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L∞-ERROR ESTIMATES FOR VARIATIONAL INEQUALITIES WITH HÖLDER CONTINUOUS OBSTACLE (*)

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Abstract — An error estimate is derived, using a linear finite element method, for the L∞-approximation of the solution of variational inequalities with Hölder continuous obstacle. If the obstacle is in \( C^{0,\alpha}(\Omega) \) (\( 0 < \alpha \leq 1 \)), then the \( L^\infty \) error for the linear element solution is in the order of \( h^{\alpha - \epsilon} \) (\( \forall \epsilon > 0 \)).

Resume. — On démontre que l'erreur d'approximation dans la norme \( L^\infty \) de la solution d'une inéquation variationnelle, avec obstacle \( \alpha \)-holdérien \( (0 < \alpha \leq 1) \), par la méthode des éléments finis linéaires, est de l'ordre \( h^{\alpha - \epsilon} \), pour tout \( \epsilon > 0 \).

1. INTRODUCTION

The interest for the study of variational inequalities (V.I.) with « irregular » obstacles has recently increased. Regularity properties of solutions have been proved for V.I. with Hölder continuous ([4], [7], [8], [12]), continuous [12], or one-sided Hölder continuous [13] obstacles.

The importance of such results lies in particular in their application to the theory of quasi-variational inequalities (Q.V.I.), namely V.I. with the obstacle depending on the solution itself. Such an implicit obstacle, in fact, is in general “fairly irregular” (see [3] for some examples connected to stochastic control theory).

From a numerical point of view, some recent results are known concerning the approximation of solutions of Q.V.I. connected to some stochastic impulse control problems (see [11], [15]), by means of finite element methods.

The aim of this paper is to show an error estimate in the \( L^\infty \) norm, for the approximation, by means of linear finite elements, of the solution of variational inequalities.
inequalities with Hölder continuous obstacle. If the obstacle is in $C^{0,a}(_\Omega)$, $0 < \alpha \leq 1$ (so that, according to the mentioned regularity results, the solution itself is in $C^{0,a}(_\Omega)$), then, under reasonable hypotheses on the triangulation, the $L^\infty$-error of such an approximation is in the order of $h^{2-a}$ (for each $\varepsilon > 0$), that is the expected order of convergence.

In § 2 we introduce some notations and we recall the regularity of solutions. In § 3 the discretization is studied, and we state our principal result (theorem 3.2) together with some remarks and corollaries. In § 4 we indicate some useful results which are needed, in § 5, to prove theorem 3.2.

2. FORMULATION OF THE PROBLEM

Let $\Omega$ be a convex bounded domain of $\mathbb{R}^N$, with sufficiently smooth boundary $\Gamma$ (we suppose for example $\Gamma \in C^2$).

With classical notations, $C^{0,a}(\Omega)$, $0 < \alpha < 1$ [$\alpha = 1$], is the space of all the Hölder [Lipschitz] continuous functions of exponent $\alpha$ over $\Omega$, with the semi-norm

$$[v]_a = \sup_{x,y \in \Omega} \frac{|v(x) - v(y)|}{|x - y|^a}.$$

For $p \geq 1$, we let $L^p(\Omega)$ denote the classical Banach space consisting of measurable functions on $\Omega$ that are $p$-integrable, with the norm

$$\| v \|_p = \left( \int_\Omega |v|^p \, dx \right)^{1/p} \quad \text{if} \quad 1 \leq p < + \infty ,$$

$$\| v \|_\infty = \text{ess. sup}_\Omega |v| \quad \text{if} \quad p = \infty .$$

Then for $p \geq 1$, $m \in \mathbb{N}$, $W^{m,p}(\Omega)$ is the classical Sobolev space defined by

$$W^{m,p}(\Omega) = \{ v : D^\gamma v \in L^p(\Omega), \text{for all } |\gamma| \leq m \} ;$$

in $W^{m,p}(\Omega)$ we introduce the norm

$$\| v \|_{m,p} = \sum_{|\gamma| \leq m} \| D^\gamma v \|_p ,$$

and we set $H^m(\Omega) = W^{m,2}(\Omega)$; then $H^1_0(\Omega)$ is the closure, in the norm of $W^{1,2}(\Omega)$, of $C^0(\Omega)$, the space of all continuous functions with compact support in $\Omega$, having all first derivatives continuous in $\Omega$.

In the following $c$ will be the notation for positive constants involved in calculation, and the terms on which $c$ depends will be clarified each time.
Let $A$ be the second order linear elliptic operator defined by

$$A = - \sum_{i,j=1}^{N} \frac{\partial}{\partial x_j} \left( a_{ij}(x) \frac{\partial}{\partial x_i} \right) + \sum_{i=1}^{N} b_i(x) \frac{\partial}{\partial x_i} + c_0(x),$$

with the following assumptions:

i) $a_{ij} \in C^1(\Omega)$, $b_i$, $c_0 \in L^\infty(\Omega)$, $i, j = 1, 2, ..., N$;

ii) There is a constant $v > 0$ such that (uniform ellipticity):

$$\sum_{i,j=1}^{N} a_{ij}(x) \xi_i \xi_j \geq v |\xi|^2, \text{ a.e. in } \Omega, \forall \xi \in \mathbb{R}^N \setminus \{0\};$$

iii) $c_0(x) \geq \bar{c} > 0, \forall x \in \Omega$, with $\bar{c}$ sufficiently large (such that $A$ is a coercive operator on the space $H^1_0(\Omega)$).

Let $a(.,.) : H^1_0(\Omega) \times H^1_0(\Omega) \rightarrow \mathbb{R}$ be the continuous and coercive bilinear form on $H^1_0(\Omega)$ associated with the operator $A$, namely, $\forall u, v \in H^1_0(\Omega),

$$a(u, v) = \sum_{i,j=1}^{N} \int_{\Omega} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \, dx + \sum_{i=1}^{N} \int_{\Omega} b_i \frac{\partial u}{\partial x_i} v \, dx + \int_{\Omega} c_0 uv \, dx.$$

Let us now consider an "obstacle problem" for the operator $A$, i.e. the following V.I. with homogeneous boundary conditions:

$$a(u, v - u) \geq (f, v - u), \quad \forall v \in \mathcal{K} \quad (2.1)$$

where $\mathcal{K} = \{ v \in H^1_0(\Omega) : v \geq \psi \text{ in } \Omega \}$ is a closed convex subset of $H^1_0(\Omega)$, and

$$f \in L^\infty(\Omega), \quad (2.2)$$

$$\psi \in C^{0,\alpha}(\overline{\Omega}), \quad 0 < \alpha \leq 1, \quad (2.3)$$

are two given functions. We assume $\psi \mid_{\Gamma} \leq 0$, in order to avoid $\mathcal{K}$ being empty. Then the following regularity result is known:

**Theorem 2.1:** Under the assumptions (2.2) and (2.3), the unique solution $u$ of problem (2.1) is in $C^{0,\alpha}(\overline{\Omega})$.

The proof in the interior of $\Omega$ can be deduced for example from Caffarelli-Kinderlehrer [7], where it is shown that the solution of problem (2.1) has the same modulus of continuity of the obstacle. For a general proof we refer to Frehse [12], where the nonlinear case has been considered. For the case $\alpha = 1$, see also Chipot [8]. Lastly we mention the result of Biroli [4] : $u \in C^{0,\alpha}(\overline{\Omega})$, $\alpha' < \alpha$, if more general boundary conditions are involved.

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3. DISCRETIZATION AND PRINCIPAL RESULT

Let $\Omega_h$ denote a polyhedral domain inscribed in $\Omega$, such that the diameter of every "face" of $\Gamma_h = \partial \Omega_h$ has length less than $h$. Let us consider that over $\Omega_h$ a "triangulation" $\mathcal{T}_h$ is defined (in the usual way, see [9]), regular, in the sense that, setting $\forall T \in \mathcal{T}_h$:

$$h_T = \text{diam} (T),$$
$$\rho_T = \sup \{ \text{diam} (B) : B \subset T \text{ is a ball in } \mathbb{R}^N \},$$

then:

i) there is a constant $\sigma$ such that, $\forall T \in \mathcal{T}_h$, $h_T \leq \sigma$;

ii) $h \geq \max_T h_T$.

A piecewise linear subspace $V_h$ can be defined on $\overline{\Omega}$ in the following way

$$V_h = \{ v \in C^0(\overline{\Omega}) : v \mid_T \text{ is a linear function}, \forall T \in \mathcal{T}_h, v \equiv 0 \text{ in } \overline{\Omega} - \Omega_h \}.$$

Let us denote by $\{ P_i \}_{i=1}^{r(h)}$ the internal nodes of $\mathcal{T}_h$. Then the functions $\{ \phi_i \}_{i=1}^{r(h)}$ of $V_h$ such that

$$\phi_i(P_j) = \delta_{ij}, \quad i, j = 1, 2, ..., r(h) \quad \text{form a basis of } V_h; \text{ in particular for every } v \in C^0(\overline{\Omega}) \cap H^1_0(\Omega) \text{ the function}$$

$$v_f(x) = \sum_{i=1}^{r(h)} v(P_i) \phi_i(x) \quad (3.1)$$

represents the interpolate of $v$ over $\mathcal{T}_h$.

Furthermore, from the definition of $\mathcal{T}_h$,

$$P_i \in \partial T \Rightarrow T \subset B(P_i, h), \quad i = 1, 2, ..., r(h), \quad \forall T \in \mathcal{T}_h,$$

where $B(P_i, h)$ is the ball of $\mathbb{R}^N$ with its center in $P_i$ and radius $h$; then

$$\text{supp } \phi_i \subset \overline{B(P_i, h)}, \quad i = 1, 2, ..., r(h). \quad (3.2)$$

Now let us consider the discrete problem associated with (2.1):

$$a(u_h, v_h - u_h) \geq (f, v_h - u_h), \quad \forall v_h \in K_h,$$

$$u_h \in K_h \quad (3.3)$$

where $K_h = \{ v_h \in V_h : v_h \geq \psi_h \}$, and $\psi_h$ is the piecewise linear function on $\Omega$.

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equal to \( \psi \) at the nodes of \( \mathcal{C}_h \) (and defined on every connected component of \( \Omega - \Omega_h \)) by a constant extension in directions normal to \( \Gamma_h \), see [6]).

**Remark 3.1**: Such a choice of \( \mathbb{K}_h \) means that the constraint \( u_h \geq \psi \) is only imposed over the internal nodes of \( \mathcal{C}_h \). It could in fact be defined in an equivalent way:

\[
\mathbb{K}_h = \{ v_h \in V_h : v_h(P_i) \geq \psi(P_i), i = 1, 2, \ldots, r(h) \}. \quad \blacksquare
\]

Let \( M_h = (m_{ij}) \) be the matrix of problem (3.3), i.e., the real \( r(h) \times r(h) \) matrix whose generic term is

\[
m_{ij} = a(\phi_i, \phi_j), \quad i, j = 1, 2, \ldots, r(h).
\]

The following assumption is needed:

\[
m_{ij} \leq 0 \quad \text{if} \quad i \neq j, \quad i, j = 1, 2, \ldots, r(h);
\]

then, by the hypotheses on the coefficients of \( A \), \( M_h \) is an \( M \)-matrix, and the discrete problem (3.3) satisfies a discrete maximum principle, in the sense of [10] (where conditions of essentially geometric type on the triangulation \( \mathcal{C}_h \) are given, under which (3.4) holds).

The solution \( u_h \) of (3.3) represents the approximation of the solution \( u \) of (2.1) in the linear finite element discretization. Under the previous assumptions we are able to obtain an error estimate, in \( L^\infty \) norm, for such an approximation.

Namely, our principal result is:

**Theorem 3.2**: If (2.2), (2.3), (3.4) hold, then \( \forall p > 1 \)

\[
\| u - u_h \|_\infty \leq ch^{p - N/p} | \log h | ,
\]

where \( c \) depends on \( \Omega, \psi, p, \) and \( \alpha, \) not on \( h \).

Estimate (3.5) is quasi-optimal. In fact the interpolation error in \( L^\infty \) for Hölder continuous functions in \( C^{0,\alpha}(\overline{\Omega}) \) is a \( 0(h^p) \). Here this result is shown under the hypotheses:

\[
u |_{\Gamma} = 0; \quad \text{dist} (\Gamma, \Gamma_h) \leq ch^2.
\]

Condition (3.6) can be easily eliminated. It should also be noted that, under the assumptions made on \( \Omega \) (convex, with \( \Gamma \in C^2 \), it is always possible to construct \( \Omega_h \) such that (3.7) holds. (We remark that, in the non-convex case, assuming condition (3.7) as an hypothesis, we still obtain an estimate such as (3.5).)
**Lemma 3.3:** If \( u \in C^{0,\alpha}(\overline{\Omega}) \), \( 0 < \alpha \leq 1 \), and conditions (3.6), (3.7) are satisfied, then

\[
\| u - u_f \|_\infty \leq ch^2,
\]

where \( c \) depends only on \( u, \alpha \) and \( \Omega \).

**Proof.** — From the definition (3.1) (since \( \sum_{i=1}^{r(h)} \phi_i(x) \leq 1, \forall x \in \overline{\Omega} \)):

\[
| u(x) - u_f(x) | \leq \left(1 - \sum_{i=1}^{r(h)} \phi_i(x)\right) | u(x) | + \sum_{i=1}^{r(h)} \phi_i(x) | u(x) - u(P_i) |; \quad (3.8)
\]

the first term in the right hand side of (3.8) is either equal to zero (when \( x \) belongs to the convex envelope of the internal nodes, \( \sum_{i=1}^{r(h)} \phi_i(x) = 1 \)), or, in the other case, it is less than \( ch^2 \alpha \) (from (3.7)). For the second term we have

\[
\sum_{i=1}^{r(h)} \phi_i(x) | u(x) - u(P_i) | \leq [u]_\alpha \sum_{i=1}^{r(h)} \phi_i(x) | x - P_i |^\alpha
\]

since, from (3.2), \( \phi_i(x) \neq 0 \) implies \( | x - P_i | < h \). ■

As a corollary of theorem 3.2 we have an approximation result for the set \( D = \{ x \in \Omega : u(x) > \psi(x) \} \), where the solution does not touch the obstacle. The boundary of \( D \) is the so-called free boundary, and it is in many cases the real unknown of problems such as (2.1). Usually the convergence of \( u_h \) to \( u \) is not enough to ensure the convergence to \( D \) (in set theoretical sense) of sets \( D_h = \{ x \in \Omega : u_h(x) \geq \psi(x) \} \). However, theorem 3.2 implies:

**Corollary 3.4:** Under the same assumptions of theorem 3.2, the sequence \( \{ D_{h,\varepsilon} \} \), where

\[
D_{h,\varepsilon} = \{ x \in \Omega : u_h(x) > \psi(x) + h^{\alpha-\varepsilon} \},
\]

"converges from the interior" to \( D \), \( \forall \varepsilon > 0 \), in the sense that:

a) \( \lim_{h \to 0^+} D_{h,\varepsilon} = D \) (in set theoretical sense);

b) \( D_{h,\varepsilon} \subseteq D \), if \( h \) is sufficiently small.

(See [2] for the proof.)

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4. PRELIMINARY RESULTS

Let us state some useful results in order to prove theorem 3.2.

— A priori estimates

The following relation between solutions and obstacles of two different V.I. is well known (see [5]):

**Lemma 4.1:** Let \( u \) [resp. \( w \)] \( \in H_0^1(\Omega) \) be the unique solution of a V.I. such as (2.1), with obstacle \( \psi \) [resp. \( \varphi \)] \( \in L^\infty(\Omega) \); then

\[
\| u - w \|_\infty \leq \| \psi - \varphi \|_\infty.
\]

The discrete analogue of lemma 4.1 is also valid (see [11]):

**Lemma 4.2:** Let \( u_h \) [resp. \( w_h \)] \( \in V_h \) denote the approximation of \( u \) [resp. \( w \)] given by problem (3.3); if \( M_h \) satisfies (3.4), then

\[
\| u_h - w_h \|_\infty \leq \| \psi_h - \varphi_h \|_\infty.
\]

— V.I. with \( W^{2,p} \)-obstacle

Let us consider a V.I. such as (2.1), with the assumption (2.2), but now let \( \psi \in W^{2,p}(\Omega) \). Then it is well known [14] that the solution \( u \) is in \( W^{2,p}(\Omega) \).

Baiocchi [1] and Nitsche [17] have already studied the approximation for the solution of this problem. In particular we have:

**Theorem 4.3:** Let \( f \in L^p(\Omega) \), \( \psi \in W^{2,p}(\Omega) \), \( \forall p < + \infty \); if (3.4) holds, then

\[
\| u - u_h \|_\infty \leq c h^{2-N/p} \log h \left\{ \| u \|_{2,p} + \| \psi \|_{2,p} \right\}, \quad \forall p < + \infty,
\]

\( c \) independent of \( h \).

Proof of theorem 4.3 can be easily derived from [1], by means of the interpolation theory (see [9]), and of error estimates in \( L^\infty \) for solutions of equations. Estimates such as (4.1) hold in fact for equations with solutions in \( W^{2,p}(\Omega) \): they can be stated using Nitsche's techniques of weighted norms; when \( A = -\Delta \), see also [18], where a quasi-optimality result in \( L^\infty \) is given for the \( H_0^1 \)-projection into finite element spaces.
5. PROOF OF THEOREM 3.2

Without loss of generality, let us consider $\|/ |_{\mathbb{R}} = 0$ (such that in problem (3.3) now $\psi_h = \psi_j$); it can be shown in fact that solution $u$ of (2.1) is equal to solution $\hat{u}$ of

$$a(\hat{u}, z - \hat{u}) \geq (f, z - \hat{u}), \quad \forall z \in H^1_0(\Omega), \quad z \geq \hat{\psi}$$

$$\hat{u} \in H^1_0(\Omega), \quad \hat{u} \geq \hat{\psi}$$

where $\hat{\psi} = \psi \lor u_0$, and $u_0$ is the solution of the related equation

$$a(u_0, v) = (f, v), \quad v \in H^1_0(\Omega)$$

$$u_0 \in H^1_0(\Omega).$$

We have $u_0 \in W^{2,p}(\Omega), \forall p < + \infty$; hence $\hat{\psi} \in C^{0,\alpha}(\overline{\Omega})$, with the same $\alpha$ of $\psi$.

The proof of theorem 3.2 is based on a regularization procedure, consisting in the "approximation" of the initial problem by means of "more regular" V.I. (namely with $W^{2,p}$-obstacle, $\forall p < + \infty$), for which we can apply theorem 4.3. We then conveniently "go back" to problem (2.1), through continuity results. This procedure can be divided into four steps.

**Step 1 : Regularization by convolution.**

**Lemma 5.1 :** There is a sequence $\{ \psi^n \}$ converging to $\psi$ in $L^\infty$, such that, $\forall n$,

$$\psi^n \in C^1(\overline{\Omega}) \quad \psi^n |_{\Gamma} = 0,$$  \hspace{1cm} (5.1)

$$\| \psi^n - \psi \|_\infty \leq c n^{-\alpha},$$  \hspace{1cm} (5.2)

$$\| \psi^n \|_{C^\alpha(\overline{\Omega})} \leq c n^{1-\alpha},$$  \hspace{1cm} (5.3)

where $c$ depends on $\psi, \alpha, \Omega$, but not on $n$.

**Proof :** See [4]; (5.1) can be shown using convolutions of $\psi$ with suitable mollifiers and cut-off functions. \(\blacksquare\)

Let us call $u^n$ the solution of the V.I. (2.1) with obstacle $\psi^n$, and $u^n_h$ the solution of the corresponding discrete problem (where now the obstacle is $\psi^n_h$).

**Step 2 : Elliptic regularization.**

**Lemma 5.2 :** For every fixed $n$, there is a sequence $\{ \psi^{n,m} \}$ converging, for $m \to + \infty$, to $\psi^n$ in $L^\infty$, such that $\forall m$, $\psi^{n,m}$ is the solution of

$$\begin{bmatrix}
m^{-1} A\psi^{n,m} + \psi^{n,m} = \psi^n \\
\psi^{n,m} |_{\Gamma} = 0
\end{bmatrix}$$
and

\[ \psi^{n,m} \in W^{2,p}(\Omega), \quad \forall p < + \infty; \]
\[ \| \psi^{n,m} - \psi^n \|_\infty \leq cm^{-1/2} \| \psi^n \|_{1,p}, \quad \forall p < + \infty, \quad (5.4) \]
\[ \| A\psi^{n,m} \|_\infty \leq cm^{1/2} \| \psi^n \|_{1,p}, \quad \forall p < + \infty, \quad (5.5) \]

where \( c \) does not depend on \( m \) and \( n \).

(For the proof see [4] again.)

As we did in Step 1, let us call \( u^{n,m} \) the solution of the V.I. (2.1) with obstacle \( \psi^{n,m} \), and \( u_h^{n,m} \) the solution of the corresponding discrete problem. Of course \( u^{n,m} \in H^1_0(\Omega) \cap W^{2,p}(\Omega), \forall p < + \infty \); it follows

\[ \| u^{n,m} \|_{2,p} \leq c \| Au^{n,m} \|_p \leq c \| A\psi^{n,m} \|_\infty. \]

Furthermore the following inequality of Lewy-Stampacchia's type holds (see e.g. [16]) :

\[ f \leq Au^{n,m} \leq (A\psi^{n,m}) \lor f; \]

this yields, recalling (5.5),

\[ \| u^{n,m} \|_{2,p} \leq c \| A\psi^{n,m} \|_\infty \leq cm^{1/2} \| \psi^n \|_{1,p}, \quad \forall p < + \infty. \]

Likewise,

\[ \| \psi^{n,m} \|_{2,p} \leq cm^{1/2} \| \psi^n \|_{1,p}, \quad \forall p < + \infty. \]

Applying theorem 4.3, then

\[ \| u^{n,m} - u_h^{n,m} \|_\infty \leq cm^{1/2} h^{2-\varepsilon(p)} \| \psi^n \|_{1,p}, \quad \forall p < + \infty, \quad (5.6) \]

where for shortness we have set : \( h^{2-\varepsilon(p)} = h^{2-N/p} | \log h |. \)

**Step 3 : Inversion of Step 2.**

**Lemma 5.3 :** The following estimate holds :

\[ \| u^n - u^n_h \|_\infty \leq ch^{1-\varepsilon(p)} \| \psi^n \|_{1,p}, \quad \forall n \in \mathbb{N}, \quad \forall p < + \infty. \quad (5.7) \]

**Proof :** For every choice of index \( m \), we have

\[ \| u^n - u^n_h \|_\infty \leq \| u^n - u^{n,m} \|_\infty + \| u^{n,m} - u_h^{n,m} \|_\infty + \| u_h^{n,m} - u_h^n \|_\infty, \]

and, by lemma 4.1 and (5.4), \( \forall p, \)

\[ \| u^n - u^{n,m} \|_\infty \leq cm^{-1/2} \| \psi^n \|_{1,p}. \]
Likewise, using lemma 4.2,
\[ \| u^n_m - u^n_h \|_\infty \leq \| \psi^m - \psi^n \|_\infty \leq c m^{-1/2} \| \psi^n \|_{1,p} ; \]
then, from (5.6), we obtain
\[ \| u^n - u^n_h \|_\infty \leq c(m^{-1/2} + m^{1/2} h^{2-\varepsilon(p)}) \| \psi^n \|_{1,p} , \forall p < + \infty . \]
If we now choose a suitable \( m \), i.e. such that \( 1/h^2 \leq m \leq (1/h^2) + 1 \), then the proof is complete.

**Step 4 : Inversion of Step 1.**

To complete the proof of theorem 3.2, let us use the same trick of Step 3, obtaining
\[ \| u - u_h \|_\infty \leq \| u - u^n \|_\infty + \| u^n - u^n_h \|_\infty + \| u^n_h - u_h \|_\infty ; \]
according to (5.3), from (5.7) we get
\[ \| u^n - u^n_h \|_\infty \leq c(n^{-\alpha} h^{1-\varepsilon(p)} ; \]
then, using lemmas 4.1 and 4.2, and (5.2),
\[ \| u - u_h \|_\infty \leq c(n^{-\alpha} + n^{1-\alpha} h^{1-\varepsilon(p)} ; \]
if we now take \( n \) such that \( 1/h \leq n \leq (1/h) + 1 \), we finally have
\[ \| u - u_h \|_\infty \leq c h^{2-\varepsilon(p)} , \forall p < + \infty , \]
that is the thesis (3.5). ■

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