Eberhard Schock

Three remarks on the use of Čebyšev polynomials for solving equations of the second kind


<http://www.numdam.org/item?id=M2AN_1981__15_3_257_0>
THREE REMARKS ON THE USE OF ČEBYŠEV POLYNOMIALS FOR SOLVING EQUATIONS OF THE SECOND KIND (*)

by Eberhard SCHOCK (1)

Communicated by R S VARGA

Abstract — Three methods are considered: the Čebyšev-Euler method, the Šchebyshev semi-iterative method and a refinement of the projection method for the approximation of the quasi inverse for self-adjoint operators A such that unity does not belong to the spectrum of A.

In this communication we consider three methods for the approximate solution of equations of the second kind

\[ x - Ax = y \]

in a (complex) Hilbert space with a selfadjoint bounded linear operator A. We only assume that unity does not belong to the spectrum of A.

We give a new proof and an error estimate for the Čebyšev-Euler method, we discuss the Čebyšev-semi-iterative method (cf. Varga [7]) and we consider a refinement of the projection method of type \( Q_\sigma \) introduced in [6].

1. INTRODUCTION

Let A be a bounded linear selfadjoint operator in a complex Hilbert space \( H \). Let \( (E_\lambda) \) be its spectral decomposition, \( \sigma \) an interval containing the spectrum \( \sigma(A) \) of A. Then

\[ A = \int_{\sigma} \lambda \, dE \]
and for each continuous function \( p : \sigma \to \mathbb{R} \)

\[
p(A) = \int_{\sigma} p(\lambda) \, dE_{\lambda}
\]

and

\[
\| p(A) \| \leq \sup_{\lambda \in \sigma} | p(\lambda) | .
\]

Especially, if \( 1 \notin \sigma \),

\[
\| (1 - A)^{-1} - p(A) \| \leq \sup_{\lambda \in \sigma} \left| \frac{1}{1 - \lambda} - p(\lambda) \right| .
\]

If \( p \) is an arbitrary polynomial of degree \( n \), then \( \| (1 - A)^{-1} - p(A) \| \) is minimal, if \( p \) is the proximum of \( r \) (with \( r(\lambda) = (1 - \lambda)^{-1} \)) in the space of all polynomials of degree \( n \) on the spectrum of \( A \) with respect of the sup-norm.

We call a method for the approximate solution of \( x - Ax = y \) polynomial, if the approximate solution \( \hat{x} \) is of the form \( \hat{x} = p(A) y \), where \( p(A) \) is an operator polynomial.

2. THE ČEBYŠEVE-EULER METHOD

If \( \sigma = [a, b], b < 1 \) then the Čebyšev-Euler method consists of determining the proximum \( p_n \) to \( r \) in \([a, b]\) by polynomials degree \( n \). Then

\[
x_n = p_n(A) \, y
\]

is the Čebyšev-Euler approximation of the solution \( x \) of \( x - Ax = y \).

This approximation is easy to calculate: it is known Čebyšev [3], Bernstein [2], Meinardus [4], that in the interval \([-1, 1]\) the proximum of

\[
s_n(\lambda) = \frac{1}{\lambda - \alpha} \quad \alpha > 1
\]

is given by the polynomials \( q_n \) of degree \( n \) which fulfill

\[
\frac{1}{\lambda - \alpha} - q_n(\lambda) = \gamma_n \cos (n \varphi + \delta)
\]

\[
\gamma_n = \frac{(\alpha - \sqrt{\alpha^2 - 1})^n}{\alpha^2 - 1}, \quad \lambda = \cos \varphi, \quad \frac{\alpha \lambda - 1}{\lambda - 1} = \cos \delta .
\]

Using the Čebyšev polynomials \( t_n \) and \( v_n \) of first resp. second kind, (2) is equivalent to

\[
(\alpha - \lambda) \, q_n(\lambda) = 1 - \gamma_n (\alpha \lambda - 1) \, t_n(\lambda) + \gamma_n \sqrt{\alpha^2 - 1} (1 - \lambda^2) \, v_{n-1}(\lambda) .
\]
The recursion formulas for $t_n$ and $v_n$ lead to the recursion formula for $q_n$

$$q_{n+1}(\lambda) = 2 \gamma \lambda q_n(\lambda) - \gamma^2 q_{n-1}(\lambda) - 2 \gamma$$

$$\gamma = \alpha - \sqrt{\alpha^2 - 1}, \quad q_0(\lambda) = \frac{\alpha}{1 - \alpha^2}, \quad q_1(\lambda) = \frac{\lambda + \sqrt{\alpha^2 - 1}}{\alpha^2 - 1}.$$

A linear transformation of the interval $[-1, 1]$ onto $[a, b]$ gives the polynomials $p_n$ of best approximation of $r$ by

$$p_n(\lambda) = -\frac{b - a}{2} q_n\left(\frac{2 \lambda - b - a}{b - a}\right)$$

which leads to the recursion formula

$$p_{n+1}(\lambda) = -\frac{2}{b - a} \left[\frac{2 \gamma}{b - a} (2 \lambda - b - a) p_n(\lambda) - \gamma^2 p_{n-1}(\lambda) - 2 \gamma\right]$$

$$p_0(\lambda) = \frac{b - a}{2} \frac{\alpha}{\alpha^2 - 1}$$

$$p_1(\lambda) = \frac{1}{\alpha^2 - 1} \left(-\lambda + \frac{b + a}{2} - \frac{b - a}{2} \sqrt{\alpha^2 - 1}\right)$$

$$\alpha = \frac{2 - b - a}{b - a}$$

$$\gamma = \alpha - \sqrt{\alpha^2 - 1}.$$

The error estimate is

$$\max_{\lambda \in [a, b]} \left|\frac{1}{1 - \lambda} - p_n(\lambda)\right| = \max_{\lambda \in [-1, 1]} \left|\frac{1}{\lambda - \alpha} - q_n(\lambda)\right|$$

$$= \frac{b - a}{2} \frac{\gamma^n}{\alpha^2 - 1} \max |\cos (n\varphi + \delta)| \leq \frac{b - a}{2(\alpha^2 - 1)} \cdot \gamma^n.$$

If we replace $\lambda$ by $A$, we obtain the following result:

If $\sigma(A) \subset [a, b], b < 1$, then the best polynomial approximation method of $x - Ax = y$ is given by the following semi-iterative method.

$$x_{n+1} = -\frac{2}{b - a} \left[\frac{2 \gamma}{b - a} (2 Ax_n - (b + a) x_n) - \gamma^2 x_{n-1} - 2 \gamma y\right]$$

$$x_0 = \frac{b - a}{2} \cdot \frac{\alpha}{\alpha^2 - 1} y, \quad x_1 = \frac{1}{\alpha^2 - 1} \left[-Ay + \left(\frac{b + a}{2} - \frac{b - a}{2} \sqrt{\alpha^2 - 1}\right) y\right]$$

$$\alpha = \frac{2 - b - a}{b - a}, \quad \gamma = \alpha - \sqrt{\alpha^2 - 1}.$$
with the error estimate

$$\|x - x_n\| \leq \frac{b - a}{2(\sigma^2 - 1)} \gamma^n.$$  

This method also can be obtained by using methods of summability theory (cf. Niethammer [5]).

3. THE ČEBYSHEV SEMI-ITERATIVE METHOD

Let $A$ be a linear selfadjoint bounded operator with $1 \notin \sigma(A)$ and $(x_n)$ the Picard iteration sequence

$$x_{n+1} = Ax_n + y, \quad x_0 = y.$$

This method calculates a linear combination

$$\tilde{x}_n = \sum_{j=0}^{n} \gamma_j x_j$$

such that $x - \tilde{x}_n$ has a small norm.

Since for

$$\tilde{w}_n = \tilde{x}_n - x = \sum_{j=0}^{n} \gamma_j (x_j - x) + \sum_{j=0}^{n} (\gamma_j - 1) x$$

it is

$$\tilde{w}_n = \sum_{j=0}^{n} \gamma_j A^j (x_0 - x) = p(A) (x_0 - x)$$

with the condition $p(1) = \sum_{j=0}^{n} \gamma_j = 1$, so

$$\tilde{w}_n = p(A) (y - (I - A)^{-1} y) = \int_{\sigma} p(\lambda) \frac{\lambda}{1 - \lambda} dE_{\lambda} y$$

and $\|\tilde{w}_n\|$ is minimal, if $p$ is a polynomial of degree $n$ with $p(1) = 1$ and

$$\max_{\lambda \in \sigma(A)} \left| p(\lambda) \frac{\lambda}{1 - \lambda} \right| \leq \max_{\lambda \in \sigma(A)} \left| q(\lambda) \frac{\lambda}{1 - \lambda} \right|,$$

where $q$ is an arbitrary polynomial of degree $n$ with $q(1) = 1$. If both 1 and 0 do not belong to the spectrum of $A$, then $p$ is up to a constant the same polynomial as the polynomial $q$ with $q(1) = 1$ and $\max_{\lambda \in \sigma(A)} |q(\lambda)|$ is minimal.
In each case, this minimal polynomial does not lead to an easy semi-iterative method, so the usual minimization condition is to determine the polynomial \( p \) of degree \( n \) with \( p(1) = 1 \) and minimal norm.

It is well known that the transformed Čebyšev polynomials have this property that their norm on an interval is minimal. So one has to consider three cases

1° \( \sigma(A) \subset [a, b], b < 1 \)
2° \( \sigma(A) \subset [a, b], a > 1 \)
3° \( \sigma(A) \subset [a_1, b_1] \cup [a_2, b_2] \).

In the first and second case

\[
\begin{align*}
\rho_n(\lambda) &= \frac{t_n(2\lambda - b - a)}{t_n(2 - b - a)} \\
p_n(\lambda) &= \frac{1}{t_n(2 - b - a)} \left| t_n\left(\frac{2 - b - a}{b - a}\right)\right|
\end{align*}
\]

is the minimal polynomial with

\[
\| p_n \| = \max_{\lambda \in [a, b]} | p_n(\lambda) | = \frac{1}{t_n(2 - b - a)} \left| t_n\left(\frac{2 - b - a}{b - a}\right)\right|
\]

Using the recursing formula for the Čebyšev polynomials we obtain for

\[
\rho_n^{-1} = \frac{2}{\rho_n^{-1}} - \frac{2 - b - a}{b - a} \rho_n^{-1} - \rho_n^{-1}, \quad \rho_0 = 1, \quad \rho_1^{-1} = \frac{2 - b - a}{b - a}
\]

and

\[
\begin{align*}
p_{n+1}(\lambda) &= \frac{2 \rho_{n+1}}{\rho_n} \frac{2 \lambda - b - a}{b - a} p_n(\lambda) - \frac{\rho_{n+1}}{\rho_n} p_{n-1}(\lambda) \\
p_0(\lambda) &= 1, \quad p_1(\lambda) = \frac{2 \lambda - b - a}{2 - b - a}
\end{align*}
\]

This gives after a short calculation, using the recursion formulas and

\[
\tilde{w}_{n+1} = \tilde{x}_{n+1} - x = p_n(A) w_0
\]
the semi-iterative method

$$\tilde{x}_{n+1} = \frac{4 \rho_{n+1}}{\rho_n(b - a)} \left( A \tilde{x}_n + y - \frac{b + a}{2} \tilde{x}_n \right) - \frac{\rho_{n+1}}{\rho_n} \tilde{x}_{n-1}$$

$$\tilde{x}_0 = y, \quad \tilde{x}_1 = \frac{2}{2 - b - a} Ay + y$$

and the error estimate

$$\| x - \tilde{x}_n \| \leq \| p_0(A) (x - y) \| \leq \| p_n \| \| x - y \| .$$

In the third case we assume that there is known a number $\eta$ such that

$$\sigma(A) \subset [-\rho, 1 - \eta] \cup [1 + \eta, \rho].$$

Since the polynomials

$$q_{2n}(\lambda) = t_n \left( \frac{2 \lambda^2 - 1 - \alpha^2}{1 - \alpha^2} \right)$$

are the polynomials of minimal norm on the intervals $[-1, -\alpha] \cup [\alpha, 1]$ of degree $2n$ with $q_{2n}(1) = 1$ (cf. Achieser [1], p. 287) a linear transformation of $[-1, -\alpha] \cup [\alpha, 1]$ onto $[-\rho, 1 - \eta] \cup [1 + \eta, \rho + 2]$ (resp. $[2 - \rho, 1 - \eta] \cup [1 + \eta, \eta]$ if more convenient) and the substitution of $\lambda$ by $A$ leads to the semi-iterative method

$$x_{n+1} = \frac{4 \tau_{n+1}}{\tau_n((\rho + 1)^2 - \eta^2)} \left[ (A^2 x_n - 2 Ax_n + x_n - Ay - y) - \frac{1}{2} ((\rho + 1)^2 + \eta^2) x_n \right] - \frac{\tau_{n+1}}{\tau_{n-1}} x_{n-1}$$

$$x_0 = y$$

$$x_1 = -\frac{2}{(\rho + 1)^2 + \eta^2} (A^2 y - Ay) + y$$

$$\tau_{n+1}^{-1} = t_n \left( \frac{1 - (\rho + 1)^2 + \eta^2}{(\rho + 1)^2 - \eta^2} \right)$$

$$\tau_{n+1}^{-1} = -2 \frac{(\rho + 1)^2 + \eta^2}{(\rho + 1)^2 - \eta^2} \tau_n^{-1} - \tau_{n-1}^{-1}$$

$$\tau_0 = 1$$

$$\tau_1^{-1} = -\frac{(\rho + 1)^2 + \eta^2}{(\rho + 1)^2 - \eta^2}.$$
The order of convergence of this method is
\[ \| x_n - x \| \leq \| p(A) (y - x) \| = 0(\tau^{-1}) \].

4. A ČEBYŠEV PROJECTION METHOD

Let \( A \) be again a bounded linear selfadjoint operator in a Hilbert space \( H \) with spectrum in \([a, b]\), \( b < 1 \).

Let \( p_n : [a, b] \rightarrow \mathbb{R} \) be the polynomials from section 2, which are the proxima of \((1 - \lambda)^{-1}\) of degree \( n \) in \([a, b]\). Then for linear independent elements \( z_1, ..., z_k \) of \( H \) we determine

\[ z = \sum_{j=1}^{n} \gamma_j z_j \]

from the system of linear equations

\[ \langle z - Az - y + (1 - A) p_n(A) y, z_j \rangle = 0 \]

for \( j = 1, 2, ..., k \). Then

\[ \hat{x}_n = p_n(A) y + z \]

is an approximation for the solution \( x \) of \( x - Ax = y \).

If \( p_n = 0 \), then this method is the usual Ritz-Galerkin method, if

\[ p_n(\lambda) = \sum_{j=0}^{n} \lambda^j, \]

then this method is the projection method of type \( Q_{n+1} \) introduced in [6].

As usual in the theory of the Ritz-Galerkin method, the optimal rate of convergence for compact \( A \) is obtained, if \( z_1, ..., z_k \) are eigenvectors of \( A \). In this case we get with

\[ x = p_n(A) y + ((1 - A)^{-1} - p_n(A)) y \]

\[ \hat{x}_n = p_n(A) y + z \]

and a simple Hilbert space calculation

\[ z = \sum_{j=1}^{k} \left( \frac{1}{1 - \lambda_j} - p_n(\lambda_j) \right) \langle y, z_j \rangle z_j \]

\[ \| x - x_n \|^2 = \sum_{j=k+1}^{\infty} \left| \frac{1}{1 - \lambda_j} - p_n(\lambda_j) \right|^2 \left| \langle y, z_j \rangle \right|^2 \]

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so
\[ \| x - \hat{x}_n \| \leq \sup_{j \geq k+1} \left| \frac{1}{1 - \lambda_j} - p_n(\lambda_j) \right| \leq \frac{b - a}{2(\alpha^2 - 1)} \gamma^n \]
where
\[ \alpha = \frac{2 - b - a}{b - a}, \quad \gamma = \alpha - \sqrt{\alpha^2 - 1} \]
as in a section 2. Also as in section 2 is shown \( p_n(A) y \) can be calculated by a semi-iterative method.

5. CONCLUDING REMARKS

Niethammer [5] has shown that the order of convergence of the Čebyšev semi-iterative method tends to the order of convergence of the Čebyšev-Euler method. In [8] M. Wolf has demonstrated that the Čebyšev projection method in general gives quite better approximations than the usual Ritz-Galerkin method.

REFERENCES