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*RAIRO. Analyse numérique*, tome 16, n° 1 (1982), p. 39-47

[http://www.numdam.org/item?id=M2AN\\_1982\\_\\_16\\_1\\_39\\_0](http://www.numdam.org/item?id=M2AN_1982__16_1_39_0)

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## ITERATIVE REFINEMENT OF FINITE ELEMENT APPROXIMATIONS FOR ELLIPTIC PROBLEMS (\*)

by Lin QUN <sup>(1)</sup>

Communiqué par J A NITSCHÉ

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Résumé — On présente une extrapolation itérative d'approximations de problèmes elliptiques par des éléments fins de bas degré

Abstract — An iterative refinement of low-degree finite element approximations for elliptic problems is presented

1. We will consider the boundary value problem

$$\Delta u + \sum a_i \frac{\partial u}{\partial x_i} + bu = -f \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \quad (1)$$

Here  $\Omega \subset \mathbb{R}^N$  is a bounded domain with boundary  $\partial\Omega$  sufficiently smooth. We will adopt the standard notations (cf. Gilbarg-Trudinger, 1977). Especially  $(., .)$  respective  $(., .)_1$  denote the  $L_2(\Omega)$ -inner-product respective the Dirichlet integral and  $\| . \|_k$  the norm in  $H_k = W_2^k(\Omega)$ .

The weak formulation of problem (1) is  $u \in \overset{\circ}{H}_1$  and

$$(u, v)_1 = (\sum a_i u |_{,i} + bu + f, v) \quad \text{for } v \in \overset{\circ}{H}_1. \quad (2)$$

Our basic assumption is : problem (1) resp. (2) has a unique solution  $u$  to  $f \in H_0$  with  $u \in \overset{\circ}{H}_1 \cap H_2$  and  $\| u \|_2 \leq c \| f \|$ . Now let  $S_h$  be the space of linear finite

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(\*) Reçu le 17 mars 1981.

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elements with isoparametric modifications in the boundary elements such that  $S_h \subset \mathring{H}_1$  holds true. Due to an argument of Schatz (1974) for  $h$  sufficiently small the Galerkin-approximation  $u^0 = u_h \in S_h$  defined by

$$(u^0, \chi)_1 = \left( \sum a_i u^0 |_{\Gamma_i} + bu^0 + f, \chi \right) \quad \text{for } \chi \in S_h \quad (3)$$

is uniquely defined. The error estimate

$$\|u - u^0\| + h \|u - u^0\|_1 \leq ch^2 \|u\|_2 \quad (4)$$

is well known.

In Lin Qun (1978), (1980) we introduced a refinement of  $u^0$  on the basis of the additional assumption: to  $F \in H_0$  given the solution of

$$\begin{aligned} -\Delta U &= F & \text{in } \Omega, \\ U &= 0 & \text{on } \partial\Omega \end{aligned} \quad (5)$$

resp.  $U \in \mathring{H}_1$  and

$$(U, v)_1 = (F, v) \quad \text{for } v \in \mathring{H}_1 \quad (6)$$

is computable. Then given  $u^0$  we can compute  $\bar{u}^0$  defined by  $\bar{u}^0 \in \mathring{H}_1$  and

$$(\bar{u}^0, v)_1 = \left( \sum a_i u^0 |_{\Gamma_i} + bu^0 + f, v \right) \quad \text{for } v \in \mathring{H}_1. \quad (7)$$

This leads to a higher accuracy in the  $H_1$ -norm:

$$\|u - \bar{u}^0\|_1 \leq ch^2 \|u\|_2. \quad (8)$$

Of course  $\bar{u}^0$  is not an element of  $S_h$ .

Following a suggestion of Nitsche (private communication) we construct starting with the pair  $(u^0, \bar{u}^0)$  iterates  $(u^{v+1}, \bar{u}^{v+1})$  for  $v \geq 0$  defined

$$u^{v+1} = \bar{u}^v + \varphi^v \quad (9)$$

with  $\varphi^v \in S_h$  and

$$\begin{aligned} (\varphi^v, \chi)_1 - \left( \sum a_i \varphi^v |_{\Gamma_i} + b\varphi^v, \chi \right) &= \\ &= \left( \sum a_i (\bar{u}^v - u^v) |_{\Gamma_i} + b(\bar{u}^v - u^v), \chi \right) \quad \text{for } \chi \in S_h \end{aligned} \quad (10)$$

and on the other hand by  $(v \geq 0)$

$$(\bar{u}^v, v)_1 = \left( \sum a_i u^v |_{\Gamma_i} + bu^v + f, v \right) \quad \text{for } v \in \mathring{H}_1. \quad (11)$$

In Section 3 we give the proof of :

**THEOREM 1 :** *Let  $(u^v, \bar{u}^v)$  be defined as above. Then*

$$\|u - u^v\| + \|u - \bar{u}^v\|_1 \leq (ch)^{v+2} \|u\|_2$$

is valid.

**2.** Our proof is based on the following operator frame work (cf. Chatelin, 1981, Hackbusch, 1981). Let us consider the equation

$$u = Ku + y \quad (12)$$

in a Banach-space  $X$  with  $K$  being a linear compact operator. Further let  $S$  be an approximating subspace and  $P : X \rightarrow S$  a bounded projection onto  $S$ . The standard Galerkin solution is defined by

$$u^0 = PKu^0 + Py. \quad (13)$$

Now we construct iterates  $\bar{u}^v$  and  $u^{v+1}$  in the way

$$\bar{u}^v = Ku^v + y, \quad (14)$$

$$u^{v+1} = \bar{u}^v + r^v \quad (15)$$

with  $r^v$  defined by

$$r^v = PKr^v + PK(\bar{u}^v - u^v). \quad (16)$$

*Remark 1 :*  $d^v = \bar{u}^v - u^v = Ku^v - u^v + y$  is the defect of the  $v$ -th iterate. Therefore  $r^v$  may be interpreted as the Galerkin-solution to the right hand side  $Kd^v$ .

*Remark 2 :* The approximations  $\bar{u}^0$  are also considered in Sloan (1976), but the higher iterates introduced there differ from ours.

**LEMMA 1 :** *Suppose that  $K$  is compact, 1 is not an eigenvalue of  $K$  and  $\kappa := \|(I - P)K\|$  is sufficiently small.*

*Then  $(I - PK)^{-1}$  exists as a bounded operator in  $X$  and the Galerkin solutions are well defined. Moreover*

$$u - u^v = (I - PK)^{-1} (I - P) K(u - u^{v-1}). \quad (17)$$

*Proof :* Since  $(I - K)^{-1}$  is bounded for  $\kappa$  small enough also  $(I - PK)^{-1}$  is bounded. As a consequence the Galerkin solution is uniquely defined. The identity

$$(I - K)^{-1} = (I - PK)^{-1} + (I - PK)^{-1} (I - P) K (I - K)^{-1} \quad (18)$$

will be useful. The solution  $u$  of (12) may be written in the form

$$u = (I - PK)^{-1} y + (I - PK)^{-1} (I - P) Ku. \quad (19)$$

Because of our construction we have

$$\begin{aligned} u^{v+1} &= Ku^v + y + (I - PK)^{-1} PK(Ku^v + y - u^v) \\ &= (I - PK)^{-1} y + (I - PK)^{-1} (I - P) Ku^v. \end{aligned} \quad (20)$$

Subtraction of (20) from (19) gives (17).

*Remark 3 :* We mention that under our assumptions also  $(I - KP)^{-1}$  exists and the recurrence relation

$$u - \bar{u}^v = (I - KP)^{-1} K(I - P)(u - \bar{u}^{v-1}) \quad (21)$$

is valid. The proof is omitted.

By our assumptions  $\|u^0\|$  is bounded by a multiple of  $\|y\|$ . Because of

$$\|(I - PK)^{-1}\| \leq \frac{\gamma}{1 - \kappa\gamma} \quad (22)$$

with  $\gamma$  being the norm of  $\|(I - K)^{-1}\|$  we conclude from lemma 1 :

**COROLLARY 1 :** Let  $\kappa = \|(I - P)K\|$  be less than the half of

$$\gamma^{-1} = \|(I - K)^{-1}\|^{-1}.$$

Then error-estimates of the type

$$\|u - u^v\| \leq c \left\{ \frac{\kappa\gamma}{1 - \kappa\gamma} \right\}^v \|y\| \quad (23)$$

hold true.

3. Now we come back to the situation discussed in section 1. We identify  $X$  with the Hilbertspace  $H_0 = L_2(\Omega)$ . Since we want to work with the Ritz-method we have to impose the condition  $S \subseteq \overset{\circ}{H}_1$ . For simplicity we focuss our attention to the case :  $S = S_h$  is the space of linear finite elements with isoparametric modifications along the boundary. Further let  $P = R_h$  be the standard Ritz-projection defined by  $Pu \in S_h$  and

$$(Pu, \chi)_1 = (u, \chi)_1 \quad \text{for } \chi \in S_h. \quad (24)$$

The operator  $K$  is defined by

$$w = Kv \Leftrightarrow w \in \mathring{H}_1 \quad \text{and} \quad (w, g)_1 = (v, -\sum (a_i g)|_i + bg) \quad \text{for} \quad g \in \mathring{H}_1. \quad (25)$$

Under suitable conditions concerning the regularity of  $a_i, b$  and since the original problem (1) resp. (2) is assumed to be uniquely solvable  $K$  is a bounded operator from  $H_0$  into  $\mathring{H}_1$  and hence compact as mapping of  $H_0$  into itself.

By duality the error-estimate

$$\|u - Pu\| \leq ch \|u\|_1 \quad (26)$$

is a consequence of (4). Because of

$$\|(I - P)Kv\| \leq ch \|Kv\|_1 \leq c' h \|v\| \quad (27)$$

we find

$$\kappa = \kappa_h = \|(I - P)K\| \leq ch \quad (28)$$

with some constant  $c$ .

The estimates derived in section 2 lead to

$$\|u - u^v\| \leq (ch)^v \|u - u^0\| \quad (29)$$

and because of (4) to

$$\|u - u^v\| \leq (ch)^{v+2} \|u\|_2. \quad (30)$$

Finally the terms  $\|u - \bar{u}^v\|_1$  are bounded in the same way since by definition

$$u - \bar{u}^v = K(u - u^v). \quad (31)$$

This completes the proof of theorem 1.

4. In this section we consider the model problem

$$\begin{aligned} -\Delta u &= f(., u) \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega \end{aligned} \quad (32)$$

in two or three space dimensions. The weak formulation of (32) is : Find  $u \in \mathring{H}_1$  such that

$$(u, v)_1 = (f(u), v) \quad \text{for} \quad v \in \mathring{H}_1. \quad (33)$$

Our assumptions are :

(i)  $f(x, z)$  is twice continuously differentiable with respect to  $z \in \mathbb{R}$  and

$$|f_{zz}(x, z)| \quad (34)$$

is uniformly bounded.

(ii) For  $z = u(x) \in C^0(\bar{\Omega})$  the functions  $f(x, u(x))$ ,  $f_x(x, u(x))$  and  $f_{zz}(x, u(x))$  are in  $C^0(\bar{\Omega})$ .

(iii)  $u$  is an isolated solution of (32), i.e. the linear problem

$$(w, g)_1 = (f'(u) w, g) \quad \text{for } g \in \mathring{H}_1 \quad (35)$$

admits only  $w = 0$  in  $\mathring{H}_1$ .

Now let  $u^0 = u_h \in S_h$  be the solution of the corresponding Galerkin-problem

$$(u^0, \chi)_1 = (f(u^0), \chi) \quad \text{for } \chi \in S_h. \quad (36)$$

Corresponding to section 1 we define the iterates  $\bar{u}^v$  for  $v \geq 0$  by

$$(\bar{u}^v, g)_1 = (f(u^v), g) \quad \text{for } g \in \mathring{H}_1, \quad (37)$$

and

$$u^{v+1} = \bar{u}^v + \varphi^v \quad (38)$$

with  $\varphi^v \in S_h$  and

$$(\varphi^v, \chi)_1 = (f'(u^0)(\varphi^v + \bar{u}^v - u^v), \chi) \quad \text{for } \chi \in S_h. \quad (39)$$

The counterpart of theorem 1 is :

**THEOREM 2 :** *Let  $(u^v, \bar{u}^v)$  be defined as above. Then*

$$\| u - u^v \| + \| u - \bar{u}^v \|_2 \leq c_1(c_2 h^2)^{v+1} \quad (40)$$

is valid. The constants  $c_1, c_2$  depend on  $u$  and bounds of  $f_x, f_{zz}$  but are independent of  $h$  and  $v$ .

*Proof :* Let  $K : H_0 \rightarrow \mathring{H}_1 \cap H_2$  be the inverse of the Laplacian defined by

$$w = Kv \Leftrightarrow (w, g)_1 = (v, g) \quad \text{for } g \in \mathring{H}_1, \quad (41)$$

and let  $P = R_h$  be the Ritz operator defined by

$$\Phi = Pv \Leftrightarrow \Phi \in S_h \quad \text{and} \quad (\Phi, \chi)_1 = (v, \chi)_1 \quad \text{for } \chi \in S_h. \quad (42)$$

Problem (32) is equivalent to  $u = Kf(u)$ . We may rewrite this in the form

$$(I - PKf'(u^0)) u = Kf(u) - PKf'(u^0) u. \quad (43)$$

In terms of  $K$  and  $P$  the iterates  $\bar{u}^v$  and  $\varphi^v$  have the representation

$$\bar{u}^v = Kf(u^v), \quad (44)$$

$$(I - PKf'(u^0)) \varphi^v = PKf'(u^0) (\bar{u}^v - u^v). \quad (45)$$

This leads to

$$(I - PKf'(u^0)) u^{v+1} = Kf(u^v) - PKf'(u^0) u^v. \quad (46)$$

By comparison of (43) and (46) and by adding and subtracting appropriate terms we come to

$$(I - PKf'(u^0)) (u^{v+1} - u) = (I - P) Kf'(u^0) (u^v - u) + K \{ f(u^v) - f(u) - f'(u) (u^v - u) + f'(u) - f'(u^0) \} (u^v - u). \quad (47)$$

The Ritz operator  $P$  is the orthogonal projection in  $H_1$  onto  $S = S_h$ . For  $v, w \in H_0$  arbitrary we get

$$\begin{aligned} ((I - P) Kv, w) &= ((I - P) Kv, Kw)_1 \\ &= ((I - P) Kv, (I - P) Kw)_1 \\ &\leq ch^2 \|Kv\|_2 \|Kw\|_2 \leq ch^2 \|v\| \|w\|. \end{aligned} \quad (48)$$

This implies that the norm of  $(I - P)K$  as mapping of  $H_0$  into  $H_0$  is bounded by  $ch^2$ . Next let  $a$  be a continuous function and  $v, w \in H_0$ . Then also  $K(avw)$  is in  $H_0$  and

$$\|K(avw)\| \leq c \|v\| \|w\|. \quad (49)$$

This follows from

$$\|K(avw)\|_0 = \sup \{ (Kavw, g) \mid \|g\| = 1 \} \quad (50)$$

and

$$(K(avw), g) = (v, \{ aKg \} w) \quad (51)$$

in combination with Sobolev's embedding lemma.

For  $h$  small enough the initial Galerkin solution  $u^0$  is "near" to  $u$ . Because of our assumption (iii) then the operator  $I - PKf'(u^0)$  will have a bounded inverse.



By the aid of these arguments we derive from the recurrence relation (47) the corresponding error bound

$$\|u^{v+1} - u\| \leq c_3 h^2 \|u^v - u\| + c_4 \|u^v - u\|^2 + c_5 \|u^0 - u\| \|u^v - u\|. \quad (52)$$

For the sake of clarity we have numbered the constants. Since an estimate of the type

$$\|u^0 - u\| \leq ch^2 \quad (53)$$

holds true anyway we derive from (52)

$$\|u^{v+1} - u\| \leq c_6 h^2 \|u^v - u\| + c_4 \|u^v - u\|^2. \quad (54)$$

Because of (53) by complete inductions there is a constant  $c_7$  such that for  $h \leq h_0$  with  $h_0$  chosen appropriate the relation

$$\|u^{v+1} - u\| \leq c_7 h^2 \|u^v - u\| \quad (55)$$

holds true (55) together with (53) lead to the error bound stated in theorem 2 for  $u^v - u$ .

Because of

$$\bar{u}^v - u = K(f(u^v) - f(u)) \quad (56)$$

we come to

$$\begin{aligned} \|\bar{u}^v - u\|_2 &\leq c \|f(u^v) - f(u)\| \\ &\leq c \|u^v - u\|. \end{aligned} \quad (57)$$

*Remark 3* : Whereas assumption (iii) is essential the two preceding ones can be reduced.

#### ACKNOWLEDGEMENT

I want to thank W. Hackbusch, J Liu and J. Nitsche for their suggestions and discussions

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