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OPTIMIZATION OF THE DOMAIN IN ELLIPTIC UNILATERAL BOUNDARY VALUE PROBLEMS BY FINITE ELEMENT METHOD (*)

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Resumé — *On considère le probleme de la minimisation d'une fonction coût par rapport au domaine, ou la fonction d'état est la solution d'une équation elliptique avec des conditions aux limites du type de Signorini sur une partie de la frontière variable On démontre* (i) *l'existence d'une solution pour quatre differentes fonctions coût et* (ii) *la convergence des approximations par éléments finis dans un certain sens*

Abstract — *The problem of the minimization of a cost functional with respect to the domain is considered, where the state variable has to solve an elliptic equation with boundary conditions of Signorini's type on a part of the variable boundary We prove* (i) *the existence of an exact solution for four different cost functionals and* (ii) *the convergence of finite element approximations in a certain sense*

INTRODUCTION

Some problems of the optimal design remain open up to this time, although they are of interest from the physical point of view. Thus for instance in some problems of the contact between elastic bodies the shape of the boundaries should be optimized to obtain minimal cost functional such as the integral of energy, contact forces or displacements.

It is the aim of the present paper to start the analysis of this class of problems on a simplified model with a unilateral problem in R^2 for the Poisson equation and boundary conditions of Signorini's type. On a given part of the boundary

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the Dirichlet homogeneous condition is prescribed and the remaining part — with unilateral conditions — has to be determined

In Section 1 we present a proof of existence of a solution for four different cost functionals and for one common state problem, which is formulated in terms of a variational inequality on a variable domain

In Section 2 a finite element approximation is proposed, following the method of Begis and Glowinski [2] who employed piecewise linear approximations of the unknown part of the boundary and piecewise bilinear finite elements on a uniform mesh in a reference square domain In Section 3 we prove that some subsequence of the approximate solutions converges to an exact solution uniformly, whereas the corresponding solutions of the state problem converge weakly in each interior subdomain of the optimal domain

1. EXISTENCE OF A SOLUTION TO THE MODEL PROBLEMS

Let us consider the following model problems Let $\Omega(v) \subset \mathbb{R}^2$ be the domain (see *fig* 1)

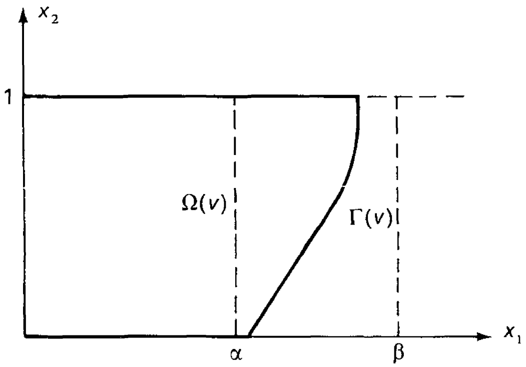

Figure 1.

$$\Omega(v) = \{ 0 < x_1 < v(x_2), 0 < x_2 < 1 \},$$

where the function v is to be determined from the problem

$$(P_i) \qquad \mathfrak{J}_i(v) = \min_{w \in \mathcal{U}_{ad}} \mathfrak{J}_i(w)$$

Here

$$\mathcal{U}_{ad} = \{ w \in C^{(0),1}([0, 1]) \quad (\text{i e Lipschitz function}),$$

$$0 < \alpha \leqq w \leqq \beta, \qquad | dw/dx_2 | \leqq C_1, \qquad \int_0^1 w(x_2)\, dx_2 = C_2 \},$$

with given constants α, β, C_1, C_2, $\mathfrak{J}_i(w) = J_i(y(w))$, i may equal any of the numbers 1, 2, 3, 4, Z_0 = const is given and

$$J_1(y(w)) = \int_{\Omega(w)} (y(w) - z_0)^2 \, dx \,, \tag{1.1}$$

$$J_2(y(w)) = \int_{\Omega(w)} | \nabla y(w) |^2 \, dx \,, \tag{1.2}$$

$$J_3(y(w)) = \int_0^1 y(w) \,|_{\Gamma(w)} \, dx_2 \,, \tag{1.3}$$

$$J_4(y(w)) = \int_0^1 (y(w) \,|_{\Gamma(w)} - z_0)^2 \, dx_2 \,, \tag{1.4}$$

where $y(w)$ denotes the solution of the following unilateral boundary value problem :

$$- \Delta y = f \quad \text{in} \quad \Omega(w) \,, \tag{1.5}$$

$$y \geqq 0 \,, \quad \frac{\partial y}{\partial \nu} \geqq 0 \,, \quad y \frac{\partial y}{\partial \nu} = 0 \quad \text{on} \quad \Gamma(w) \,,$$

$$y = 0 \quad \text{on} \quad \partial\Omega(w) \dot{-} \Gamma(w) \,.$$

Here $f \in L^2(\Omega_\beta)$ is given, $\Omega_\beta = (0, \beta) \times (0, 1)$ and $\partial y / \partial \nu$ denotes the derivative with respect to the outward normal to $\Gamma(w)$. In the following, we denote by $H^k(\Omega)$ the Sobolev space $W_2^{(k)}(\Omega)$ with the usual norm $\| \, . \, \|_{k,\Omega}$, $H^0 \equiv L^2$, and by $| \, . \, |_{k,\Omega}$ the seminorm generated by all derivatives of the k-th order.

It is well-known that the state problem (1.5) can be formulated in terms of a variational inequality, as follows :

$$K(w) = \{ \, z \in H^1(\Omega(w)) \mid z = 0 \quad \text{on} \quad \partial\Omega(w) \dot{-} \Gamma(w) \,,$$
$$z \geqq 0 \quad \text{on} \quad \Gamma(w) \, \} \,;$$

find $y \in K(w)$ such that for any $z \in K(w)$

$$\int_{\Omega(w)} \nabla y . \nabla (z - y) \, dx \geqq \int_{\Omega(w)} f(z - y) \, dx \,. \tag{1.6}$$

The problem (1.6) has a unique solution for any $w \in \mathcal{U}_{ad}$.

We are going to prove the main result of the section, i.e.

THEOREM 1 : *The problem* (P_i) *has at least one solution for any of the four cost functionals* $\mathfrak{J}_i$, $i = 1, 2, 3, 4$.

Proof: Let us consider a minimizing sequence $\{ w_n \}$, $\mathfrak{J}_\iota(w_n) \to \inf_{w \in \mathcal{U}_{ad}} \mathfrak{J}_\iota(w)$ for $n \to \infty$. Since the set $\mathcal{U}_{ad}$ is compact in $C([0, 1])$, we may choose a subsequence, denoted again by $\{ w_n \}$, such that $w_n \to v$ in $C([0, 1])$. It is readily seen that $v \in \mathcal{U}_{ad}$. Let us denote by $\Omega = \Omega(v)$ the domain bounded by $\Gamma = \Gamma(v)$. For any positive integer m let G_m be the domain bounded by γ_m, where

$$\gamma_m = \left\{ (x_1, x_2) \mid x_1 = v(x_2) - \frac{1}{m} \right\}.$$

Furthermore, let Ω_n be the domain bounded by the graph of the function w_n and y_n the corresponding solution of the state problem (1.5) or (1.6), respectively, where $\Omega(w_n) = \Omega_n$, $\Gamma(w_n) = \Gamma_n$ and $K(w_n)$ is inserted.

Choosing $z = 0$ and $z = 2\, y_n$ in (1.6), we obtain

$$\int_{\Omega_n} | \nabla y_n |^2 \, dx = \int_{\Omega_n} f y_n \, dx \leqq \| f \|_{0,\Omega_n} \| y_n \|_{0,\Omega_n} . \tag{1.7}$$

By a standard argument, we may write

$$\| y_n \|^2_{0,\Omega_n} \leqq \beta^2 \int_{\Omega_n} | \nabla y_n |^2 \, dx . \tag{1.8}$$

Combining (1.7) and (1.8), we are led to the estimate

$$\| y_n \|_{1,\Omega_n} \leqq C_0 \qquad \forall n , \tag{1.9}$$

with C_0 independent of n.

Next let us consider a fixed domain G_m. There exists $n_0(m)$ such that

$$n > n_0(m) \Rightarrow G_m \subset \Omega_n .$$

Then

$$\| y_n \|_{1,G_m} \leqq \| y_n \|_{1,\Omega_n} \leqq C_0 \qquad \forall n . \tag{1.10}$$

Consequently, a subsequence $\{ y_{n_1} \}$ exists such that

$$y_{n_1} \rightharpoonup y^{(m)} \text{ (weakly) in } H^1(G_m) , \quad y^{(m)} \in H^1(G_m) .$$

For G_{m+1} we obtain a similar assertion, if we choose the proper subsequence $\{ y_{n_2} \}$ of the sequence $\{ y_{n_1} \}$, etc.

Consider the diagonal subsequence $\{ y^D_n \}$ of all subsequences

$$\{ y_{n_1} \}, \{ y_{n_2} \}, \dots .$$

It is easy to prove that *a function $y \in H^1(\Omega)$ exists such that*

$$y_n^D \rightharpoonup y\,|_{G_m} = y^{(m)} \tag{1.11}$$

weakly in $H^1(G_m)$ holds for any $m > \alpha^{-1}$.

In fact, the existence of generalized derivatives $\partial y/\partial x_i$ follows from the definition of the Sobolev space. Moreover, we have

$$\| y^{(m)} \|_{1,G_m} \leqq C_0 \qquad \forall m\,,$$

since any ball in $H^1(G_m)$ is weakly closed. Hence defining y such that

$$y\,|_{G_m} = y^{(m)} \qquad \forall m > \alpha^{-1}\,,$$

we obtain

$$\| y \|_{1,\Omega}^2 = \lim_{m\to\infty} \| y^{(m)} \|_{1,G_m}^2 \leqq C_0^2 < \infty\,.$$

LEMMA 1.1 : *The trace of y on Γ is non-negative.*

Proof : Assume that $y < 0$ on a set $M_0 \subset \Gamma$, mes $M_0 > 0$. Let M denote the projection of M_0 into the x_2-axis. Hence we have

$$\int_M y\,|_\Gamma\, dx_2 = c_0 < 0\,.$$

Denote $y\,|_{\gamma_m} = \eta_m$, $y\,|_\Gamma = \eta$,

$$c_m = \int_M \eta_m\, dx_2\,,$$

and

$$V_m = \left\{ (x_1, x_2) \mid v - \frac{1}{m} < x_1 < v,\, x_2 \in M \right\}.$$

Then we have

$$\left| \int_M (\eta - \eta_m)\, dx_2 \right| = \left| \int_M dx_2 \int_{v-1/m}^{v} \frac{\partial y}{\partial x_1}(\xi, x_2)\, d\xi \right| =$$

$$= \left| \int_{V_m} \frac{\partial y}{\partial x_1}\, dx \right| \leqq \| y \|_{1,\Omega} (\text{mes } V_m)^{1/2},\ \text{mes } V_m = \frac{1}{m} \text{mes } M\,. \tag{1.12}$$

Consequently, $\lim c_m = c_0$ and it holds

$$c_m \leqq \frac{1}{2} c_0 \tag{1.13}$$

for sufficiently great m.

Let us denote $V_{nm} = \{ (x_1, x_2) \in \Omega_n - G_m, x_2 \in M \}$,

$$d_n = \int_M y_n^D |_{\Gamma_n} dx_2 \geqq 0$$

(since $y_n^D \in K(w_n)$) and

$$d_{nm} = \int_M y_n^D |_{\gamma_m} dx_2$$

By the same argument as in (1 12) we obtain for $n \geqq n_0(m)$

$$\left| \int_M (y_n^D |_{\Gamma_n} - y_n^D |_{\gamma_m}) \, dx_2 \right| \leqq C_0 (\text{mes } V_{nm})^{1/2}$$

Since lim mes $V_{nm} = 0$ for $n \to \infty, m \to \infty, n \geqq n_0(m)$,

$$d_{nm} \geqq \frac{1}{4} c_0 \tag{1 14}$$

follows for sufficiently great n, m, $n \geqq n_0(m)$

From the weak convergence $y_n^D \rightharpoonup y$ in $H^1(G_m)$ we deduce

$$\lim_{n \to \infty} d_{nm} = \lim_{n \to \infty} \int_M y_n^D |_{\gamma_m} dx_2 = \int_M \eta_m \, dx_2 = c_m$$

By virtue of (1 14), we therefore have

$$c_m \geqq \frac{1}{4} c_0$$

for sufficiently great m, which is a contradiction with (1 13)

LEMMA 1 2 *The function y belongs to $K(v)$ and satisfies the variational inequality* (1 6) *on* $\Omega = \Omega(v)$

Proof (i) Let us consider an arbitrary G_m and denote

$$V(G_m) = \{ z \in H^1(G_m) \mid z = 0 \quad \text{on} \quad \partial G_m - \gamma_m \}$$

Since $V(G_m)$ is weakly closed, $y^{(m)} \in V(G_m)$ follows from (1 11) Hence we have $y = 0$ on $\partial\Omega - \Gamma$ and combining this result with Lemma 1 1, we obtain $y \in K(v)$

(ii) For any fixed m we introduce the set

$$M_m = \{ \zeta \in H_0^1(\Omega_\beta) \mid \zeta = 0 \quad \text{on} \quad \Omega_\beta - G_m \}$$

For any $n > n_0(m)$ and $z_k \in M_m$ it holds (we omit the superscript D in what follows) :

$$\int_{\Omega_n} \nabla y_n . \nabla (z_k - y_n) \, dx \geqq \int_{\Omega_n} f(z_k - y_n) \, dx , \tag{1.15}$$

since $z_k \geqq 0$ on Γ_n and therefore $z_k \in K(w_n)$.

Let us pass to the limit for $n \to \infty$ in (1.15). We have

$$\int_{\Omega_n} \nabla y_n . \nabla z_k \, dx = \int_{G_m} \nabla y_n . \nabla z_k \, dx + \int_{\Omega_n - \Omega} \nabla y_n . \nabla z_k \, dx + $$
$$+ \int_{(\Omega - G_m) \cap \Omega_n} \nabla y_n . \nabla z_k \, dx ,$$

$$\lim_{n \to \infty} \int_{G_m} \nabla y_n . \nabla z_k \, dx = \int_{G_m} \nabla y . \nabla z_k \, dx ,$$

$$\lim_{n \to \infty} \int_{\Omega_n - \Omega} \nabla y_n . \nabla z_k \, dx = 0 .$$

The last result follows from the estimate

$$\left| \int_{\Omega_n - \Omega} \nabla y_n . \nabla z_k \, dx \right| \leqq \| y_n \|_{1,\Omega_n} \| \nabla z_k \|_{0,\Omega_n - \Omega} ,$$

using (1.9) and $\lim_{n \to \infty} [\text{mes} (\Omega_n - \Omega)] = 0$.

By a similar argument we obtain

$$\left| \int_{(\Omega - G_m) \cap \Omega_n} \nabla y_n . \nabla z_k \, dx \right| \leqq C_0 \mid z_k \mid_{1,\Omega - G_m} .$$

Thus we may write

$$\limsup_{n \to \infty} \int_{\Omega_n} \nabla y_n . \nabla z_k \, dx \leqq \int_{G_m} \nabla y . \nabla z_k \, dx + C_0 \mid z_k \mid_{1,\Omega - G_m} . \tag{1.16}$$

The same approach leads to the inequality

$$\liminf_{n \to \infty} \left(- \int_{\Omega_n} f y_n \, dx \right) \geqq - \int_{G_m} f y \, dx - C_0 \| f \|_{0,\Omega - G_m} . \tag{1.17}$$

Moreover, we obtain

$$\lim_{n \to \infty} \int_{\Omega_n} f z_k \, dx = \int_{\Omega} f z_k \, dx \tag{1 18}$$

The inequality (1 15) yields

$$- \int_{G_m} | \nabla y_n |^2 \, dx \geqq - \int_{\Omega_n} \nabla y_n . \nabla z_k \, dx + \int_{\Omega_n} f(z_k - y_n) \, dx$$

For $n \to \infty$ (and a proper subsequence) we deduce on the basis of (1 16), (1 17) and (1 18) that

$$\begin{aligned} - \int_{G_m} | \nabla y |^2 \, dx &\geqq \lim_{n \to \infty} \left(- \int_{G_m} | \nabla y_n |^2 \, dx \right) \\ &\geqq - \int_{G_m} \nabla y . \nabla z_k \, dx - C_0 \, | z_k |_{1 \, \Omega - G_m} + \\ &\quad + \int_{\Omega} f z_k \, dx - \int_{G_m} f v \, dx - C_0 \, \| f \|_{0 \, \Omega - G_m} \end{aligned}$$

Consequently, we may write

$$\begin{aligned} &\int_{G_m} \nabla y . \nabla (z_k - y) \, dx \geqq \\ &\qquad \geqq \int_{\Omega} f z_k \, dx - \int_{G_m} f y \, dx - C_0 (| z_k |_{1 \, \Omega - G_m} + \| f \|_{0 \, \Omega - G_m}) \end{aligned} \tag{1 19}$$

Let a $z \in K(v)$ be given There exists a function $\omega \in H^1(\Omega_\beta)$ such that $\omega = z$ on $\partial\Omega$ and $\omega \geqq 0$ a e in Ω_β Then $Z = z - \omega |_\Omega \in H_0^1(\Omega)$ and therefore a sequence $\{ Z_k \}, Z_k \in C_0^\infty(\Omega)$ exists such that if we define

$$z_k |_\Omega = \omega + Z_k , \quad z_k |_{\Omega_\beta - \Omega} = \omega ,$$

then it holds

$$1/m < d_k \equiv \operatorname{dist} (\Gamma, \operatorname{supp} Z_k) \Rightarrow z_k \in M_m ,$$

$$\| z_k - z \|_{1,\Omega} = \| Z_k - Z \|_{1 \, \Omega} \to 0 \quad \text{for} \quad k \to \infty$$

Passing to the limit in (1 19) for $m \to \infty, k \to \infty, m > 1/d_k$, we obtain

$$| z_k |_{1 \, \Omega - G_m} = | \omega |_{1 \, \Omega - G_m} \to 0 ,$$

$$\left| \int_{G_m} \nabla y . \nabla z_k \, dx - \int_{G_m} \nabla y . \nabla z \, dx + \int_{G_m} \nabla y . \nabla z \, dx - \int_{\Omega} \nabla y . \nabla z \, dx \right| \leqq$$

$$\leqq \left| \int_{G_m} \nabla y . \nabla (z_k - z) \, dx \right| + \left| \int_{\Omega - G_m} \nabla y . \nabla z \, dx \right|$$

$$\leqq | y |_{1,\Omega} \, | z_k - z |_{1,\Omega} + \int_{\Omega - G_m} | \nabla y . \nabla z | \, dx \to 0 \, .$$

Thus we arrive at the inequality

$$\int_{\Omega} \nabla y . \nabla z \, dx - \int_{\Omega} | \nabla y |^2 \, dx \geqq \int_{\Omega} fz \, dx - \int_{\Omega} fy \, dx \, . \qquad \text{Q.E.D.}$$

LEMMA 1.3 : *For any* $i = 1, 2, 3, 4$ *there exists a subsequence of the minimizing sequence, denoted again by* $\{ w_n \}$, *such that* $w_n \to v$ *in* $C([0, 1])$, $v \in \mathcal{U}_{ad}$ *and*

$$\lim_{n \to \infty} \mathfrak{J}_i(w_n) = \mathfrak{J}_i(v) \equiv J_i(y(v)) \, . \tag{1.20}$$

Proof : Let $i = 1$. For any $m > \alpha^{-1}$ it holds

$$\mathfrak{J}_1(w_n) = \int_{G_m} (y_n - z_0)^2 \, dx + \int_{\Omega_n - G_m} (y_n - z_0)^2 \, dx \geqq \int_{G_m} (y_n - z_0)^2 \, dx \, . \tag{1.21}$$

Considering the subsequence $\{ y_n^D \}$ and using (1.11), we obtain, by virtue of the Rellich's theorem

$$\lim_{n \to \infty} \mathfrak{J}_1(w_n) \geqq \int_{G_m} (y - z_0)^2 \, dx \, .$$

For $m \to \infty$ in the right-hand side we get

$$\lim_{n \to \infty} \mathfrak{J}_1(w_n) \geqq \int_{\Omega} (y - z_0)^2 \, dx \, .$$

Since $v \in \mathcal{U}_{ad}$, $y = y(v)$ on the basis of Lemma 1.2, and

$$\lim_{n \to \infty} \mathfrak{J}_1(w_n) = \inf_{w \in \mathcal{U}_{ad}} \mathfrak{J}_1(w) \, , \tag{1.22}$$

we have

$$\mathfrak{J}_1(v) = \int_{\Omega} (y - z_0)^2 \, dx \leqq \inf \mathfrak{J}_1(w) \Rightarrow \mathfrak{J}_1(v) = \inf \mathfrak{J}_1(w) \, .$$

From this and (1.22) we conclude that (1.20) holds.

Let $i = 2$. For $m > \alpha^{-1}$ we have

$$\mathfrak{J}_2(w_n) \geqq \int_{G_m} |\nabla y_n|^2 \, dx .$$

Since the latter integral is a weakly lower semi-continuous functional on $H^1(G_m)$, considering the subsequence $\{ y_n^D \}$, we may write

$$\varliminf_{n \to \infty} \mathfrak{J}_2(w_n) \geqq \int_{G_m} |\nabla y|^2 \, dx .$$

Passing to the limit for $m \to \infty$, we obtain

$$\inf_{w \in \mathcal{U}_{ad}} \mathfrak{J}_2(w) \geqq \int_{\Omega} |\nabla y|^2 \, dx . \tag{1.23}$$

Using Lemma 1.2 we conclude that the equality takes place in (1.23) and (1.20) follows.

Let $i = 3$. Using Lemma 1.2, we may write

$$\mathfrak{J}_3(w_n) - \mathfrak{J}_3(v) = \int_0^1 (y_n|_{\Gamma_n} - y|_{\Gamma}) \, dx_2 = I_1 + I_2 + I_3 , \tag{1.24}$$

where

$$I_1 = \int_0^1 (y_n|_{\Gamma_n} - y_n|_{\gamma_m}) \, dx_2 , \quad I_2 = \int_0^1 (y_n|_{\gamma_m} - y|_{\gamma_m}) \, dx_2 ,$$

$$I_3 = \int_0^1 (y|_{\gamma_m} - y|_{\Gamma}) \, dx_2 .$$

The following estimates hold :

$$\begin{aligned} |I_1| = \left| \int_0^1 dx_2 \int_{v - \frac{1}{m}}^{w_n} \frac{\partial y_n}{\partial x_1}(\xi, x_2) \, d\xi \right| &\leqslant \\ &\leqslant \left(\frac{1}{m} + \beta_n \right)^{1/2} \int_0^1 dx_2 \left(\int_{v - \frac{1}{m}}^{w_n} \left(\frac{\partial y_n}{\partial x_1} \right)^2 d\xi \right)^{1/2} \\ &\leqslant \left(\frac{1}{m} + \beta_n \right)^{1/2} \left[\int_0^1 \int_0^{w_n} \left(\frac{\partial y_n}{\partial x_1} \right)^2 dx_1 \, dx_2 \right]^{1/2} \\ &\leqslant \left(\frac{1}{m} + \beta_n \right)^{1/2} \| y_n \|_{1,\Omega_n} , \end{aligned} \tag{1.25}$$

where

$$\beta_n = \max_{x_2 \in [0,1]} | w_n(x_2) - v(x_2) | ;$$

$$\lim_{n \to \infty} I_2 = 0 \tag{1.26}$$

by virtue of (1.11); finally

$$\lim_{m \to \infty} I_3 = 0 \tag{1.27}$$

follows from the argument of (1.12), applied to $M \equiv [0, 1]$.

Combining (1.25), (1.9), (1.26), (1.27) and (1.24), we obtain (1.20).

Let $i = 4$. We may write, using Lemma 1.2 :

$$\mathfrak{J}_4(w_n) - \mathfrak{J}_4(v) = K_1 + K_2 + K_3 , \tag{1.28}$$

where

$$K_1 = \int_0^1 (y_n|_{\Gamma_n} - z_0)^2 \, dx_2 - \int_0^1 (y_n|_{\gamma_m} - z_0)^2 \, dx_2 ,$$

$$K_2 = \int_0^1 (y_n|_{\gamma_m} - z_0)^2 \, dx_2 - \int_0^1 (y|_{\gamma_m} - z_0)^2 \, dx_2 ,$$

$$K_3 = \int_0^1 (y|_{\gamma_m} - z_0)^2 \, dx_2 - \int_0^1 (y|_{\Gamma} - z_0)^2 \, dx_2 , \quad n > n_0(m) .$$

Let us derive estimates for K_i. Thus we have

$$| K_1 | \leqq \int_0^1 | y_n|_{\Gamma_n} - y_n|_{\gamma_m} | . | y_n|_{\Gamma_n} + y_n|_{\gamma_m} - 2 z_0 | \, dx_2 \leqq$$

$$\leqq \left[\int_0^1 (y_n|_{\Gamma_n} - y_n|_{\gamma_m})^2 \, dx_2 \right]^{1/2} \left[\int_0^1 (y_n|_{\Gamma_n} + y_n|_{\gamma_m} - 2 z_0)^2 \, dx_2 \right]^{1/2} , \tag{1.29}$$

$$\int_0^1 (y_n|_{\Gamma_n} - y_n|_{\gamma_m})^2 \, dx_2 = \int_0^1 dx_2 \left(\int_{v - \frac{1}{m}}^{w_n} \frac{\partial y_n}{\partial x_1} \, dx_1 \right)^2 \leqq$$

$$\leqq \left(\frac{1}{m} + \beta_n \right) \int_0^1 dx_2 \int_{v - \frac{1}{m}}^{w_n} \left(\frac{\partial y_n}{\partial x_1} \right)^2 dx_1 \leqq \left(\frac{1}{m} + \beta_n \right) \| y_n \|_{1,\Omega_n}^2 , \tag{1.30}$$

$$\int_0^1 (y_n|_{\Gamma_n} + y_n|_{\gamma_m} - 2 z_0)^2 \, dx_2 \leqq$$

$$\leqq 3 \int_0^1 [(y_n|_{\Gamma_n})^2 + (y_n|_{\gamma_m})^2 + 4 z_0^2] \, dx_2 \leqq C , \tag{1.31}$$

where C is independent of all sufficiently great $n, m, n > n_0(m)$. Indeed,

$$\int_0^1 (y_n|_{\Gamma_n})^2\, dx_2 \leqq 2\left[\int_0^1 (y_n|_{\Gamma_n} - z_0)^2\, dx_2 + \int_0^1 z_0^2\, dx_2\right] \leqq C_4\,,$$

since y_n generate a minimizing sequence $J_4(y_n)$. Furthermore, using (1.30), we may write

$$\int_0^1 (y_n|_{\gamma_m})^2\, dx_2 \leqq$$

$$\leqq 2\left(\int_0^1 (y_n|_{\Gamma_n})^2\, dx_2 + \int_0^1 (y_n|_{\gamma_m} - y_n|_{\Gamma_n})^2\, dx_2\right) \leqq 2\, C_4 + C_5$$

and we arrive at (1.31).

Consequently,

$$\lim K_1 = 0 \qquad \text{for} \quad n \to \infty\,, \quad m \to \infty\,, \quad n > n_0(m) \tag{1.32}$$

follows from (1.29), (1.30), (1.31) and (1.9).

Since the integral

$$K_0(\eta) \equiv \int_0^1 (\eta|_{\gamma_m} - z_0)^2\, dx_2$$

is a weakly lower semi-continuous functional in $H^1(G_m)$, we have for any $m > \alpha^{-1}$

$$\liminf_{n\to\infty} K_2 \geqq 0\,. \tag{1.33}$$

Finally,

$$|K_3| = \left|\int_0^1 (y|_{\gamma_m} - y|_\Gamma)(y|_{\gamma_m} + y|_\Gamma - 2\,z_0)\, dx_2\right| \leqq$$

$$\leqq C\left[\int_0^1 (y|_{\gamma_m} - y|_\Gamma)^2\, dx_2\right]^{1/2} \to 0 \quad \text{for} \quad m \to \infty \tag{1.34}$$

follows from Theorem 4.6 in chapter 2 of the book [3].

Using (1.28), (1.32), (1.33) and (1.34), we obtain

$$\lim_{n\to\infty} (\mathfrak{J}_4(w_n) - \mathfrak{J}_4(v)) \geqq 0$$

for a subsequence $\{ w_n \}$.

Hence

$$\mathfrak{J}_4(v) \leqq \lim \mathfrak{J}_4(w_n) = \inf_{\mathcal{U}_{ad}} \mathfrak{J}_4(w)$$

and only the equality sign may take place, since $v \in \mathcal{U}_{ad}$. Q.E.D.

Theorem 1 is now an easy consequence of Lemma 1.3 and of the equation

$$\lim_{n \to \infty} \mathfrak{J}_i(w_n) = \inf_{w \in \mathcal{U}_{ad}} \mathfrak{J}_i(w) .$$

2. APPROXIMATE SOLUTION BY FINITE ELEMENTS

The problems (P_i) have to be solved approximately. To this end we follow the approach of Begis and Glowinski [2], transforming each of the problems (P_i) into an equivalent one with the state problem defined on a fixed square domain and then employing bilinear finite elements on a uniform mesh. The unknown part of the boundary is sought among continuous piecewise linear functions.

Thus let N be a positive integer and $h = 1/N$. Denote by $e_j, j = 1, \ldots, N$, the interval $[(j-1)\,h, jh]$ and introduce the set

$$\mathcal{U}_{ad}^h = \{ w_h \in \mathcal{U}_{ad} \mid w_h|_{e_j} \in P_1 \quad \forall j \}$$

where P_1 denotes the space of linear polynomials.

Let Ω_h denote the domain bounded by the graph Γ_h of the function $w_h \in \mathcal{U}_{ad}^h$, i.e. $\Omega_h \equiv \Omega(w_h)$.

We define : $\hat{\Omega} = (0, 1) \times (0, 1)$,

$$\begin{aligned}
\hat{K}_{ij} &= [(i-1)\,h, ih] \times [(j-1)\,h, jh] , \\
\hat{\mathcal{K}}_h &= \{ \hat{K}_{ij} \}_{i,j=1}^N , \\
F_h : \hat{\Omega} \to \Omega_h , &\quad F_h = (F_{1h}, F_{2h}) , \\
F_{1h}(\hat{x}_1, \hat{x}_2) = \hat{x}_1\, w_h(\hat{x}_2) , &\quad F_{2h}(\hat{x}_1, \hat{x}_2) = \hat{x}_2 , \\
K_{ij} = F_h(\hat{K}_{ij}) \quad \forall i, j , &\quad \mathcal{K}_h = \{ K_{ij} \}_{i,j=1}^N .
\end{aligned} \tag{2 1}$$

Note that each K_{ij} is a trapezoid and

$$F_h|_{K_{ij}} \in Q_1 \times Q_1 ,$$

where $Q_1 = \{ p \mid p = p(\hat{x}_1, \hat{x}_2) = a_{00} + a_{10}\,\hat{x}_1 + a_{01}\,\hat{x}_2 + a_{11}\,\hat{x}_1\,\hat{x}_2 \}$ denotes the space of bilinear polynomials.

Let us consider the problem (1.6) on the domain Ω_h. To approximate $K(w_h)$ we introduce the set

$$K_h = \{ z_h \mid z_h \in K(w_h) \cap C(\overline{\Omega}_h),\ z_h \circ F_h |_{\hat{K}_{ij}} \in Q_1 \quad \forall i, j \} .$$

Let us define the finite element approximations of (1.6) as the solution $y_h \in K_h$ of (1.6) on Ω_h for any $z_h \in K_h$. Instead of (1.6), however, it is more suitable to solve numerically an equivalent problem on $\hat{\Omega}$, which is obtained by the transformation (2.1) of the integrals in (1.6). Thus we arrive at the inequality

$$a(w_h ; \hat{y}_h, \hat{z}_h - \hat{y}_h) \geqq L(w_h ; \hat{z}_h - \hat{y}_h) \quad \forall \hat{z}_h \in \hat{K}_h , \tag{2.2}$$

where

$$\hat{y}_h = y_h \circ F_h \in \hat{K}_h ,$$

$$a(w_h ; \hat{y}_h, \hat{t}_h) = \int_{\hat{\Omega}} \left[\frac{1}{w_h^2} \frac{\partial \hat{y}_h}{\partial x_1} \frac{\partial \hat{t}_h}{\partial x_1} + \left(\hat{x}_1 \frac{w_h'}{w_h} \frac{\partial \hat{y}_h}{\partial \hat{x}_1} - \frac{\partial \hat{y}_h}{\partial \hat{x}_2} \right) \cdot \right.$$

$$\left. \cdot \left(\hat{x}_1 \frac{w_h'}{w_h} \frac{\partial \hat{t}_h}{\partial \hat{x}_1} - \frac{\partial \hat{t}_h}{\partial \hat{x}_2} \right) \right] w_h \, d\hat{x}, \quad w_h' = dw_h/dx_2 ,$$

$$L(w_h ; \hat{t}_h) = \int_{\hat{\Omega}} \hat{f} \hat{t}_h \, w_h \, d\hat{x} , \quad \hat{f} = f \circ F_h ,$$

$$\hat{K}_h = \{ \hat{z}_h \mid \hat{z}_h \in C(\overline{\hat{\Omega}}),\ \hat{z}_h \mid_{\hat{K}_{ij}} \in Q_1\ \forall i, j,\ \ \hat{z}_h \geqq 0 \text{ on } \hat{\Gamma}, \quad z_h = 0 \text{ on } \partial\hat{\Omega} - \hat{\Gamma} \} ,$$

$$\hat{\Gamma} = \{ (\hat{x}_1, \hat{x}_2) \mid \hat{x}_1 = 1,\ 0 \leqq \hat{x}_2 \leqq 1 \} .$$

To simplify further the calculations, we introduce approximate forms a_h and L_h, as follows :

$$a_h(w_h ; \hat{y}_h, \hat{t}_h) = \sum_{i,j=1}^{N} \int_{\hat{K}_{ij}} \left[\frac{1}{w_h^2(\xi_j)} \frac{\partial \hat{y}_h}{\partial \hat{x}_1} \frac{\partial \hat{t}_h}{\partial \hat{x}_1} + \right.$$

$$\left. + \left(\xi_i \frac{w_h'}{w_h(\xi_j)} \frac{\partial \hat{y}_h}{\partial \hat{x}_1} - \frac{\partial \hat{y}_h}{\partial \hat{x}_2} \right) \left(\xi_i \frac{w_h'}{w_h(\xi_j)} \frac{\partial \hat{t}_h}{\partial \hat{x}_1} - \frac{\partial \hat{t}_h}{\partial \hat{x}_2} \right) \right] \cdot w_h(\xi_j) \, d\hat{x} ,$$

where

$$\xi_i = \left(i - \frac{1}{2} \right) h , \quad \xi_j = \left(j - \frac{1}{2} \right) h ,$$

$$L_h(w_h ; \hat{t}_h) = \frac{h^2}{4} \sum_{i,j=1}^{N} w_h(\xi_j) \sum_{k=1}^{4} \hat{f}(P_{ij}^k) \, \hat{t}_h(P_{ij}^k) ,$$

where P_{ij}^k $k = 1, \ldots, 4$, are the vertices of $\hat{K}_{ij}$ [1].

---

[1] In case that f is not continuous, we replace $f(P_{ij}^k)$ by the mean value of $\hat{f}$ over the set

$$\mathcal{O}(P_{ij}^k) \equiv \hat{\Omega} \cap \left\{ \mid \hat{x}_r - (P_{ij}^k)_r \mid < \frac{1}{2} h, r = 1, 2 \right\}$$

We replace (2.2) by the inequality

$$a_h(w_h\,;\hat{y}_h,\hat{z}_h-\hat{y}_h)\geqq L_h(w_h\,;\hat{z}_h-\hat{y}_h)\quad\forall\hat{z}_h\in\hat{K}_h\,. \tag{2.3}$$

Moreover, the cost functionals $\mathfrak{J}_i$ will be replaced by the approximate functionals :

$$\mathfrak{J}_{1h}(w_h)=J_{1h}(w_h\,;\hat{y}_h)=\frac{h^2}{4}\sum_{i,j=1}^{N}w_h(\xi_j)\sum_{k=1}^{4}(\hat{y}_h(P_{ij}^k)-z_0)^2\,, \tag{2.4}$$

$$\mathfrak{J}_{2h}(w_h)=J_{2h}(w_h\,;\hat{y}_h)=a_h(w_h\,;\hat{y}_h,\hat{y}_h)\,,$$

$$\mathfrak{J}_{3h}(w_h)=J_{3h}(w_h\,;\hat{y}_h)=h\sum_{j=1}^{N}\hat{y}_h(1,\xi_j)\,.$$

Note that among the functionals $\mathfrak{J}_i$ only $\mathfrak{J}_3$ is integrated without loss of accuracy, i.e.

$$\mathfrak{J}_{3h}(w^h)=J_3(y_h)=\int_0^1\hat{y}_h(1,\hat{x}_2)\,d\hat{x}_2=\int_0^1 y_h\,|_{\Gamma_h}\,dx_2\,. \tag{2.4'}$$

We then solve the problem :

$$(P_{ih})\qquad\qquad \mathfrak{J}_{ih}(v_h)=\min_{w_h\in\mathcal{U}_{ad}}\mathfrak{J}_{ih}(w_h)\,,\quad i=1,2,3\,,$$

where $\mathfrak{J}_{ih}$ are defined by means of the formulae (2.4) and $\hat{y}_h\in\hat{K}_h$ are solutions of (2.3).

3. CONVERGENCE OF FINITE ELEMENT APPROXIMATIONS

We are going to show that a subsequence $\{u_h\}$ of the approximate solutions exists, which converges in some sense to a solution of the continuous problem (P_i), $i=1,2,3$. In several points of the argument we use the results of Begis and Glowinski [1].

First we prove an important lemma.

LEMMA 3.1 : *Suppose that $f\in C^1(\overline{\Omega}_\beta)$ and a sequence $\{v_h\}$, $v_h\in\mathcal{U}_{ad}^h$, converges uniformly to a function v. Let $\hat{y}_h$ be the solution of the inequality (2.3) for $w_h\equiv v_h$ and $y_h=\hat{y}_h\circ F_h^{-1}$. Moreover, let $y(v)$ be the solution of the problem (1.6) for $w\equiv v$ on the domain $\Omega\equiv\Omega(v)$.*

Then for any integer $m>\alpha^{-1}$ a subsequence $\{y_h\}$ exists such that

$$y_h\to y(v)\ (\textit{weakly})\ \textit{in}\ H^1(G_m)\,, \tag{3.1}$$

where

$$G_m=\left\{(x_1,x_2)\,|\,0<x_1<v(x_2)-\frac{1}{m},0<x_2<1\right\}.$$

Proof : First let us show that (2.3) has a unique solution. In fact, one can derive that (see [1], Proposition 6.1)

$$a_h(w_h ; \hat{z}_h, \hat{z}_h) \geqq \frac{\alpha}{1 + \beta^2 + C_1^2} | \hat{z}_h |_{1,\hat{\Omega}}^2 \quad \forall h, \forall w_h \in \mathcal{U}_{ad}^h, \forall \hat{z}_h \in H^1(\hat{\Omega}) . \quad (3.2)$$

Since the set $\hat{K}_h$ is convex, the inequality (2.3) is equivalent with the problem of minimizing the functional

$$I_h(\hat{z}_h) = \frac{1}{2} a_h(w_h ; \hat{z}_h, \hat{z}_h) - L_h(w_h ; \hat{z}_h)$$

over the set $\hat{K}_h$. From (3.2), (3.5) and the Friedrichs inequality

$$\| \hat{z}_h \|_{0,\hat{\Omega}} \leqq C | \hat{z}_h |_{1,\hat{\Omega}} \; \forall \hat{z}_h \in H^1(\hat{\Omega}) \text{ such that } \hat{z}_h(0, \hat{x}_2) = 0 , \quad (3.3)$$

we conclude that I_h is coercive on $\hat{K}_h$ and convex. Using also its continuity and the closedness of $\hat{K}_h$, the existence of a solution follows. The uniqueness is a consequence of (3.2) and the Friedrichs inequality.

Let us insert $\hat{z}_h = 0$ and $\hat{z}_h = 2 \hat{y}_h$ into (2.3), respectively. Thus we obtain

$$a_h(v_h ; \hat{y}_h, \hat{y}_h) = L_h(v_h ; \hat{y}_h) . \quad (3.4)$$

It is easy to show that if $\hat{f} \in C(\overline{\hat{\Omega}})$, then

$$| L_h(v_h, \hat{y}_h) | \leqq \beta \| \hat{f} \|_C \| \hat{y}_h \|_{0,\Omega} . \quad (3.5)$$

Using (3.2), (3.4), (3.5) and (3.3), we obtain

$$| \hat{y}_h |_{1,\hat{\Omega}} \leqq C \quad \forall h ; \quad (3.6)$$

Since it holds for any $h > 0$

$$C_0 | y_h |_{1,\Omega_h}^2 \leqq | \hat{y}_h |_{1,\hat{\Omega}}^2 \leqq C_2 | y_h |_{1,\Omega_h}^2 \quad (3.7)$$

(see [1], Proposition 5.1), we may also write

$$| y_h |_{1,\Omega_h}^2 \leqq C \quad \forall h .$$

The Friedrichs inequality in Ω_h yields

$$\| y_h \|_{1,\Omega_h}^2 \leqq C \quad \forall h , \quad (3.8)$$

where C is independent of h.

Arguing as in the proof of Theorem 1, we deduce that a subsequence of $\{ y_h \}$, which will be denoted by the same symbol, and a function $y^0 \in H^1(\Omega)$ exist such that for any $m > \alpha^{-1}$

$$y_h \rightharpoonup y^0 |_{G_m} \quad \text{in} \quad H^1(G_m) \quad \text{(weakly)} \tag{3.9}$$

The same argument as the proof of Lemma 1.1 leads to the conclusion that the trace of y^0 on $\Gamma = \Gamma(v)$ is non-negative.

Let us show that $y^0 = y(v)$, i.e. y^0 satisfies the inequality (1.6) on $\Omega = \Omega(v)$ and $y^0 \in K(v)$. The latter assertion follows from the closedness of $V(G_m)$ (see the proof of Lemma 1.2 (i)) and (3.9). It remains to prove (1.6).

Let $z \in K(v)$. A function $w \in H^1(\Omega_\beta)$ exists such that

$$w = z \quad \text{on} \quad \partial\Omega\,, \quad w = 0 \quad \text{on} \quad \partial\Omega_\beta\,,$$

$$w \geqq 0 \quad \text{a.e.} \quad \text{in} \quad \Omega_\beta$$

Extending w by zero outside Ω_β, shrinking and regularizing, we obtain functions $R_{\mathscr{H}} w$ such that

$$R_{\mathscr{H}} w \in C_0^\infty(\Omega_\beta)\,, \quad R_{\mathscr{H}} w \geqq 0 \quad \text{in} \quad \Omega_\beta\,,$$

$$\| R_{\mathscr{H}} w - w \|_{1,\Omega_\beta} \to 0 \quad \text{for} \quad \mathscr{H} \to 0$$

Then $Z \equiv z - w \in H_0^1(\Omega)$ and therefore functions $Z_k \in C_0^\infty(\Omega)$ exist such that

$$\| Z_k - Z \|_{1,\Omega} \to 0 \quad \text{for} \quad k \to \infty$$

Let us define functions $\omega_k^{\mathscr{H}}$ in Ω_β such that

$$\omega_k^{\mathscr{H}} = R_{\mathscr{H}} w |_\Omega + Z_k \quad \text{in} \quad \Omega\,,$$

$$\omega_k^{\mathscr{H}} = R_{\mathscr{H}} w \quad \text{in} \quad \Omega_\beta - \Omega$$

Then obviously

$$\omega_k^{\mathscr{H}} \geqq 0 \quad \text{in} \quad \Omega_\beta - G_m\,, \quad \text{if} \quad 1/m < d_k = \text{dist}\,(\Gamma, \text{supp}\, Z_k)\,, \tag{3.10}$$

$$\begin{aligned} \| \omega_k^{\mathscr{H}} - z \|_{1,\Omega} &= \| R_{\mathscr{H}} w + Z_k - Z - w \|_{1,\Omega} \leqq \\ &\leqq \| R_{\mathscr{H}} w - w \|_{1,\Omega} + \| Z_k - Z \|_{1,\Omega} \to 0 \quad \text{for} \quad \mathscr{H} \to 0,\, k \to \infty \end{aligned} \tag{3.11}$$

Next let ω_h be the interpolate of $\omega_k^{\mathscr{H}}$ on the mesh $\mathscr{K}_h$, i.e. a function such that

$$\omega_h(P) = \omega_k^{\mathscr{H}}(P) \text{ at all nodes of the mesh } \mathscr{K}_h\,,$$

$$\omega_h \circ F_h |_{K_{ij}} \in Q_1 \quad \forall i, j$$

It is readily seen that $\varphi_h \in K_h$ for sufficiently small h, since $\Gamma_h \subset \Omega_\beta - G_m$ holds for such h.

From the regularity of the family $\{\mathscr{K}_h\}$, $0 < h < 1$ of meshes (see [1], Lemma 7.2) it follows (see [2]) that

$$\| \varphi_h - \varphi_k^{\mathscr{H}} \|_{1,\Omega_h} \leqq Ch \| \varphi_k^{\mathscr{H}} \|_{2,\Omega_\beta} . \tag{3.12}$$

Next let us estimate the difference

$$\left| L_h(v_h ; \hat{z}_h) - \int_{\hat{\Omega}} v_h \hat{f} \hat{z}_h \, d\hat{x} \right| \leqq$$
$$\leqq \sum_{i,j=1}^{N} \left| v_h(\xi_j) \sum_{k=1}^{4} \frac{h^2}{4} \mathscr{F}(P_{ij}^k) - \int_{\hat{K}_{ij}} v_h(\hat{x}_2) \mathscr{F} \, d\hat{x} \right|$$
$$\leqq \sum_{i,j=1}^{N} \left[\int_{\hat{K}_{ij}} | g_h \mathscr{F} - \mathscr{F} | \, v_h(\xi_j) \, d\hat{x} + \int_{\hat{K}_{ij}} | \mathscr{F} | \, | v_h(\xi_j) - v_h(\hat{x}_2) | \, d\hat{x} \right],$$

where $\mathscr{F} \equiv \hat{f} \hat{z}_h$ and $g_h \mathscr{F}$ is the piecewise constant function, equal to $\mathscr{F}(P_{ij}^k)$ over each of the subdomain $\mathscr{O}(P_{ij}^k)$. Since

$$| g_h \mathscr{F} - \mathscr{F} | \leqq Ch \, | \nabla \mathscr{F} | , \quad | v_h(\xi_j) - v_h(\hat{x}_2) | \leqq C_1 \, h/2 , \tag{3.12}$$

and

$$\| \nabla \mathscr{F} \|_{0,\hat{\Omega}} \leqq C \| \hat{f} \|_{C^1(\overline{\Omega}_\beta)} \| \hat{z}_h \|_{1,\hat{\Omega}} ,$$
$$\| \mathscr{F} \|_{0,\hat{\Omega}} \leqq \| \hat{f} \|_{C^1(\Omega_\beta)} \| z_h \|_{0,\hat{\Omega}} ,$$

we obtain, using also (3.3),

$$\left| L_h(v_h, \hat{z}_h) - \int_{\hat{\Omega}} v_h \hat{f} \hat{z}_h \, d\hat{x} \right| \leqq Ch \| \hat{f} \|_{C^1(\overline{\hat{\Omega}}_\beta)} | \hat{z}_h |_{1,\hat{\Omega}} \tag{3.13}$$
$$\forall \hat{z}_h \in H^1(\hat{\Omega}), \, \hat{z}_h(0, \hat{x}_2) = 0 .$$

An analogous argument yields

$$| a_h(v_h ; \hat{y}_h, \hat{z}_h) - a(v_h ; \hat{y}_h, \hat{z}_h) | \leqq Ch \, | \hat{y}_h |_{1,\hat{\Omega}} \, | \hat{z}_h |_{1,\hat{\Omega}} \tag{3.14}$$
$$\forall \hat{y}_h, \hat{z}_h \in H^1(\hat{\Omega}) \quad \text{such that} \quad \hat{y}_h(0, \hat{x}_2) = \hat{z}_h(0, \hat{x}_2) = 0 .$$

Since $\hat{\varphi}_h = \omega_h \circ F_h \in \hat{K}_h$, we have

$$a_h(v_h ; \hat{y}_h, \hat{\varphi}_h - \hat{y}_h) \geqq L_h(v_h ; \hat{\omega}_h - \hat{y}_h) , \tag{3.15}$$

$$a(v_h ; \hat{y}_h, \hat{\varphi}_h - \hat{y}_h) = \int_{\Omega_h} \nabla y_h . \nabla(\varphi_h - y_h) \, dx . \tag{3.16}$$

Therefore we may write

$$\int_{\Omega_h} \nabla y_h . \nabla(\omega_h - y_h)\, dx = a_h(v_h, \hat{y}_h, \hat{\omega}_h - \hat{y}_h) +$$

$$+ [a(v_h, \hat{y}_h, \hat{\omega}_h - \hat{y}_h) - a_h(v_h, \hat{y}_h, \hat{\omega}_h - \hat{v}_h] \geqq$$

$$\geqq \int_{\Omega_h} f(\omega_h - y_h)\, dx + \left[L_h(v_h, \hat{\omega}_h - \hat{y}_h) - \int_{\Omega} v_h \hat{f}(\hat{\omega}_h - \hat{y}_h)\, d\hat{x} \right] +$$

$$+ [a(v_h, \hat{v}_h, \hat{\omega}_h - \hat{y}_h) - a_h(v_h, \hat{y}_h, \hat{\omega}_h - \hat{y}_h)] \tag{3 17}$$

Passing to the limit with $h \to 0$, we obtain for any $m > \alpha^{-1}$

$$\int_{G_m} \nabla y_h . \nabla \omega_h\, dx \to \int_{G_m} \nabla y^0 . \nabla \omega_k^{\mathscr{H}}\, dx\,, \tag{3 18}$$

by virtue of (3 9) and (3 12), furthermore

$$\left| \int_{\Omega_h \quad \Omega} \nabla y_h . \nabla \omega_h\, dx \right| \leqq | y_h |_{1\, \Omega_h} . | \omega_h |_{1\, \Omega_h - \Omega} \leqq$$

$$\leqq C \,\| R_{\mathscr{H}} w \|_{C^1(\bar{\Omega}_\beta)} (\text{mes}\,(\Omega_h - \Omega))^{1/2}\,, \tag{3 19}$$

since it is readily seen that in $\Omega_h - \Omega$ we have

$$\left| \frac{\partial \omega_h}{\partial x_r} \right| \leqq \max_{\bar{\Omega}_\beta} \left| \frac{\partial R_{\mathscr{H}} w}{\partial x_r} \right| \leqq \| R_{\mathscr{H}} w \|_{C^1(\bar{\Omega}_\beta)}\,, \quad r = 1, 2$$

Finally, we can write for $m > 1/d_k$

$$\left| \int_{(\Omega - G_m) \cap \Omega_h} \nabla y_h . \nabla \varphi_h\, dx \right| \leqq C \,| \omega_h |_{1\, \Omega - G_m} \leqq$$

$$\leqq C(1/m)^{1/2} \,\| R_{\mathscr{H}} w \|_{C^1(\bar{\Omega}_\beta)} \tag{3 20}$$

Combining (3 18), (3 19) and (3 20), we obtain for any $m > 1/d_k$, any $\mathscr{H}$ and k

$$\limsup_{h \to 0} \left(- \int_{\Omega_h} \nabla y_h . \nabla \omega_h\, dx \right) = \limsup_{h \to 0} \left\{ - \int_{G_m} \nabla y_h . \nabla \omega_h\, dx - \right.$$

$$\left. - \int_{\Omega_l - \Omega} \nabla y_h . \nabla \omega_h\, dx - \int_{(\Omega - G_m) \cap \Omega_h} \nabla y_h . \nabla \omega_h\, dx \right\} \geqq$$

$$\geqq - \int_{G_m} \nabla y^0 . \nabla \omega_k^{\mathscr{H}}\, dx - C_{\mathscr{H}} / \sqrt{m}\,, \tag{3 21}$$

where $C_{\mathscr{H}}$ depends on $\mathscr{H}$ but not on m

In a similar way we get

$$\limsup_{h\to 0} \int_{\Omega_h} - f v_h \, dx \geqq - \int_{G_m} f y^0 \, dx - C_0 \| f \|_{0\,\Omega - G_m}, \tag{3 22}$$

$$\lim_{h\to 0} \int_{\Omega_h} f \omega_h \, dx = \lim_{h\to 0} \int_{\Omega \cap \Omega_h} f \omega_h \, dx = \int_\Omega f \omega_k^{\mathscr{H}} \, dx, \tag{3 23}$$

where the estimate (3 12) has also been used

From (3 17) it follows

$$- \int_{G_m} | \nabla y_h |^2 \, dx \geqq - \int_{\Omega_h} \nabla y_h . \nabla \omega_h \, dx + \int_{\Omega_h} f(\omega_h - y_h) \, dx +$$
$$+ \left[L_h(v_h, \hat{\omega}_h - \hat{y}_h) - \int_\Omega v_h \hat{f}(\hat{\omega}_h - \hat{y}_h) \, dx \right]$$
$$+ \left[a(v_h, \hat{y}_h, \hat{\omega}_h - \hat{y}_h) - a_h(v_h, \hat{\omega}_h - \hat{y}_h) \right]$$

For $h \to 0$ we obtain, on the basis of (3 13), (3 14), (3 9), (3 21), (3 22) and (3 23)

$$- \int_{G_m} | \nabla y^0 |^2 \, dx \geqq \limsup \left[- \int_{G_m} | \nabla y_h |^2 \, dx \right] \geqq$$
$$\geqq - \int_{G_m} \nabla y^0 . \nabla \omega_k^{\mathscr{H}} \, dx - C_{\mathscr{H}} / \sqrt{m}$$
$$+ \int_\Omega f \omega_k^{\mathscr{H}} \, dx - \int_{G_m} f y^0 \, dx - C_0 \| f \|_{0\,\Omega - G_m}$$

Applying (3 13) and (3 14), we have used the estimate

$$| \hat{\omega}_h - \hat{y}_h |_{1\,\Omega} \leqq C_{k\mathscr{H}} < \infty \quad \forall h,$$

which in turn is an easy consequence of (3 6), (3 7) (for ω_h) and (3 12)

Passing to the limit with $m \to \infty$, we obtain

$$\int_\Omega \nabla y^0 . \nabla (\omega_k^{\mathscr{H}} - y^0) \, dx \geqq \int_\Omega f(\omega_k^{\mathscr{H}} - y^0) \, dx$$

Finally, we let $\mathscr{H} \to 0$ and $k \to \infty$ to get from (3 11)

$$\int_\Omega \nabla y^0 . \nabla (z - y^0) \, dx \geqq \int_\Omega f(z - y^0) \, dx \qquad \text{Q E D}$$

LEMMA 3.2 : *Let the assumptions of Lemma* 3.1 *hold. Then a subsequence of* $\{ v_h \}$ *exists such that*

$$\lim_{h \to 0} \mathfrak{J}_{ih}(v_h) = \mathfrak{J}_i(v), \quad i = 1, 2, 3 .$$

Proof : Let $i = 1$. Using the functions $g_h \hat{y}_h$ and piecewise constant functions $p_h v_h$, which equal to $v_h(\xi_j)$ in every subinterval e_j, we may write

$$J_{1h}(v_h ; \hat{y}_h) = \int_{\hat{\Omega}} p_h v_h (g_h \hat{y}_h - z_0)^2 \, d\hat{x} .$$

From the estimate (see [1], (7.34))

$$\| g_h \hat{y}_h - \hat{y}_h \|_{0,\hat{\Omega}} \leqq h\sqrt{2} \, | \hat{y}_h |_{1,\hat{\Omega}}$$

and (3.6) we derive easily that

$$\lim_{h \to 0} J_{1h}(v_h ; \hat{y}_h) = \lim_{h \to 0} \int_{\hat{\Omega}} | \hat{y}_h - z_0 |^2 v_h \, d\hat{x} . \tag{3.24}$$

On the other hand, recall that

$$\int_{\hat{\Omega}} | \hat{y}_h - z_0 |^2 v_h \, d\hat{x} = \int_{\Omega_h} (y_h - z_0)^2 \, dx =$$

$$= \int_{G_m} (y_h - z_0)^2 \, dx + \int_{\Omega_h - G_m} (y_h - z_0)^2 \, dx . \tag{3.25}$$

From the Rellich's theorem and Lemma 3.1 we get

$$\lim_{h \to 0} \int_{G_m} (y_h - z_0)^2 \, dx = \int_{G_m} (y(v) - z_0)^2 \, dx . \tag{3.26}$$

Second, we prove that

$$\lim_{\substack{h \to 0 \\ m \to \infty}} \int_{\Omega_h - G_m} (y_h - z_0)^2 \, dx = 0 . \tag{3.27}$$

Recall that the curve $F_h^{-1}(\gamma_m)$ is the graph of the function

$$\psi_{hm}(\hat{x}_2) = \frac{v(\hat{x}_2) - \dfrac{1}{m}}{v_h(\hat{x}_2)}$$

and

$$\lim_{\substack{h \to 0 \\ m \to \infty}} |\psi_{hm} - 1| \to 0 \quad \text{uniformly}. \tag{3.28}$$

Then

$$\int_{\Omega_h - G_m} (y_h - z_0)^2 \, dx = \int_0^1 v_h \, d\hat{x}_2 \int_{\psi_{hm}(x_2)}^1 (\hat{y}_h - z_0)^2 \, d\hat{x}_1 \leqq$$

$$\leqq \beta \int_{\hat{V}_{hm}} (\hat{y}_h - z_0)^2 \, d\hat{x} = C\beta(\text{mes } \hat{V}_{hm})^{1/2} \left[\int_{\hat{\Omega}} (\hat{y}_h - z_0)^4 \, d\hat{x} \right]^{1/2},$$

where $\hat{V}_{hm} \equiv F_h^{-1}(\Omega_h - G_m)$. Since

$$\| \hat{y}_h - z_0 \|_{L^4(\hat{\Omega})} \leqq C \| \hat{y}_h - z_0 \|_{1,\hat{\Omega}} \leqq C_3 < \infty \quad \forall h$$

and mes $\hat{V}_{hm}$ tends to zero by virtue of (3.28), (3.27) is true. From (3.24), (3.25), (3.26) and (3.27), the assertion of the Lemma follows.

Let $i = 2$. According to (2.4) and (3.4), (3.13) we may write

$$\lim_{h \to 0} \mathfrak{J}_{2h}(v_h) = \lim a_h(v_h, \hat{y}_h, \hat{y}_h) = \lim L_h(v_h, \hat{y}_h) =$$

$$= \lim \int_{\hat{\Omega}} v_h \hat{f} \hat{y}_h \, d\hat{x} = \lim \int_{\Omega_h} f y_h \, dx .$$

Since

$$\int_{\Omega_h} f y_h \, dx = \int_{G_m} f y_h \, dx + \int_{\Omega_h - G_m} y_h \, dx$$

and from (3.8) it follows that

$$\lim_{\substack{h \to 0 \\ m \to \infty}} \left| \int_{\Omega_h - G_m} f y_h \, dx \right| \leqq C \lim_{\substack{m \to \infty \\ h \to 0}} \| f \|_{0,\Omega_h - G_m} = 0 ,$$

we obtain

$$\lim_{h \to 0} \int_{\Omega_h} f y_h \, dx = \int_{\Omega(v)} f y(v) \, dx = \int_{\Omega(v)} | \nabla y |^2 \, dx = \mathfrak{J}_2(v) .$$

Let $i = 3$. Recall that we have (see (2.4'))

$$\mathfrak{J}_{3h}(v_h) = J_3(y_h) = \int_0^1 y_h |_{\Gamma_h} \, dx_2 .$$

Following the argument of Lemma 1.3 with y_h instead of y_n and using (3.8), (3.9), we obtain the assertion of the Lemma 3.2. Q.E.D.

THEOREM 3.1 : *Assume that* $f \in C^1(\overline{\Omega}_\beta)$. *Let* $\{ u_h \}$, $h \to 0$, *be a sequence of solutions of the approximate problems* (P_{ih}), $i = 1, 2, 3$ *and let* $\hat{Y} = \hat{Y}_h(u_h)$ *be the corresponding solutions of* (2.3) (*with* $w_h \equiv u_h$), $Y_h = \hat{Y}_h \circ F_h^{-1}$.

Then a subsequence of $\{ u_h \}$ *exists such that for* $h \to 0$

$$u_h \to u \quad in \quad C([0, 1]) , \tag{3.29}$$

$$Y_h \rightharpoonup y(u)\ (weakly)\ in\ H^1(G_m) \quad \forall m > \alpha^{-1} , \tag{3.30}$$

where u *and* $y(u)$ *is a solution of the problem* (P_i) *and of* (1.6) *with* $w \equiv u$, *respectively,* G_m *is the domain bounded by the graph of* $u - 1/m$.

Any subsequence of $\{ u_h \}$, *converging in* $C([0, 1])$, *has the property* (3.30).

Proof : Consider a function $v \in \mathcal{U}_{ad}$. There exists a sequence $\{ v_h \}$, $h \to 0$ such that $v_h \in \mathcal{U}_{ad}^h$, $v_h \to v$ in $C([0, 1])$ (see e.g. [1], Lemma 7.1). Let $\hat{y}_h$ be the solutions of (2.3) for $w_h \equiv v_h$ and $y_h = \hat{y}_h \circ F_h^{-1}$.

Since $\mathcal{U}_{ad}$ is compact in $C([0, 1])$, a subsequence of $\{ u_h \}$ exists such that $u_h \to u$ in $C([0, 1])$ and $u \in \mathcal{U}_{ad}$.

The definition of the problem (P_{ih}) yields

$$\mathfrak{J}_{ih}(u_h) \leqslant \mathfrak{J}_{ih}(v_h) \quad \forall h , \quad i = 1, 2, 3 .$$

Let us apply Lemmas 3.1 and 3.2 to both sequences $\{ u_h \}$ and $\{ v_h \}$, to obtain

$$\mathfrak{J}_i(u) \leqq \mathfrak{J}_i(v) .$$

Consequently, u is a solution of (P_i). The convergence (3.30) follows from Lemma 3.1.

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