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NUMERICAL STABILITY IN DYNAMIC ELASTIC-PLASTIC PROBLEMS (*)

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Abstract — *The purpose of this paper is to prove the conjecture based on the engineering intuition that the plasticity has no influence on the numerical stability of step-by-step integration schemes to solve dynamic problems We consider a typical formulation of the elastic-plastic vibration problem and give a rigorous proof of this conjecture for a certain approximating scheme which is widely used in the practical computation*

Résumé — *Le but de cet article est de démontrer la conjecture (basée sur l'intuition des Ingénieurs) selon laquelle la plasticité n'a pas d'influence sur la stabilité numérique des schémas d'intégration pas à pas pour résoudre les problèmes dynamiques On considère une formulation type du problème de vibration élastoplastique, et on démontre rigoureusement cette conjecture pour un schéma d'approximation très couramment utilisé dans les calculs pratiques*

INTRODUCTION

For simplicity we consider two-dimensional problem. Let Ω be a bounded domain in (x_1, x_2) plane with boundary $\partial\Omega$ and T the time interval $(0, T)$. We use the notations $u_{,j}$ and $\dot{u}$ to denote the derivatives with respect to x_j and time t, respectively.

By u, ε, σ and α we denote the displacement (u_1, u_2), the strain $(\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{12})$, the stress $(\sigma_{11}, \sigma_{22}, \sigma_{12})$ and the parameter representing the center of the yield surface $(\alpha_{11}, \alpha_{22}, \alpha_{12})$.

The dynamic elastic-plastic problem is formulated by the following conditions.

(1) *Equation of motion:*

$$\rho \ddot{u}_i - \sum_j \sigma_{ij,j} = 0 \quad \text{in } T \times \Omega,$$

where ρ is a positive constant and $\sigma_{21} = \sigma_{12}$.

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(2) *Initial-boundary condition:* For simplicity we assume

$$u_i(0, x)=0, \qquad \dot{u}_i(0, x)=u_i^0(x),$$
$$u_i(t, x)=0, \quad x\in\Gamma_0,$$
$$\sum_j \sigma_{ij}(t, x)\cos(n, x_j)=0, \quad x\in\Gamma_1,$$

where $\partial\Omega=\Gamma_0\cup\Gamma_1$ and n is the outward normal to $\partial\Omega$. We assume that Γ_0 consists of closed curves with positive length. $\{u_i^0(x)\}$ are given functions vanishing on Γ_0.

(3) *Strain-displacement relation:* We assume that the strain is small.

$$\varepsilon_{11}=u_{1,1}, \qquad \varepsilon_{22}=u_{2,2}, \qquad \varepsilon_{12}=u_{1,2}+u_{2,1}.$$

(4) *Yield condition:* We employ von Mises' condition:

$$f^2(\sigma-\alpha)=(\sigma_{11}-\alpha_{11})^2+(\sigma_{22}-\alpha_{22})^2 -(\sigma_{11}-\alpha_{11})(\sigma_{22}-\alpha_{22})+3(\sigma_{12}-\alpha_{12})^2=r_0^2,$$

where r_0 is a given positive constant.

To introduce the flow-hardening rule we use the following notation.

$$\partial f=\left(\frac{\partial f}{\partial\sigma_{11}}, \frac{\partial f}{\partial\sigma_{22}}, \frac{\partial f}{\partial\sigma_{12}}\right).$$

Also we use * to denote the transpose of a vector. $\beta^*\beta'$ and $\|\beta\|_*$ denote the inner product and the norm of vectors. The inner product and norm of (vector) functions in $L_2(\Omega)$ are denoted by (β, β') and $\|\beta\|$, respectively.

Now let $d\varepsilon$, $d\varepsilon_p$, $d\varepsilon_e$ and $d\sigma$ be the total strain increments, plastic strain increments, elastic strain increments and the stress increments. Then we assume the followings.

(5) *Flow rule* [4]:

$$d\varepsilon_p=d\varepsilon-d\varepsilon_e=\frac{1}{\eta}\partial f\,\partial f^*\,d\sigma,$$

where η (>0) is assumed to be constant, for simplicity. The relation between $d\sigma$ and $d\varepsilon_e$ is given by Hooke's rule $d\sigma=Dd\varepsilon_e$, where

$$D=\frac{E}{1-\nu^2}\begin{pmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \nu'\end{pmatrix} \qquad \left(\nu'=\frac{1-\nu}{2}\right).$$

(6) *Hardening rule:* We assume Ziegler's hardening rule [5]. In this rule the center α must satisfy

$$d\alpha=(\sigma-\alpha)d\mu,$$

where $d\mu$ is a non-negative function.

The function $d\mu$ is determined automatically by the condition that the stress point must be on the yield surface during the plastic flow. Because, if σ remains on the yield surface, then the vector $d\sigma-d\alpha$ must be orthogonal to the normal of the surface at this point:

$$\partial f^*(d\sigma-d\alpha)=0.$$

Substituting the equation on $d\alpha$ into this equality we have

$$d\mu=\frac{\partial f^*\, d\sigma}{(\sigma-\alpha)^*\,\partial f}.$$

Therefore the hardening rule is written as follows:

$$d\alpha=(\sigma-\alpha)\frac{\partial f^*\, d\sigma}{(\sigma-\alpha)^*\,\partial f}.$$

The matrix D is symmetric and positive definite. Hence $d\sigma$ can be explicitly given in terms of $d\varepsilon$ as

$$d\sigma=(D-D')\,d\varepsilon, \qquad D'=\frac{D\,\partial f\,\partial f^*\,D}{\eta+\partial f^*\,D\,\partial f}.$$

The identity $(\sigma-\alpha)^*\,\partial f=f$ holds. Hence, taking time t as the parameter of the increments, our problem is to solve:

$$\begin{cases} \rho\,\ddot{u}_i-\sum_j \sigma_{ij,j}=0 \quad \text{in } T\times\Omega, \\ \dot{\sigma}=D\dot{\varepsilon}, \qquad \dot{\alpha}=0 \quad \text{in elastic region}, \\ \dot{\sigma}=(D-D')\,\dot{\varepsilon}, \qquad \dot{\alpha}=(\sigma-\alpha)\dfrac{\partial f^*\,\dot{\sigma}}{f} \quad \text{in plastic region}, \end{cases}$$

under the strain-displacement relation and the initial-boundary condition.

1. A FINITE ELEMENT APPROXIMATION

The concepts "elastic" and "plastic" are not yet defined for the problem formulated above. We can, of course, introduce some physical definitions.

However, mathematically, these concepts are very difficult to define Because, to define them we need a certain information about the solution before seeking it On the other hand, for getting information about the solution, the problem itself must be posed already This dilemma arises especially when we pose the problem beyond the moment when yielding (or unloading) occurs This will be one reason why the variational inequality is introduced in [1] (*see*, however, [3]) For a certain discrete systems, however, this dilemma does not occur

We consider one of the simplest finite element methods For simplicity, we assume that Ω is a polygonal domain By $\hat{\Omega}$ we denote a regular triangulation of the closure of Ω We assume that the end point of Γ_0 is always the node of $\hat{\Omega}$ Let $\{ \varphi_p \}$ be the piecewise linear finite element basis which takes 1 at the node p The approximate value of u_i at the time step $n(=0, 1, 2, \quad)$ is sought in the form

$$u_i(n) = \sum_{p \in P} u_i^p(n) \varphi_p,$$

where P is the set of nodes in $\hat{\Omega} - \Gamma_0$

The *equation of motion* is approximated as follows

$$(\rho D_t D_{\bar{t}} u_i(n), \varphi_p) + \sum_j (\sigma_{ij}(n), \varphi_{p\,j}) = 0, \qquad p \in P, \qquad (1\ 1)$$

where D_t and $D_{\bar{t}}$ denote the forward and backward difference operators with time increment Δt, respectively

The *initial condition* is approximated by $u_i(0)=0$, $u_i(1)|_p = \Delta t u_i^0(p)$ The *strain-displacement relation* is the same as that given before, so that the approximate strain is constant over each element Approximate stress and function α are also constant on each element, as defined later

To introduce a discrete stress-strain relation, let us take an arbitrary element e and consider things on it

DEFINITION 1 (1) By the yield surface of step (0) (or the initial yield surface) for element e, we mean the ellipsoid in E^3 defined by

$$\{ \tau \in E^3, f^2(\tau - \alpha(0)) = r_0^2 \}$$

(2) We say $\{ \sigma(n+1), \alpha(n+1) \}$ is determined by the *elastic rule*, if $\sigma(n)$ is in or on the yield surface of step (n), and $\{ \sigma(n+1, \alpha(n+1) \}$ and the yield surface of step $(n+1)$ are determined by the following rule

(2_a) $$D_t \varepsilon(n) = C D_t \sigma(n) \ (C = D^{-1}),$$

(2_b) $$D_t \alpha(n) = 0,$$

(2_c) The function defining the yield surface of step $(n+1)$ is

$$f^2(\tau - \alpha(n+1)) = \mathrm{Max}(r_0^2, \mathrm{Max}_{m \leq n} [f^2(\sigma(m+1) - \alpha(m+1))])$$

(3) We say $\{ \sigma(n+1), \alpha(n+1) \}$ is determined by the *plastic rule*, if $\sigma(n)$ is on the yield surface of step (n), and if $\{ \sigma(n+1), \alpha(n+1) \}$ and the yield surface of step $(n+1)$ are determined by the following rule.

$$(3_a) \qquad D_t \varepsilon(n) = \partial f_n \frac{\partial f_n^* D_t \sigma(n)}{\eta} + C D_t \sigma(n),$$

$$(3_b) \qquad D_t \alpha(n) = (\sigma(n) - \alpha(n)) \frac{\partial f_n^* D_t \sigma(n)}{f_n},$$

(3_c) The function defining the yield surface of step $(n+1)$ is

$$f^2(\tau - \alpha(n+1)) = f^2(\sigma(n+1) - \alpha(n+1)),$$

where f_n and ∂f_n are the values of f and ∂f at $\{ \sigma, \alpha \} = \{ \sigma(n), \alpha(n) \}$, respectively.

(4) Assume that $\sigma(n)$ is in the yield surface of step (n) and the point $\tilde{\sigma}(n+1)$ determined by the elastic rule from $\{ \sigma(n), \alpha(n) \}$ is outside of the yield surface of step (n). If $\partial f_n^* (\tilde{\sigma}(n+1) - \sigma(n)) \geqq 0$, then $\{ \sigma(n+1), \alpha(n+1) \}$ is determined by the following rule. In this case we say that it is determined by the *elastic-plastic rule:* Choose θ_n (> 0) so that the point

$$\bar{\sigma}(n) = \sigma(n) + \theta_n(\tilde{\sigma}(n+1) - \sigma(n)), \tag{1.2}$$

comes on the yield surface of step (n). Define

$$\begin{aligned} \bar{f}_n &= f|_{\{\sigma, \alpha\} = \{\bar{\sigma}(n), \alpha(n)\}}, \\ \partial \bar{f}_n &= \partial f|_{\{\sigma, \alpha\} = \{\bar{\sigma}(n), \alpha(n)\}}. \end{aligned} \tag{1.3}$$

Clearly $\partial \bar{f}_n^* (\tilde{\sigma}(n+1) - \sigma(n)) > 0$. Then $\{ \sigma(n+1), \alpha(n+1) \}$ is determined by the following rule. Put $\bar{\varepsilon}(n) = \varepsilon(n) + C(\bar{\sigma}(n) - \sigma(n))$.

$$(4_a) \quad \varepsilon(n+1) - \bar{\varepsilon}(n) = \partial \bar{f}_n \frac{\partial \bar{f}_n^* (\sigma(n+1) - \bar{\sigma}(n))}{\eta} + C(\sigma(n+1) - \bar{\sigma}(n)),$$

$$(4_b) \qquad D_t \alpha(n) = (\bar{\sigma}(n) - \alpha(n)) \frac{\partial \bar{f}_n^* (\sigma(n+1) - \bar{\sigma}(n))}{\Delta t \bar{f}_n},$$

(4_c) The function defining the yield surface of step $(n+1)$ is

$$f^2(\tau - \alpha(n+1)) = f^2(\sigma(n+1) - \alpha(n+1)).$$

REMARK 1: Let the symmetric matrix $\bar{D}'(n)$ be defined by

$$\bar{D}'(n) = \frac{D \partial \bar{f} \partial \bar{f}^* D}{\eta + \partial \bar{f}^* D \partial \bar{f}}.$$

Then, (4_a) is written also as follows.

$$(4_{a'}) \qquad D_t\sigma(n) = [D - (1-\theta_n)\bar{D}'(n)]\, D_t\varepsilon(n).$$

To see this, we first remember the following relations:

$$\varepsilon(n+1) - \varepsilon(n) = C(\tilde{\sigma}(n+1) - \sigma(n)) = \frac{1}{\theta_n} C(\bar{\sigma}(n) - \sigma(n)) = \frac{1}{\theta_n}(\bar{\varepsilon}(n) - \varepsilon(n)).$$

Inverting relation (4_a) we have

$$\sigma(n+1) - \bar{\sigma}(n) = (D - \bar{D}'(n))\,(\varepsilon(n+1) - \bar{\varepsilon}(n)).$$

Substituting the identity $\bar{\sigma}(n) - \sigma(n) = D(\bar{\varepsilon}(n) - \varepsilon(n))$ into this equality, we have

$$\sigma(n+1) - \sigma(n) = D(\varepsilon(n+1) - \varepsilon(n)) - \bar{D}'(n)\,(\varepsilon(n+1) - \bar{\varepsilon}(n)).$$

Since $\varepsilon(n+1) - \bar{\varepsilon}(n) = (1-\theta_n)\,(\varepsilon(n+1) - \varepsilon(n))$, $(4_{a'})$ follows.

REMARK 2: The cases $\theta_n = 1$ and $\theta_n = 0$ stand for the elastic rule and the plastic rule, respectively.

REMARK 3: When the elastic-plastic rule is applied, the vector $\tilde{\sigma}(n+1) - \bar{\sigma}(n)$ is transversal to the yield surface at the stress point $\bar{\sigma}(n)$. This is the case also for the vector $\sigma(n+1) - \bar{\sigma}(n)$. This is proved as follows:

$$\begin{aligned}
\operatorname{sgn} \partial\bar{f}_n^*(\tilde{\sigma}(n+1) - \bar{\sigma}(n)) &= \operatorname{sgn} \partial\bar{f}_n^*(\tilde{\sigma}(n+1) - \sigma(n)) \\
&= \operatorname{sgn} \partial\bar{f}_n^* D D_t\varepsilon(n) = \operatorname{sgn} \partial\bar{f}_n^* \left[D - \frac{D\,\partial\bar{f}_n\, \partial\bar{f}_n^* D}{\eta + \partial\bar{f}_n^* D\, \partial\bar{f}_n} \right] D_t\varepsilon(n) \\
&= \operatorname{sgn} \partial\bar{f}_n^* [D - \bar{D}'(n)]\, D_t\varepsilon(n) \\
&= \operatorname{sgn} \partial\bar{f}_n^* [D - \bar{D}'(n)]\, (\varepsilon(n+1) - \bar{\varepsilon}(n)) \\
&= \operatorname{sgn} \partial\bar{f}_n^* (\sigma(n+1) - \bar{\sigma}(n)).
\end{aligned}$$

The relation between the stress and strain increments is given by:

Discrete stress-strain relation:

Naturally, $\sigma(0) = \alpha(0) = 0$.

(A) Let $\{\tilde{\sigma}(n+1), \tilde{\alpha}(n+1)\}\,(n \geqq 0)$ be determined by the elastic rule as far as $\sigma(n)$ is in the yield surface of step(n). If $\tilde{\sigma}(n+1)$ is still in the yield surface of step(n), then, we define

$$\{\sigma(n+1), \alpha(n+1)\} = \{\tilde{\sigma}(n+1), \tilde{\alpha}(n+1)\}.$$

(B) Assume that $\sigma(n)$ is in the yield surface of step(n) and that $\tilde{\sigma}(n+1)$ determined by the elastic rule comes on or outside of the yield surface of step(n). Then, $\{\sigma(n+1), \alpha(n+1)\}$ is determined by,

(B_a) the elastic-plastic rule if $\partial f_n^*(\tilde{\sigma}(n+1)-\sigma(n)) \geqq 0$, and otherwise by the elastic rule, i.e.,

(B_b) $$\{\sigma(n+1), \alpha(n+1)\} = \{\tilde{\sigma}(n+1), \tilde{\alpha}(n+1)\}.$$

The subsequent relations are given by the following procedure.

(C) If $\sigma(n+1)$ is on the yield surface of step$(n+1)$, then determine $\{\tilde{\sigma}(n+2), \tilde{\alpha}(n+2)\}$ by the plastic rule and

(C_a) define $\{\sigma(n+2), \alpha(n+2)\} = \{\tilde{\sigma}(n+2), \tilde{\alpha}(n+2)\}$ if

$$\partial f_{n+1}^*(\tilde{\sigma}(n+2)-\sigma(n+1)) \geqq 0,$$

and otherwise

(C_b) determine $\{\sigma(n+2), \alpha(n+2)\}$ anew by the elastic rule.

(D) If $\sigma(n+1)$ is in the yield surface of step$(n+1)$, then return to the procedure (A) $\rightarrow$ (B), replacing n by $n+1$.

REMARK 4: There are three cases that the elastic rule is applied.

(1) $\sigma(n)$ is in the yield surface of step (n) and $\tilde{\sigma}(n+1)$ determined by the elastic rule is so too.

(2) $\sigma(n)$ is on the yield surface of step (n) and $\tilde{\sigma}(n+1)$ determined by the plastic rule satisfies

$$\partial f_n^*(\tilde{\sigma}(n+1)-\sigma(n)) < 0. \tag{1.4}$$

This case corresponds to unloading.

(3) $\sigma(n)$ is in the yield surface of step (n) and $\tilde{\sigma}(n+1)$ determined by the elastic rule is outside of this yield surface. However, the following inequality holds.

$$\partial f_n^*(\tilde{\sigma}(n+1)-\sigma(n)) < 0.$$

REMARK 5: Suppose that $\{\tilde{\sigma}(n+1), \tilde{\alpha}(n+1)\}$ is determined by the plastic rule and the condition (1.4) holds. Then unloading occurs and $\{\sigma(n+1), \alpha(n+1)\}$ is determined by the elastic rule. In this case the following inequality again holds.

$$\partial f_n^*(\sigma(n+1)-\sigma(n)) < 0.$$

2. STABILITY OF THE FINITE ELEMENT SCHEME

In the preceding section we posed a discrete initial value problem. A discrete solution $\{u(n), \sigma(n), \alpha(n)\}$ $(n \geqq 1)$ is thus determined step-by-step. We seek a criterion to ensure the stability of this solution, by means of the energy method.

LEMMA 2.1: (1) *The vectors* ∂f *and* $\sigma-\alpha$ *satisfy*

$$f\partial f=S(\sigma-\alpha), \qquad 2S=\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 6 \end{pmatrix}.$$

(2) *For any* $n\geqq 0$, *holds*

$$D_t\varepsilon(n)=\frac{1}{\eta}SD_t\alpha(n)+CD_t\sigma(n).$$

(3) *If* $\sigma(n+1)$ *is determined by the plastic rule, then*

$$\partial f_n^*(D_t\sigma(n)-D_t\alpha(n))=0.$$

(4) *If* $\sigma(n+1)$ *is determined by the elastic-plastic rule, then*

$$\partial\bar{f}_n^*(\sigma(n+1)-\bar{\sigma}(n)-[\alpha(n+1)-\alpha(n)])=0.$$

where $\bar{\sigma}(n)$ *and* $\partial\bar{f}_n$ *are those defined by* (1.2) *and* (1.3).

(5) *The yield surface expands monotonically.*

(6) *Let* f' *and* $\partial f'$ *be the values of* f *and* ∂f *at* $\{\sigma,\alpha\}=\{\sigma',\alpha'\}$ *respectively, and put* $k=f/f'$. *Then*

$$(\sigma-\alpha)^*\partial f'=k(\sigma'-\alpha')^*\partial f.$$

Proof: We prove (5) and (6). During the elastic deformation, the yield surface is unchanged or expanding. If $\{\sigma(n+1),\alpha(n+1)\}$ is determined by the plastic rule, then we have

$$f_{n+1}^2-f_n^2=\partial(f_n^2)^*(D_t\sigma(n)-D_t\alpha(n))\Delta t+f^2(D_t\sigma(n)-D_t\alpha(n))\Delta t^2.$$

The first term of the right side vanishes by (3) of this lemma. Hence $f_{n+1}^2\geqq f_n^2$. The situation is the same in the elastic-plastic case too. This proves (5). By (1):

$$\partial f'=\frac{1}{f'}S(\sigma'-\alpha').$$

Therefore we have

$$(\sigma-\alpha)^*\partial f'=\frac{1}{f'}(\sigma-\alpha)^*S(\sigma'-\alpha')=\frac{1}{f'}(\sigma'-\alpha')^*S(\sigma-\alpha)=\frac{f}{f'}(\sigma'-\alpha')^*\partial f,$$

which proves (6).

REMARK 6: The expansion of the yield surface is inevitable as far as we employ Ziegler's formula as it is, exept one-dimentional case (vibration of a rod). The situation is the same for Prager's rule (*see* the last remark on this rule). Since we assumed so called the kinematic hardening, this may have some influence on the accuracy of the computed solutions. We think, however, that it does not harm the solution desperately by the reasons that (*a*) if the computed solution is stable, the expansion of the radius of the yield surface is at most of $O(\sqrt{\Delta t})$ in totality (*b*) it is proved that, as $\Delta t \to 0$, the computed solution will converge to the solution of a *semi-discrete* system (the case that "time" is continuous). We also remark that the expansion due to *elastic* deformation [the case (3) and possibly (2) of Remark 4] will be negligible, since such case will be rare in the practical computation.

We want to show that a certain quadratic quantity with respect to $D_t u$ and the strain is bounded by the initial energy and this quantity can be regarded as an measure of the energy of the system. By $\|u\|_\rho^2$ we denote $\sum(\rho u_i, u_i)$.

LEMMA 2.2: *For any n ($\geqq 1$) holds*

$$D_{\bar{t}}\left[\|D_t u(n)\|_\rho^2+\frac{1}{\eta}(S\alpha(n), \alpha(n+1))+(C\sigma(n), \sigma(n+1))\right]\leqq 0. \tag{2.1}$$

Proof: Replacing φ_p by $(D_t+D_{\bar{t}})u_i(n)$ in the both sides of (1.1) and adding on i, we have

$$D_{\bar{t}}\|D_t u(n)\|_\rho^2+(\sigma(n), (D_t+D_{\bar{t}})\,\varepsilon(n))=0.$$

Thus, taking into account (2) of lemma 2.1, we have

$$\begin{aligned}0&=-(\sigma(n), (D_t+D_{\bar{t}})\,\varepsilon(n))+\frac{1}{\eta}(\sigma(n), S(D_t+D_{\bar{t}})\,\alpha(n))\\&\qquad+(\sigma(n), C(D_t+D_{\bar{t}})\,\sigma(n))\\&=D_{\bar{t}}\left[\|D_t u(n)\|_\rho^2+\frac{1}{\eta}(S\alpha(n), \alpha(n+1))+(C\sigma(n), \sigma(n+1)\right]\\&\qquad+\frac{1}{\eta}(\sigma(n)-\alpha(n), S(D_t+D_{\bar{t}})\,\alpha(n)).\end{aligned}$$

The problem is thus to show that the second term of the right side is non-negative. Define $(u, v)_e=\int_e u^* v\,dx$ and put

$$Q_e=(\sigma(n)-\alpha(n), S(D_t+D_{\bar{t}})\,\alpha(n))_e,$$

for an arbitrary element e. We shall show that $Q_e\geqq 0$ holds always.

(1) Both $\sigma(n)$ and $\sigma(n+1)$ are determined by the elastic rule. In this case, clearly $Q_e=0$.

(2) Both $\sigma(n)$ and $\sigma(n+1)$ are determined by the plastic rule. By (3) of lemma 2.1, we have

$$(\sigma(n)-\alpha(n), \partial f_{n-1})_e=(\sigma(n-1)-\alpha(n-1), \partial f_{n-1})_e \geqq 0.$$

Therefore

$$Q_e=(\sigma(n)-\alpha(n), \partial f_n)_e \partial f_n^* D_t \sigma(n) + (\sigma(n)-\alpha(n), \partial f_{n-1})_e \partial f_{n-1}^* D_t \sigma(n-1) \geqq 0.$$

(3) $\sigma(n)$ is determined by the elastic (resp. plastic) rule and $\sigma(n+1)$ by the plastic (resp. elastic) rule. $Q_e \geqq 0$ is proved samely as in the case (2).

(4) $\sigma(n)$ is determined by the elastic rule and $\sigma(n+1)$ by the elastic-plastic rule. By (6) of lemma 2.1, we have

$$(\sigma(n)-\alpha(n), \partial \bar{f}_n)_e=k_n(\bar{\sigma}(n)-\alpha(n), \partial f_n)_e = k_n(\sigma(n)-\alpha(n), \partial f_n)_e+k_n(\bar{\sigma}(n)-\sigma(n), \partial f_n)_e \geqq 0.$$

Therefore, by remark 3, we have

$$Q_e=(\sigma(n)-\alpha(n), \partial \bar{f}_n)_e \frac{\partial \bar{f}_n^*(\sigma(n+1)-\bar{\sigma}(n))}{\Delta t} \geqq 0.$$

(5) The case that $\sigma(n)$ and $\sigma(n+1)$ are determined by the plastic and elastic-plastic rule, respectively, does not occur.

(6) $\sigma(n)$ is determined by the elastic-plastic rule and $\sigma(n+1)$ by the plastic rule. We have the identity

$$(\sigma(n)-\alpha(n), \partial \bar{f}_{n-1})_e=(\bar{\sigma}(n-1)-\alpha(n-1), \partial \bar{f}_{n-1})_e + (\sigma(n)-\bar{\sigma}(n-1)-[\alpha(n)-\alpha(n-1)], \partial \bar{f}_{n-1})_e.$$

This is non-negative, since the first term of the right side is non-negative and the second term vanishes by (4) of lemma 2.1. Therefore

$$Q_e=(\sigma(n)-\alpha(n), \partial \bar{f}_{n-1})_e \frac{\partial \bar{f}_{n-1}^*(\sigma(n)-\bar{\sigma}(n-1))}{\Delta t} + (\sigma(n)-\alpha(n), \partial f_n)_e \partial f_n^* D_t \sigma(n) \geqq 0.$$

(7) The case that $\sigma(n)$ and $\sigma(n+1)$ are determined by the elastic-plastic and elastic rule, respectively, is now evident. Hence the lemma is proved.

Let us introduce E_n and R_n defined by

$$E_n=\|D_t u(n)\|_\rho^2+\frac{1}{2\eta}(\|\alpha(n)\|_S^2+\|\alpha(n+1)\|_S^2)+\frac{1}{2}(\|\sigma(n)\|_C^2+\|\sigma(n+1)\|_C^2),$$

$$R_n=\frac{1}{2\eta}\|D_t \alpha(n)\|_S^2 \Delta t^2+\frac{1}{2}\|D_t \sigma(n)\|_C^2 \Delta t^2,$$

where $\| \sigma \|_C^2 = (C\sigma, \sigma)$, $\| \alpha \|_S^2 = (S\alpha, \alpha)$. Then inequality (2.1) is written as

$$D_{\bar{t}} (E_n - R_n) \leqq 0. \tag{2.2}$$

In order to prove that $E_n - R_n$ is a positive quadratic form, we first prove the following inequality.

$$R_n \leqq \frac{\Delta t^2}{2} (D D_t \varepsilon(n), D_t \varepsilon(n)). \tag{2.3}$$

Let us introduce the quantity $\bar{\bar{D}}'(n)$ defined by

$$\bar{\bar{D}}'(n) = (1 - \theta_n) \bar{D}'(n).$$

Then the stress-strain relation and α-strain relation are written respectively as

$$D_t \sigma(n) = (D - \bar{\bar{D}}'(n)) D_t \varepsilon(n),$$
$$D_t \alpha(n) = \eta S^{-1} C \bar{\bar{D}}'(n) D_t \varepsilon(n),$$

if we take θ_n and $\bar{\sigma}(n)$ suitably (*see* remarks 1 and 2). We shall substitute these relations into R_n and derive the desired estimate. Let us start from the identity

$$\frac{2}{\Delta t^2} R_n = \eta \| S^{-1} C \bar{\bar{D}}'(n) D_t \varepsilon(n) \|_S^2 + \| (D - \bar{\bar{D}}'(n)) D_t \varepsilon(n) \|_C^2.$$

Expanding the second term of the right side we have

$$\begin{aligned}(C(D - \bar{\bar{D}}'(n)) D_t \varepsilon(n), (D - \bar{\bar{D}}'(n)) D_t \varepsilon(n)) \\ = (D D_t \varepsilon(n), D_t \varepsilon(n)) - 2 (\bar{\bar{D}}'(n) D_t \varepsilon(n), D_t \varepsilon(n)) \\ + (\bar{\bar{D}}'(n) C \bar{\bar{D}}'(n) D_t \varepsilon(n), D_t \varepsilon(n)).\end{aligned}$$

Let us put

$$\begin{aligned}S_n = \eta \| S^{-1} C \bar{\bar{D}}'(n) D_t \varepsilon(n) \|_S^2 - 2 (\bar{\bar{D}}'(n) D_t \varepsilon(n), D_t \varepsilon(n)) \\ + (\bar{\bar{D}}'(n) C \bar{\bar{D}}'(n) D_t \varepsilon(n), D_t \varepsilon(n)). \\ = ([\eta \bar{\bar{D}}'(n) C S^{-1} C \bar{\bar{D}}'(n) - 2 \bar{\bar{D}}'(n) \\ + \bar{\bar{D}}'(n) C \bar{\bar{D}}'(n)] D_t \varepsilon(n), D_t \varepsilon(n)),\end{aligned}$$

so that

$$\frac{2}{\Delta t^2} R_n = (DD_t \varepsilon(n),\ D_t \varepsilon(n)) + S_n.$$

S_n is non-positive. To prove this, put $A_n = \eta + \partial \bar{f}_n^* D \partial \bar{f}_n$. Then, since

$$\overline{\overline{D}}'(n) = (1-\theta_n) \frac{D \partial \bar{f}_n \partial \bar{f}_n^* D}{A_n} \tag{2.4}$$

and $\partial \bar{f}_n^* S^{-1} \partial \bar{f}_n = \partial \bar{f}_n^* (\bar{\sigma}(n) - \alpha(n))/\bar{f}_n = 1$, we have

$$\eta \overline{\overline{D}}'(n)\ CS^{-1}\ C\overline{\overline{D}}'(n) = \eta (1-\theta_n) \frac{\overline{\overline{D}}'(n)}{A_n}.$$

Therefore S_n can be written as follows:

$$S_n = \left(\left[\frac{\eta(1-\theta_n)}{A_n} - 1 \right] \overline{\overline{D}}'(n)\ D_t \varepsilon(n),\ D_t \varepsilon(n) \right) + ([\overline{\overline{D}}'(n)\ C\overline{\overline{D}}'(n) - \overline{\overline{D}}'(n)]\ D_t \varepsilon(n),\ D_t \varepsilon(n)).$$

Since $\overline{\overline{D}}'(n)$ is non-negative definite, the first term of the right side is non-positive. This is the case for the second term too. Because, substitute (2.4) into this term. Then we have

$$\text{second term} = \frac{1-\theta_n}{A_n} \left([1-\theta_n] \frac{\partial \bar{f}_n^* D \partial \bar{f}_n}{A_n} - 1 \right) [\partial \bar{f}_n^* DD_t \varepsilon(n)]^2 \leqq 0.$$

S_n is hence non-positive, and inequality (2.3) holds well.

The next theorem is our final conclusion.

THEOREM: *Let h be the maximum length of the sides of all triangles of $\hat{\Omega}$. There are positive constants ξ and c such that if*

$$1 - \xi \frac{\Delta t^2}{h^2} > 0,$$

then for any n ($\geqq 1$) holds the following inequality:

$$c \| D_t u(n) \|_\rho^2 + \frac{1}{2\eta} (\| \alpha(n) \|_S^2 + \| \alpha(n+1) \|_S^2) + \frac{1}{2} (\| \sigma(n) \|_C^2 + \| \sigma(n+1) \|_C^2) \leqq \| D_t u(0) \|_\rho^2.$$

Proof: It is known that there is a positive constant ξ such that

$$\frac{\Delta t^2}{2}(DD_t\varepsilon(n),\, D_t\varepsilon(n)) \leqq \xi\frac{\Delta t^2}{h^2}\|D_t u(n)\|_\rho^2. \tag{2.5}$$

Therefore

$$\|D_t u(n)\|_\rho^2 - R_n \geqq \left(1-\xi\frac{\Delta t^2}{h^2}\right)\|D_t u(n)\|_\rho^2.$$

Put $c=1-\xi\Delta t^2/h^2$ and use (2.2) and lemma 2.2 to get the conclusion.

REMARK 7: As is already seen in the above proof, the stability in the sense of energy is independent of the existence of the plasticity. The constant ξ appearing in the condition (2.5) is estimated in [2].

REMARK: On Prager's hardening rule.

If we assume Prager's hardening rule

$$d\alpha=\gamma d\varepsilon_p, \tag{2.6}$$

where γ is a positive constant which characterizes the material, then the associated flow rule changes the form slightly.

Let us assume that the plastic part of the strain increment is parallel to the normal of the yield surface, i.e.,

$$d\varepsilon_p=\partial f d\lambda. \tag{2.7}$$

The function $d\lambda$ can not be chosen arbitrary. Because, if the stress point remains on the yield surface, the following condition must be satisfied.

$$\partial f^*(d\sigma-d\alpha)=0. \tag{2.8}$$

Therefore, substituting (2.6) and (2.7) into this identity, we have

$$d\lambda=\frac{\partial f^* d\sigma}{\gamma\|\partial f\|_*^2}.$$

The flow-hardening rule takes the following form in this case.

$$d\sigma=\left[D-\frac{D\,\partial f\partial f^* D}{\gamma\|\partial f\|_*^2+\partial f^* D\,\partial f}\right]d\varepsilon, \qquad d\alpha=\partial f\frac{\partial f^* d\sigma}{\|\partial f\|_*^2}.$$

We can derive an finite element scheme based on this rule which is similar to that analyzed in this paper. It is important that the stability criterion given in the above theorem is a sufficient condition also for this scheme. The plasticity has no influence on the numerical stability (in energy) in this case too.

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