequality. Otherwise we set $w(x) = \int_{-1}^{x} z(\xi) \, d\xi$ ; clearly $w \in H^1_\omega(I)$ and by (2.8), we have

$$\left| \int_I z(\phi\omega)_x \, dx \right| = \left| \int_I w_x(\phi\omega)_x \, dx \right| \leqslant \gamma \| w_x \|_{0,\omega} \| \phi \|_{1,\omega}$$

and (3.16) holds.

(ii) At this sep we want to evaluate the quantities

$$E(\chi, \phi) = | (\chi, \phi)_{N,\omega} - (\chi, \phi)_\omega |$$

for any $\chi, \phi \in C^0(\bar{I})$. Let us recall the following result : setting

$$\| \phi \|^2_{N,\omega} = (\phi, \phi)_{N,\omega} ,$$

we have [6]

$$\forall \phi \in \mathbb{P}_N(I) \quad \frac{1}{2} \| \phi \|_{N,\omega} \leqslant \| \phi \|_{0,\omega} \leqslant \| \phi \|_{N,\omega} . \quad (3.17)$$

Moreover we get (see [6, Theorems 3.1 and 3.2]) for any $u \in H^\sigma_\omega(I)$ with $\sigma > \frac{1}{2}$, and for any $\mu \in [0, \sigma]$

$$\| u - P_c u \|_{\mu,\omega} \leqslant C \| u \|_{\sigma,\omega} \begin{cases} N^{2\mu - \sigma} & \text{if} \quad \omega(x) = (1 - x^2)^{-1/2} \\ N^{2\mu - \sigma + 1/2} & \text{if} \quad \omega \equiv 1 . \end{cases} \quad (3.18)$$

Let us denote by $\Pi_{0\,N}$ $L^2_\omega(I) \to \mathbb{P}_N(I)$ the $L^2_\omega$ projection operator upon $\mathbb{P}_N(I)$, i e

$$\forall \phi \in \mathbb{P}_N(I) \quad (u - \Pi_{0\,N}\, u, \phi)_\omega = 0 \tag{3 19}$$

We have (see [6, Theorems 2 1 and 2 3]) for any $u \in H^\sigma_\omega(I)$ with $\sigma \geqslant 0$

$$\| u - \Pi_{0\,N}\, u \|_{0\,\omega} \leqslant CN^{-\sigma} \| u \|_{\sigma\,\omega} \tag{3 20}$$

First we assume that $\phi \in \mathbb{P}_N(I)$ Using (3 5) and (3 6) we have

$$\begin{aligned} E(\chi, \phi) &= | (P_c\, \chi, \phi)_{N\,\omega} - (\Pi_{0\,N-1}\, \chi, \phi)_{N\,\omega} + (\Pi_{0\,N-1}\, \chi, \phi)_\omega - (\chi, \phi)_\omega | \\ &= | ((P_c - \Pi_{0\,N-1})\, \chi, \phi)_{N\,\omega} + (\Pi_{0\,N-1}\, \chi - \chi, \phi)_\omega | \leqslant \\ &\leqslant \| (P_c - \Pi_{0\,N-1})\, \chi \|_{N\,\omega} \| \phi \|_{N\,\omega} + \| \Pi_{0\,N-1}\, \chi - \chi \|_{0\,\omega} \| \phi \|_{0\,\omega}, \end{aligned} \tag{3 21}$$

then using (3 17) we have

$$\begin{aligned} &\forall \chi \in C^0(\bar{I}),\ \forall \phi \in \mathbb{P}_N(I), \quad | (\chi, \phi)_{N\,\omega} - (\chi, \phi)_\omega | \leqslant \\ &\qquad \leqslant C \| \phi \|_{0\,\omega} (\| \chi - P_c\, \chi \|_{0\,\omega} + \| \chi - \Pi_{0\,N-1}\, \chi \|_{0\,\omega}) \end{aligned} \tag{3 22}$$

If $\phi$ does not belong to $\mathbb{P}_N(I)$, arguing as before we get

$$\begin{aligned} E(\chi, \phi) &= | (P_c\, \chi, P_c\, \phi)_{N\,\omega} - (\Pi_{0\,N-1}\, \chi, P_c\, \phi)_{N\,\omega} + (\Pi_{0\,N-1}\, \chi, P_c\, \phi)_\omega - \\ &\quad - (\Pi_{0\,N-1}\, \chi, \phi)_\omega + (\Pi_{0\,N-1}\, \chi, \phi)_\omega - (\chi, \phi)_\omega | \leqslant \\ &\leqslant C \{ \| (P_c - \Pi_{0\,N-1})\, \chi \|_{0\,\omega} \| P_c\, \phi \|_{0\,\omega} + \\ &\quad + \| \Pi_{0\,N-1}\, \chi \|_{0\,\omega} \| \phi - P_c\, \phi \|_{0\,\omega} + \| \chi - \Pi_{0\,N-1}\, \chi \|_{0\,\omega} \| \phi \|_{0\,\omega} \}, \end{aligned}$$

using (3 18) and (3 20) we obtain that for any $\frac{1}{2} < \mu \leqslant 1$

$$\begin{aligned} &\forall \chi \in H^1_\omega(I),\ \forall \phi \in H^\mu_{0\,\omega}(I), \quad | (\chi, \phi)_{N\,\omega} - (\chi, \phi)_\omega | \leqslant \\ &\qquad \leqslant CN^{(1/2)-\mu} \| \chi \|_{1\,\omega} \| \phi \|_{\mu\,\omega} \end{aligned} \tag{3 23}$$

(iii) Now we want to show that the difference between $G_N$ and $G$ vanishes when $N$ tends to infinity

For any $\lambda \in \mathbb{R}$ and $v \in V_N$, (2 11) and (3 15) lead to

$$\begin{aligned} \forall \phi \in V \quad \langle (G - G_N)(\lambda, v), \phi \rangle = \lambda \Big\{ \frac{1}{2} \int_I (v^2 - P_c\, v^2)_x\, \phi\omega\, dx\, + \\ + (f, \phi)_\omega - (f, \phi)_{N\,\omega} \Big\} \end{aligned} \tag{3 24}$$

Integrating by parts and using (3 16) and (3 18) we get

$$\left| \int_I (v^2 - P_c v^2)_x \phi\omega \, dx \right| = \left| - \int_I (v^2 - P_c v^2)(\phi\omega)_x \, dx \right| \leqslant$$
$$\leqslant C \| v^2 - P_c v^2 \|_{0\,\omega} \| \phi \|_{1\,\omega} \leqslant CN^{-1/2} \| v^2 \|_{1\,\omega} \| \phi \|_{1\,\omega} \leqslant$$
$$\leqslant CN^{-1/2} \| v \|_{1\,\omega}^2 \| \phi \|_{1\,\omega} \qquad (3\ 25)$$

where the last inequality is due to the fact that $H^1_\omega(I)$ is an algebra (see [1, Theorem 5 23] and [11, Theorem 1 2]) Finally, using (3 24), (3 25) and (3 23) with $\chi = f$ for any $\lambda \in \mathbb{R}$ we obtain that

$$\forall v \in V_N, \ \forall \phi \in V, \quad | \langle (G - G_N)(\lambda, v), \phi \rangle | \leqslant$$
$$\leqslant CN^{-1/2} | \lambda | (\| v \|_{1\,\omega}^2 + \| f \|_{1\,\omega}) \| \phi \|_{1\,\omega} \qquad (3\ 26)$$

(iv) Since $W$ is topologically imbedded in $V$, using (2 12) we have that $G(\lambda, v)$ belongs to $V'$ for any $\lambda \in \mathbb{R}$ and any $v \in V_N$ Hence using (3 26) the lemma is proved □

Let us go back to problem (3 7) Using (3 6) and integrating by parts we have

$$\forall \phi \in V_N \ - (u_{N_{xx}}, \phi)_{N\,\omega} = - (u_{N_{xx}}, \phi)_\omega = c(u_N, \phi) \qquad (3\ 27)$$

Then by the definition (3 15), it follows from (3 7) and (3 27) that

$$\forall \phi \in V_N \quad c(u_N, \phi) = - \langle G_N(\lambda, u_N), \phi \rangle$$

Finally, setting $F_N^* \ \mathbb{R} \times V_N \to V_N$,

$$F_N^*(\lambda, u_N) = u_N + T_N \, G_N(\lambda, u_N), \qquad (3\ 28)$$

we get from (3 9) that the pseudo-spectral problem (3 7) is equivalent to finding $u_N \in V_N$ such that

$$F_N^*(\lambda, u_N) = 0 \qquad (3\ 29)$$

In order to apply the abstract Theorem 1 2 to problem (3 29) we need to prove some further results

LEMMA 3 2 *The operator $G_N$ defined by (3 15) is a $C^\infty$ mapping from $\Lambda \times V_N$ into $V$ Moreover there exists a positive increasing function $K$ $\mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$\forall l \geqslant 1 \quad \| D^l G_N[\lambda, v] \|_{\mathscr{L}([\mathbb{R} \times V]^l \, W)} \leqslant K(| \lambda | + \| v \|_{1\,\omega}) \qquad (3\ 30)$$

*Proof* : The first assertion is obvious. To prove (3.30) we proceed by steps.

(i) Consider first the derivative with respect to $\lambda$; using (3.15) we have for any $\mu \in \Lambda$

$$\langle D_\lambda G_N[\lambda, v](\mu), \phi \rangle = \frac{1}{2}\mu \int_I (P_c v^2)_x \phi\omega \, dx - \mu(f, \phi)_{N,\omega} . \quad (3.31)$$

It follows from the Cauchy-Schwarz inequality that

$$\forall \phi \in L^2_\omega(I) \quad \left| \int_I (P_c v^2)_x \phi\omega \, dx \right| \leqslant C_1 \| P_c v^2 \|_{1,\omega} \| \phi \|_{0,\omega} ; \quad (3.32)$$

On the other hand, integrating by parts and using (3.16) and (3.31) we have

$$\forall \phi \in V \quad \left| \int_I (P_c v^2)_x \phi\omega \, dx \right| \leqslant C_2 \| P_c v^2 \|_{0,\omega} \| \phi \|_{1,\omega} . \quad (3.33)$$

Finally, by interpolation between (3.32) and (3.33) (see [3, Theorem 4.4.1]) we obtain

$$\forall \phi \in H^{3/4}_{0,\omega}(I) \left| \int_I (P_c v^2)_x \phi\omega \, dx \right| \leqslant C \| P_c v^2 \|_{1/4,\omega} \| \phi \|_{3/4,\omega} . \quad (3.34)$$

Using (3.18) and the inequality $\| v^2 \|_{1,\omega} \leqslant C \| v \|^2_{1,\omega}$ we have that

$$\| P_c v^2 \|_{1/4,\omega} \leqslant \| v^2 \|_{1/4,\omega} + \| v^2 - P_c v^2 \|_{1/4,\omega} \leqslant C \| v^2 \|_{1,\omega} \leqslant C \| v \|^2_{1,\omega} ;$$

in addition, since $\| \phi \|_{L^r(I)} \leqslant C \| \phi \|_{3/4,\omega}$ (see [11, Theorem 1.2]), we have

$$\forall \phi \in H^{3/4}_{0,\omega}(I) \quad |(f, \phi)_{N,\omega}| \leqslant C \| f \|_{L^{r'}(I)} \| \phi \|_{L^r(I)} \leqslant$$
$$\leqslant C \| f \|_{1,\omega} \| \phi \|_{3/4,\omega} .$$

Summarizing the previous inequality and using (3.31) and (3.34) we get

$$\| D_\lambda G_N[\lambda, v] \|_{\mathscr{L}(\mathbb{R},W)} \leqslant C(\| f \|_{1,\omega} + \| v \|^2_{1,\omega}) . \quad (3.35)$$

Clearly, higher order derivatives with respect to $\lambda$ vanish identically.

(ii) We consider now $D_u G_N$, arguing as in (i) we have

$$| \langle D_u G[\lambda, v] (w), \phi \rangle | = \left| - \lambda \int_I [P_c(vw)]_x \phi\omega \, dx \right| \leqslant$$

$$\leqslant C | \lambda | \begin{cases} \| P_c(vw) \|_{1,\omega} \| \phi \|_{0,\omega}, & \forall \phi \in L^2_\omega(I) \\ \| P_c(vw) \|_{0,\omega} \| \phi \|_{1,\omega}, & \forall \phi \in V \end{cases}$$

Then by interpolation it follows from (3 18) that

$$\forall w \in V, \ \forall \phi \in H^{3/4}_{0,\omega}(I), \quad | \langle D_u G[\lambda, v] (w), \phi \rangle | \leqslant$$
$$\leqslant C | \lambda | \ \| v \|_{1,\omega} \| w \|_{1,\omega} \| \phi \|_{3/4,\omega}$$

whence

$$\| D_u G_N[\lambda, v] \|_{\mathscr{L}(\mathbb{R} \times V, W)} \leqslant C | \lambda | \ \| v \|_{1,\omega} . \tag{3 36}$$

(iii) Finally consider the second order derivatives of $G_N$ For any $\mu \in \mathbb{R}$ we have

$$\forall w \in V \quad | \langle D_{u\lambda} G_N[\lambda, v] (\{ \mu, w \}) \phi \rangle | = \left| - \mu \int_I [P_c(vw)]_x \phi\omega \, dx \right|,$$

then proceeding as in (ii) this term can be bounded by

$$| \mu | \ \| v \|_{1,\omega} \| w \|_{1,\omega} \| \phi \|_{3/4,\omega},$$

so we get

$$\| D_{u\lambda} G_N[\lambda, v] \|_{\mathscr{L}([\mathbb{R} \times V]^2, W)} \leqslant C \| v \|_{1,\omega} . \tag{3 37}$$

Finally

$$\forall w_1, w_2 \in V \quad \langle D_{uu} G_N[\lambda, v] (\{ w_1, w_2 \}), \phi \rangle = - \lambda \int_I [P_c(w_1, w_2)]_x \phi\omega \, dx$$

so arguing as usual we obtain

$$\| D_{uu} G_N[\lambda, v] \|_{\mathscr{L}([\mathbb{R} \times V]^2, W)} \leqslant C | \lambda | . \tag{3 38}$$

Higher order derivatives vanish identically, so (3 30) follows from (3 35), . , (3 38) □

LEMMA 3 3 : *We have*

$$\lim_{N \to \infty} \sup_{\lambda \in \Lambda} \| D_u(G - G_N) [\lambda, \Pi_N u(\lambda)] \|_{\mathscr{L}(V, V)} = 0 \tag{3 39}$$

*Proof* Using (3 26), (3 16), (3 18) and setting $v \equiv \Pi_N u(\lambda)$ we have for any $w \in V$

$$\begin{aligned}
\forall \phi \in V \quad & |\langle D_u(G - G_N)[\lambda, v](w), \phi \rangle| = \\
&= \left| \lambda \int_I \{ (vw)_x - [P_c(vw)]_x \} \phi\omega \, dx \right| \\
&= \left| - \lambda \int_I [(vw) - P_c(vw)] (\phi\omega)_x \, dx \right| \leqslant \\
&\leqslant C |\lambda| \, \| vw - P_c(vw) \|_{0\,\omega} \| \phi \|_{1\,\omega} \\
&\leqslant C |\lambda| N^{-1/2} \| vw \|_{1\,\omega} \| \phi \|_{1\,\omega} \\
&\qquad\qquad \leqslant C |\lambda| N^{-1/2} \| v \|_{1\,\omega} \| w \|_{1\,\omega} \| \phi \|_{1\,\omega}
\end{aligned}$$

Then noting that $\| v \|_{1\,\omega} = \| \Pi_N u(\lambda) \|_{1\,\omega} \leqslant C \| u(\lambda) \|_{1\,\omega}$ by (3 20), the property (3 39) follows □

LEMMA 3 4 *We have*

$$\lim_{N \to \infty} \sup_{\lambda \in \Lambda} \| F_N^*(\lambda, \Pi_N u(\lambda)) \|_{1\,\omega} = 0 \tag{3 40}$$

*Proof* From (2 14), (2 15) and (3 28) we get

$$\begin{aligned}
F_N^*(\lambda, \Pi_N u(\lambda)) &= F_N^*(\lambda, \Pi_N u(\lambda)) - F(\lambda, u(\lambda)) = \Pi_N u(\lambda) - u(\lambda) + \\
&+ (T_N - T) G(\lambda, u(\lambda)) + T_N(G(\lambda, \Pi_N u(\lambda)) - G(\lambda, u(\lambda))) + \\
&\qquad + T_N(G_N - G)(\lambda, \Pi_N u(\lambda))
\end{aligned} \tag{3 41}$$

Using (2 14) and (3 11) we have

$$\| (T_N - T) G(\lambda, u(\lambda)) \|_{1\,\omega} = | u(\lambda) - \Pi_N u(\lambda) \|_{1\,\omega}, \tag{3 42}$$

owing to the uniform continuity of $\Pi_N$ in $V$ (see (3 20)), to the differentiability of $G$, to the continuity of $T$ from $W$ into $V$ and to (1 5) we have

$$\begin{aligned}
\| T_N(G(\lambda, \Pi_N u(\lambda)) - G(\lambda, u(\lambda))) \|_{1\,\omega} &\leqslant C \| T(G(\lambda, \Pi_N u(\lambda)) - \\
- G(\lambda, u(\lambda))) \|_{1\,\omega} &\leqslant C \| G(\lambda, \Pi_N u(\lambda)) - G(\lambda, u(\lambda)) \|_W \leqslant \\
&\leqslant C \| \Pi_N u(\lambda) - u(\lambda) \|_{1\,\omega}
\end{aligned} \tag{3 43}$$

Finally, using (3.16), (3.21) and (3.26) we have

$$\| T_N(G_N - G)(\lambda, \Pi_N u(\lambda)) \|_{1,\omega} = \sup_{\phi \in V_N \ \|\phi\|_{1,\omega}=1} | \langle (G_N - G)(\lambda, \Pi_N u(\lambda)), \phi \rangle | =$$

$$= \sup_{\phi \in V_N \ \|\phi\|_{1,\omega}=1} \left| \frac{1}{2} \int_I [(\Pi_N u(\lambda))^2 - P_c(\Pi_N u(\lambda))^2]_x \phi\omega \, dx + (f, \phi)_\omega - (f, \phi)_{N,\omega} \right| \leqslant$$

$$\leqslant C \{ \| (\Pi_N u(\lambda))^2 - P_c(\Pi_N u(\lambda))^2 \|_{0,\omega} + \| f - P_c f \|_{0,\omega} + \| f - \Pi_{0,N-1} f \|_{0,\omega} \} \leqslant C \{ \| u(\lambda)^2 - P_c u(\lambda)^2 \|_{0,\omega} + \| (\mathrm{Id} - P_c)(u(\lambda)^2 - [\Pi_N u(\lambda)]^2) \|_{0,\omega} + \| f - P_c f \|_{0,\omega} + \| f - \Pi_{0,N-1} f \|_{0,\omega} \} . \quad (3.44)$$

Now (3 40) holds thanks to (3.41), ..., (3.44) and to (3.18) and (3.20). □

Finally we have :

THEOREM 3.1 : *Let* $\{ \{ \lambda, u(\lambda) \}, \lambda \in \Lambda \}$ *be a branch of non singular solutions of* (2.15), *and let N be a sufficiently large number. There exist a neighborhood* $\theta$ *of* 0 *independent of N, and, for any* $\lambda \in \Lambda$, *a unique* $C^1$ *mapping* $\lambda \to u_N(\lambda)$ *such that*

$$F_N^*(\lambda, u_N(\lambda)) = 0 , \quad u_N(\lambda) - \Pi_N u(\lambda) \in \theta . \quad (3.45)$$

*Moreover, if* $f \in H_\omega^\sigma(I) \left( \sigma > \frac{1}{2} \right)$, *then for any* $\lambda \in \Lambda$ $u(\lambda) \in V \cap H_\omega^{\sigma+2}(I)$, *and the following error estimate holds* :

$$\| u_N(\lambda) - u(\lambda) \|_{1,\omega} \leqslant C \{ N^{-(\sigma+1)} \| u(\lambda) \|_{\sigma+2,\omega} + N^{e(\omega)-(\sigma+2)} \| u(\lambda) \|^2_{\sigma+2,\omega} + N^{e(\omega)-\sigma} \| f \|_{\sigma,\omega} \} \quad (3.46)$$

*where* $e(\omega) = 0$ *if* $\omega(x) = (1 - x^2)^{-1/2}$, *and* $e(\omega) = \frac{1}{2}$ *if* $\omega \equiv 1$.

*Proof* : Due to (3.13), (3.14), (2.13) and Lemmas (3.1), (3.2), (3.3) and (3.4) we can apply Theorem 1.2 with $Z = V$. Note that the hypothesis (1.9) is trivially satisfied with $r = 0$. Then by (1.16) we immediately get (3.45). Moreover it follows from (1.17), (3.42) and (3.44) that for any $\lambda \in \Lambda$

$$\| u_N(\lambda) - u(\lambda) \|_{1,\omega} \leqslant C \{ \| (\mathrm{Id} - \Pi_N) u(\lambda) \|_{1,\omega} + \| (\mathrm{Id} - P_c)(u(\lambda)^2 - [\Pi_N u(\lambda)]^2) \|_{0,\omega} + \| (\mathrm{Id} - P_c) u(\lambda)^2 \|_{0,\omega} + \| (\mathrm{Id} - P_c) f \|_{0,\omega} + \| (\mathrm{Id} - \Pi_{0,N-1}) f \|_{0,\omega} \} . \quad (3.47)$$

Using the continuity of the operator $T$ from $H^s_\omega(I)$ into $V \cap H^{s+2}_\omega(I)$ if $s \geqslant 0$ (see [11, Theorem 1 4 and Remark 1 2]), it is easy to see that if $f \in H^\sigma_\omega(I)$, then for any $\lambda$ the solution of (2 3) belongs to $H^{\sigma+2}_\omega(I)$, so $u(\lambda) \in V \cap H^{\sigma+2}_\omega(I)$ for any $\lambda$ Using (3 18) and the fact that $H^{\sigma+2}_\omega(I)$ is an algebra we get

$$\| (\mathrm{Id} - P_c)\, u^2(\lambda) \|_{0\ \omega} \leqslant C N^{e(\omega)-2-\sigma} \| u^2(\lambda) \|_{\sigma+2\ \omega} \leqslant \\ \leqslant C N^{e(\omega)-2-\sigma} \| u(\lambda) \|^2_{\sigma+2\ \omega} \qquad (3\ 48)$$

Moreover, using (3 12) and (3 18) we have

$$\| (\mathrm{Id} - P_c)\, (u^2(\lambda) - [\Pi_N u(\lambda)]^2) \|_{0\ \omega} \leqslant \\ \leqslant C N^{e(\omega)-1} \| u(\lambda) - \Pi_N u(\lambda) \|_{1\ \omega} \| u(\lambda) + \Pi_N u(\lambda) \|_{1\ \omega} \\ \leqslant C N^{e(\omega)-2-\sigma} \| u(\lambda) \|_{\sigma+2\ \omega} \| u(\lambda) \|_{1\ \omega} \qquad (3\ 49)$$

Finally, using (3 18) and (3 20), (3 46) follows from (3 47), , (3 49) □

We want now to obtain an $L^2_\omega$ error estimate that improves the one which can be trivially deduced from (3 46)

To this and let $M > N$ be an integer and define the discrete inner product $(.,\ .)_{M\ \omega}$ as in (3 2) by formally replacing $N$ with $M$

Let $\tilde{P}_c$ denote the interpolation operator with respect to the points $x_\nu$, $0 \leqslant \nu \leqslant M$ Using (3 18) we then obtain

$$\forall \sigma > \frac{5}{2} \quad \| f - \tilde{P}_c f \|_{0\ \omega} \leqslant C \| f \|_{\sigma-2\ \omega} \begin{cases} M^{2-\sigma} & \text{if } \omega(x) = (1 - x^2)^{-1/2} \\ M^{(5/2)-\sigma} & \text{if } \omega(x) \equiv 1 \end{cases}$$

hence, if $M > N^{\sigma(\sigma-2)}$ when $\omega(x) = (1 - x^2)^{-1/2}$, and $M > N^{\frac{2\sigma-1}{2\sigma-5}}$ when $\omega = 1$, we deduce the inequality

$$\| f - \tilde{P}_c f \|_{0\ \omega} \leqslant C \| f \|_{\sigma-2\ \omega} N^{e(\omega)-\sigma} \qquad (3\ 50)$$

Define now a new pseudo spectral problem as follows find $\tilde{u}_N \in V_N$ such that

$$\tilde{F}_N(\lambda, \tilde{u}_N(\lambda)) = 0 \qquad (3\ 51)$$

Here we set, for any $\lambda \in \Lambda$, $v \in V_N$ and $\phi \in V$

$$\langle \tilde{G}_N(\lambda, v), \phi \rangle = \frac{\lambda}{2} ([P_c v^2]_x, \phi)_\omega - \lambda (f, \phi)_{M\ \omega} \qquad (3\ 52)$$

and

$$\tilde{F}_N(\lambda, v) = v + T_N \tilde{G}_N(\lambda, v)$$

We note that problem (3 51) differs from problem (3 29) only for a more precise integration formula used for the computation of the contributions of $f$.

It is an easy matter to check that the Theorem 3 1 still holds if $u_N$ is replaced by $\tilde{u}_N$ and $F_N^*$ by $\tilde{F}_N$

THEOREM 3 2 . *Assume that, for some $\sigma > 2$, $f \in H_\omega^{\sigma-2}(I)$ and that the mapping $\lambda \in \Lambda \to u(\lambda) \in V \cap H_\omega^\sigma(I)$ is continuous.*

*Then for any $\lambda \in \Lambda$ there exists a positive constant $C(\lambda)$ depending on $\| u(\lambda) \|_{\sigma,\omega}$ and on $\| f \|_{\sigma-1,\omega}$ such that*

$$\| \tilde{u}_N(\lambda) - u(\lambda) \|_{0,\omega} \leqslant C(\lambda) \, N^{e(\omega)-\sigma} . \qquad (3\ 54)$$

*Proof* : To achieve (3 54) it is sufficient to verify the hypotheses of the Corollary 1 1

For that we set $K = V$, $H = L_\omega^2(I)$ and $Y = V'$.

First, we deduce from (2.11) that

$$D_u \, G[\lambda, v] \, w = \frac{1}{2} (vw)_x$$

If $v \in V$ and $w \in H$, then $vw \in H$, and by (3 16) we get that

$$(vw)_x \in (H_{0,\omega}^1(I))' = Y$$

This proves (1.39).

Next, it is an easy consequence of (2 10) that (1 48) holds. Then, (1.41) is a simple consequence of (2 10), (3 11) and (3 12)

Finally, as previously seen, $\| \tilde{u}_N(\lambda) - u(\lambda) \|_{1,\omega} = 0(1/N)$, hence (1 42) holds

Then using (1 43) we obtain for any $\lambda \in \Lambda$

$$\| \tilde{u}_N(\lambda) - u(\lambda) \|_{0,\omega} \leqslant C \, \| u(\lambda) + T_N \, G(\lambda, u(\lambda)) \|_{0,\omega} + \\ + \| T_N(G - \tilde{G}_N) \, (\lambda, \tilde{u}_N(\lambda)) \|_{0,\omega} . \qquad (3\ 55)$$

Using the equality $u(\lambda) + TG(\lambda, u(\lambda)) = 0$, and $T_N = \Pi_N \circ T$ we get

$$\| u(\lambda) + T_N \, G(\lambda, u(\lambda)) \|_{0,\omega} \leqslant CN^{-\sigma} \, \| u(\lambda) \|_{\sigma,\omega} \qquad (3\ 56)$$

Next let us estimate the last term of (3.55).

By Lemma 2 1 there exists $\phi \in V_N$ such that for any $v \in V_N$

$$c(v, \phi) = (v, T_N(G - \tilde{G}_N) \, (\lambda, \tilde{u}_N(\lambda)))_\omega . \qquad (3.57)$$

Moreover there exists a positive constant $C$ such that

$$\| \phi \|_{1\,\omega} \leqslant C \| T_N(G - \tilde{G}_N)(\lambda, \tilde{u}_N(\lambda)) \|_{0\,\omega}$$

Then, taking $v = T_N(G - \tilde{G}_N)(\lambda, \tilde{u}_N(\lambda))$ by (3 57) we get

$$\| T_N(G - \tilde{G}_N)(\lambda, \tilde{u}_N(\lambda)) \|_{0\,\omega}^2 = c(T_N(G - \tilde{G}_N)(\lambda, \tilde{u}_N(\lambda)), \phi)$$

Using now (3 9), (3 16) and (3 21) we have

$$\begin{aligned} \| T_N(G - \tilde{G}_N)(\lambda, \tilde{u}_N(\lambda)) \|_{0\,\omega}^2 &= | \langle (G - \tilde{G}_N)(\lambda, \tilde{u}_N(\lambda)), \phi \rangle | = \\ &= \left| \lambda \left\{ \frac{1}{2} \int_I [(\mathrm{Id} - P_c)\, \tilde{u}_N^2(\lambda)]_x\, \phi\omega\, dx + (f, \phi)_\omega - (f, \phi)_{M\,\omega} \right\} \right| \leqslant \\ &\leqslant C\, |\lambda|\, \{ \| (\mathrm{Id} - P_c)\, \tilde{u}_N^2(\lambda) \|_{0\,\omega} + \| (\mathrm{Id} - \tilde{P}_c) f \|_{0\,\omega} + \\ &\qquad + \| (\mathrm{Id} - \Pi_{0\,M-1}) f \|_{0\,\omega} \}\, \| \phi \|_V \qquad (3\ 59) \end{aligned}$$

Using (3 18) we estimate the first term of the right hand side as follows

$$\begin{aligned} \| (\mathrm{Id} - P_c)\, \tilde{u}_N^2(\lambda) \|_{0\,\omega} &\leqslant \| (\mathrm{Id} - P_c)\, u^2(\lambda) \|_{0\,\omega} + \\ &+ \| (\mathrm{Id} - P_c)(u^2(\lambda) - \tilde{u}_N^2(\lambda)) \|_{0\,\omega} \leqslant C \{ N^{e(\omega)-\sigma} \| u(\lambda) \|_{\sigma\,\omega}^2 + \\ &\qquad + N^{e(\omega)-1} \| \tilde{u}_N(\lambda) - u(\lambda) \|_{1\,\omega} \| \tilde{u}_N(\lambda) + u(\lambda) \|_{1\,\omega} \} \end{aligned}$$

Using the $H^1$-error estimate concerning $\tilde{u}_N$, and (3 50), (3 58) and (3 59) we then get

$$\begin{aligned} \| T_N(G - \tilde{G}_N)(\lambda, \tilde{u}_N(\lambda)) \|_{0\,\omega} &\leqslant C\, |\lambda|\, \{ N^{e(\omega)-\sigma} \| u(\lambda) \|_{\sigma\,\omega} + \\ &+ N^{2e(\omega)-\sigma-1} \| u(\lambda) \|_{\sigma\,\omega}^2 + N^{e(\omega)-\sigma} \| f \|_{\sigma\ 2\,\omega} \} \qquad (3\ 60) \end{aligned}$$

Finally, from (3 55), (3 56) and (3 60) we conclude that

$$\begin{aligned} \| \tilde{u}_N(\lambda) - u(\lambda) \| &\leqslant C\, |\lambda|\, N^{e(\omega)-\sigma} \{ \| u(\lambda) \|_{\sigma\,\omega} + \\ &\qquad + N^{e(\omega)-1} \| u(\lambda) \|_{\sigma\,\omega}^2 + \| f \|_{\sigma-2\,\omega} \} \end{aligned}$$

So (3 54) holds $\square$

We finish this paper by making some remarks about the practical solution of the approximate problem As an example let us consider the pseudo-spectral problem (3 29) Following the Section 1 3, according to (1 55) we can define a sequence $(v^n)$ of functions of $V_N$ by solving

$$(\mathrm{Id} + T_N\, D_u\, G_N[v^n])(v^{n+1} - v^n) = -(v^n + T_N\, G_N(v^n)), \quad n \geqslant 0 \qquad (3\ 55)$$

(the parameter $\lambda$ is taken as fixed, so it does not appear in (3 55)) An equivalent form of (3 55) is as follows find $v^{n+1} \in V_N$ such that

$$\forall \phi \in V_N \quad c(v^{n+1}, \phi) + \lambda \int_I [P_c(v^n v^{n+1})]_x \phi\omega \, dx =$$

$$\frac{\lambda}{2} \int_I [P_c(v^n)^2]_x \phi\omega \, dx + (f, \phi)_{N \omega} \qquad (3\ 56)$$

To apply Theorem 1 4 we need only to check that (1 50) holds

Using Lemma 2 1 and (3 9), (3 11), (3 12) it follows

$$\begin{aligned} \forall v, w \in V_N \quad \| T_N D_u^2 G_N[u_N](v, w) \|_V^2 &\leqslant \beta^{-1} c(T_N D_u^2 G_N[u_N](v, w), \\ T_N D_u^2 G_N[u_N](v, w)) &\leqslant \beta^{-1} | \langle D_u^2 G_N[u_N](v, w), \\ T_N D_u^2 G_N[u_N](v, w) \rangle | &\leqslant C\beta^{-1} [ \sup_{\substack{\phi \in 1 \\ \|\phi\|_1 \quad 1}} (P_c(vw)_x, \phi)_\omega]^2 \end{aligned}$$

Finally, by (3 16), (3 18) and the inequality $\| vw \|_{1\ \omega} \leqslant C \| v \|_{1\ \omega} \| w \|_{1\ \omega}$ we get

$$\| T_N D_u^2 G_N[u_N](v, w) \|_V^2 \leqslant C \| v \|_{1\ \omega}^2 \| w \|_{1\ \omega}^2$$

hence (3 50) holds

Then by Theorem 1 4 we can conclude that if $v^0$ is suitably chosen, then the Newton iterates $(v^n)$ converge quadratically to $u_N(\lambda)$ for any $\lambda \in \Lambda$

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