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ON THE APPROXIMATION OF THE SOLUTION OF AN OPTIMAL CONTROL PROBLEM GOVERNED BY AN ELLIPTIC EQUATION (*)

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Communiqué par J.-L. LIONS

Abstract. — *We obtain error estimates for the approximate solutions of an optimal control problem in which the state is governed by a Neumann problem. To establish the estimates the solution is characterized in terms of the saddle point of a Lagrangian obtained by using Fenchel-Rockafellar duality theory.*

Résumé. — *Nous obtenons une estimation de l'erreur pour les solutions approchées d'un problème de contrôle optimal, dans lequel l'état est gouverné par un problème de type Neumann. Pour établir cette estimation, la solution est caractérisée en terme d'un point-selle d'un Lagrangien, obtenu en faisant appel à la théorie de la dualité de Fenchel-Rockafellar.*

INTRODUCTION

Few results exist concerning the rate of convergence of approximate solutions of optimal control problems governed by partial differential equations [3, 9]. Characterization of the solution in terms of the saddle point of a Lagrangian in order to obtain error estimates, the approach employed in [8] and [9] by Lasiecka and Malanowski, appears to yield results in a more direct manner than the techniques employed in [3] by Falk. On the other hand, the existence and regularity of appropriate multipliers are demonstrated with some difficulty in [8].

In this paper we appeal to Fenchel-Rockafellar duality [2], as first employed in a similar context by Mossino [12], in order to characterize the solutions of an optimal control problem in a way that leads to error estimates. We believe that this approach is more direct and more general than the previous ones.

We consider the problem

$$\text{(P)} \qquad \operatorname*{Inf}_{\substack{v \in L^2(\Gamma) \\ |v(x)| \leq 1 \text{ a.e}}} \left[\frac{1}{2} \| y(v) + y_d \|^2_{L^2(\Omega)} + \frac{\nu}{2} \| v \|^2_{L^2(\Gamma)} \right],$$

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where Ω is a given convex polygonal domain in $\mathbb{R}^2$; $\Gamma=\partial\Omega$, and $y_d\in H^1(\Omega)$, and $\nu>0$ are given. The state $y(v)$ is determined by

$$-\Delta y(v)+y(v)=0 \quad \text{in } \Omega,$$

$$\frac{\partial}{\partial n}y(v)=v \quad \text{on } \Gamma.$$

In section 1 we give a precise description of the problem (P) and its discretization, and collect together relevant results concerning the regularity and finite element approximation of the solutions of the equations involved.

In section 2 we characterize (lemma 2.1) the solution of (P) in terms of the saddle point of a Lagrangian. Lemma 2.2 presents the corresponding characterization of the solution of the approximating problem. We then obtain (in lemma 2.3) the regularity result for the optimal control u ($u\in H^1(\Gamma)$), as in Lions [11].

In section 3 we derive $O(h)$ estimates for the error in the approximation of the optimal control and of the corresponding state, in L^2-norm for the former and H^1-norm for the latter (h is the largest of the sides of the triangles of a certain triangulation of Ω).

It must be pointed out that, as in [3] and [9], there seem to be no obvious modifications of our approach which would make it possible to treat cases where state constraints are present. Hints as to the complexity of such cases are available, for instance in the papers by Hager and Mitter [6], Mossino [12], and Rockafellar [15].

1. PROBLEM (P), APPROXIMATING PROBLEM (P_h), AND SOME RESULTS FROM THE THEORY OF FINITE ELEMENT APPROXIMATION

Assume Ω is a bounded convex polygonal domain in $\mathbb{R}^2$, and the boundary of Ω is $\Gamma=\overline{\Gamma}_1\cup\overline{\Gamma}_2\cup\ldots\cup\overline{\Gamma}_N$, each Γ_i being an open line segment, and $\overline{\Gamma}_i\cap\overline{\Gamma}_j$ is either empty or a common end-point for $i\neq j$. The set of admissible controls is

$$K=\left\{v\in L^2(\Gamma):|v(x)|\leqq 1 \text{ a.e. on } \Gamma\right\},$$

so that K is a closed, convex subset of $L^2(\Gamma)$. The state $y=y(v)\in H^1(\Omega)$ is determined by the variational equation

$$a(y(v),\varphi)=(v,\gamma_0\varphi)_{L^2(\Gamma)}, \qquad \forall\varphi\in H^1(\Omega), \tag{1.1}$$

where $\gamma_0\varphi$ is the trace of φ on Γ,

$$a(y,\varphi)=\int_\Omega\sum_{i=1}^{2}\frac{\partial y}{\partial x_i}\frac{\partial\varphi}{\partial x_i}dx+\int_\Omega y\varphi\,dx,$$

and

$$(v, \gamma_0 \varphi)_{L^2(\Gamma)} = \int_\Gamma v \gamma_0 \varphi \, d\Gamma.$$

Thus if $y(v)$ is sufficiently regular, $y(v)$ is the solution of the Neumann problem

$$\left.\begin{aligned} -\Delta y + y = 0 \quad \text{in } \Omega, \\ \frac{\partial y}{\partial n} = v \quad \text{on } \Gamma_i, \qquad i = 1, 2, \ldots, N, \end{aligned}\right\} \tag{1.1'}$$

where Δ is the Laplacian and n denotes the unit normal vector directed towards the exterior of Ω.

The cost functional is

$$J(v) = \frac{1}{2} \| y(v) - y_d \|^2_{L^2(\Omega)} + \frac{\nu}{2} \| v \|^2_{L^2(\Gamma)}, \tag{1.2}$$

where $y_d \in H^1(\Omega)$ and $\nu > 0$ are given.

The optimal control problem is

(P) $$\operatorname*{Inf}_{v \in K} J(v),$$

and a unique solution exists, as discussed in Lions [11].

We shall consider an approximation

(P_h) $$\operatorname*{Inf}_{v_h \in K_h} J_h(v_h) \quad (h \text{ a positive parameter}),$$

such that (P_h) can be solved numerically. Specifically, assume τ_h is a "classical" triangulation of Ω with the angles of all triangles $T \in \tau_h$ uniformly bounded below by $\theta_0 > 0$ for all $h (h \downarrow 0)$, as discussed, for example, in [1], [4] and [14]. Here, h is the longest side of all $T \in \tau_h$. We consider the space $H^1_h(\Omega) \subset H^1(\Omega)$ of functions which are continuous in $\overline{\Omega}$ and linear on each $T \in \tau_h$, the space $L^2_h(\Omega) \subset L^2(\Omega)$ of functions which are constant on each $T \in \tau_h$, and the space $L^2_h(\Omega) \subset L^2(\Gamma)$ of functions defined on Γ which are constant on each segment determined by an edge of a $T \in \tau_h$ that lies on Γ.

Let

$$K_h = \{ v_h \in L^2_h(\Gamma) : | v_h(x) | \leqq 1, \forall x \in \Gamma \}, \qquad (K_h \subset K).$$

The approximate problem is

(P_h) $$\operatorname*{Inf}_{v_h \in K_h} J_h(v_h),$$

where

$$J_h(v_h) = \frac{1}{2} \| s_h y_h(v_h) - s_h y_d \|^2_{L^2(\Omega)} + \frac{\nu}{2} \| v_h \|^2_{L^2(\Gamma)}, \tag{1.3}$$

$y_h(v) \in H_h^1(\Omega)$ is determined for any $v \in L^2(\Gamma)$ by the variational equation

$$a(y_h(v), \varphi_h) = (v, \gamma_0 \varphi_h)_{L^2(\Gamma)}, \qquad \forall \varphi_h \in H_h^1(\Omega), \tag{1.4}$$

and $s_h : L^2(\Omega) \to L_h^2(\Omega)$ is the projection;

$$\text{for } y \in L^2(\Omega), \qquad (s_h y)(x) = \frac{1}{Area(T)} \int_T y, \qquad x \in \operatorname{Int}(T), \quad T \in \tau_h. \tag{1.5}$$

Since $L_h^2(\Omega)$, $L_h^2(\Gamma)$ and $H_h^1(\Omega)$ are finite-dimensional spaces. (P_h) is a nonlinear programming problem in finite dimensions, and can be solved by means of various techniques. Assuming that the minimizer $u_h \in K_h$, the existence and uniqueness of which are readily established as in the case of (P), has been computed, we shall estimate $\| u_h - u \|_{L^2(\Gamma)}$ and $\| y_h(u_h) - y(u) \|_{H^1(\Omega)}$ as $O(h)$, where u is the solution of (P).

We need some results about the regularity and approximation of the solution of (1.1). We shall say that $v = (v_1, v_2, \ldots, v_N) \in H^2(\Gamma)$ $(\alpha > 0)$ iff $v_i \in H^2(\Gamma_i)$, $i = 1, 2, \ldots, N$; then we set

$$\| v \|_{H^\alpha(\Gamma)} = \left(\sum_{i=1}^{N} \| v \|_{H^\alpha(\Gamma_i)}^2 \right)^{1/2},$$

i.e., $H^\alpha(\Gamma) = \prod_{i=1}^{N} H^\alpha(\Gamma_i)$.

It is known ([5, 13], and the references given there) that the solution $y(v)$ of (1.1) is in $H^2(\Omega)$ if $v \in H^{1/2}(\Gamma)$, and also that

$$\| y(v) \|_{H^2(\Omega)} \leqq C \| v \|_{L^2(\Omega)}. \tag{1.6}$$

(Here, and in the sequel, C will denote a generic constant, not necessarily the same in any two places.) This regularity result enables us to interpret $y(v)$ as being the solution of (1.1′).

We also need to consider the "adjoint" equation

$$a(\tilde{y}(p), \varphi) = (p, \varphi)_{L^2(\Omega)}, \qquad \forall \varphi \in H^1(\Omega), \tag{1.7}$$

$\tilde{y}(p) \in H^1(\Omega)$, for $p \in L^2(\Omega)$. Again from [5] and [13], $\tilde{y}(p) \in H^2(\Omega)$ and

$$\| \tilde{y}(p) \|_{H^2(\Omega)} \leqq C \| p \|_{L^2(\Omega)}, \tag{1.8}$$

and $y(p)$ may be interpreted as being the solution of

$$\begin{aligned} -\Delta \tilde{y} + \tilde{y} &= p \quad \text{in } \Omega, \\ \frac{\partial \tilde{y}}{\partial n} &= 0 \quad \text{on } \Gamma. \end{aligned} \tag{1.7′}$$

In this context, $\gamma_0 \tilde{y}(p) \in H^{3/2}(\Gamma)$, *a fortiori* $\gamma_0 \tilde{y}(p) \in H^1(\Gamma)$, and

$$\| \gamma_0 \tilde{y}(p) \|_{H^1(\Gamma)} \leqq C \| \tilde{y}(p) \|_{H^2(\Omega)} \leqq C' \| p \|_{L^2(\Omega)}. \tag{1.9}$$

The corresponding equation for (P_h) is

$$a(\tilde{y}_h(p), \varphi_h) = (p, \varphi_h)_{L^2(\Omega)}, \qquad \forall \varphi_h \in H_h^1(\Omega), \qquad \tilde{y}_h(p) \in H_h^1(\Omega). \tag{1.10}$$

Let us denote the linear map $v \to y(v)$ by

$$L: \quad L^2(\Gamma) \to L^2(\Omega), \tag{1.11}$$

and the map $v \to y_h(v)$ by

$$L_h: \quad L^2(\Gamma) \to L^2(\Omega). \tag{1.12}$$

Corresponding to $s_h : L^2(\Omega) \to L_h^2(\Omega)$ (1.5), we shall utilize the projection $t_h : L^2(\Gamma) \to L_h^2(\Gamma)$, and an interpolation operator $r_h : H^2(\Omega) \to H_h^1(\Omega)$ with the property

$$\| y - r_h y \|_{H^1(\Omega)} \leqq C \| y \|_{H^2(\Omega)} . h, \qquad \forall y \in H^2(\Omega) \tag{1.13}$$

(*see* [1, 4, 14]).

s_h and t_h have the properties

$$\| y - s_h y \|_{L^2(\Omega)} \leqq C \| y \|_{H^1(\Omega)} . h, \tag{1.14}$$

$$\| v - t_h v \|_{L^2(\Gamma)} \leqq C \| v \|_{H^1(\Gamma)} . h, \tag{1.15}$$

for each $y \in H^1(\Omega)$ and each $v \in H^1(\Gamma)$, respectively (*see* [1, 16]).

We collect together as a lemma the required approximation results concerning the state and the adjoint state.

LEMMA 1.1 : $L^* : L^2(\Omega) \to L^2(\Gamma)$, *the adjoint of* L, *is given by* $L^* p = \gamma_0 \tilde{y}(p)$, $\forall p \in L^2(\Omega)$; $L_h^* : L^2(\Omega) \to L^2(\Gamma)$, *the adjoint of* L_h, *is given by* $L^* p = \gamma_0 \tilde{y}_h(p)$, $\forall p \in L^2(\Omega)$; *and for* $v_1 \in H^{1/2}(\Gamma)$, $v_2 \in L^2(\Gamma)$, p_1, $p_2 \in L^2(\Omega)$, *we have the following estimates:*

$$\| L v_1 - L_h v_2 \|_{H^1(\Omega)} \leqq C (\| v_1 - v_2 \|_{L^2(\Gamma)} + \| v_1 \|_{H^{1/2}(\Gamma)} . h), \tag{1.16}$$

$$\| L^* p_1 - L_h^* p_2 \|_{H^1(\Omega)} \leqq C (\| p_1 - p_2 \|_{L^2(\Omega)} + \| p_1 \|_{L^2(\Omega)} . h). \tag{1.17}$$

Proof: By the definition of L, $\forall v \in L^2(\Gamma)$,

$$a(Lv, \varphi) = (v, \gamma_0 \varphi)_{L^2(\Gamma)}, \qquad \forall \varphi \in H^1(\Omega),$$

so that, in particular

$$a(Lv, y(p)) = (v, \gamma_0 \tilde{y}(p))_{L^2(\Gamma)}, \qquad \forall p \in L^2(\Omega). \tag{1.18}$$

But by the definition of $\tilde{y}(p)$,

$$a(Lv, \tilde{y}(p)) = a(\tilde{y}(p), Lv) = (p, Lv)_{L^2(\Omega)} \tag{1.19}$$

so that

$$(Lv, p)_{L^2(\Omega)} = (v, \gamma_0 \tilde{y}(p))_{L^2(\Gamma)},$$
$$\forall v \in L^2(\Gamma), \quad \forall p \in L^2(\Omega), \quad \text{and} \quad L^* p = \gamma_0 \tilde{y}(p).$$

An identical argument yields the statement concerning L_h^*.

Estimates (1.16) and (1.17) are obtained by the standard techniques of finite element approximation. We include the short derivation of (1.16) for convenience:

$$\| Lv_1 - L_h v_1 \|^2_{H^1(\Omega)} = a(Lv_1 - L_h v_1, Lv_1 - L_h v_1)$$
$$= a(Lv_1 - L_h v_1, r_h Lv_1 - L_h v_1) + a(Lv_1 - L_h v_1, Lv_1 - r_h Lv_1)$$
$$= a(Lv_1 - L_h v_1, Lv_1 - r_h Lv_1),$$

by the definition of $Lv_1 = y(v_1)$ [*see* (1.1)] and $L_h v_1 = y_h(v_1)$ [*see* (1.4)]; $r_h : H^2(\Omega) \to H_h^1(\Omega)$ is the interpolation operator with property (1.13). Thus

$$\| Lv_1 - L_h v_1 \|^2_{H^1(\Omega)} \leqq \| Lv_1 - L_h v_1 \|_{H^1(\Omega)} \cdot \| Lv_1 - r_h Lv_1 \|_{H^1(\Omega)}$$
$$\leqq C \| Lv_1 - L_h v_1 \|_{H^1(\Omega)} \cdot \| Lv_1 \|_{H^2(\Omega)} \cdot h,$$

and we obtain

$$\| Lv_1 - L_h v_1 \|_{H^1(\Omega)} \leqq C \| Lv_1 \|_{H^2(\Omega)} \cdot h \leqq C \| v_1 \|_{H^{1/2}(\Gamma)} \cdot h, \tag{1.20}$$

by (1.6).

(1.20) and the definition (1.4) of $y_h(v)$ yield (1.16).

Q.E.D.

2. SADDLE POINT CHARACTERIZATION OF THE SOLUTIONS OF (P) AND (P_h) AND REGULARITY OF OPTIMAL CONTROL

We start by characterizing the solution of (P).

LEMMA 2.1: *If u is the solution of* (P), *there exist* $p_1 \in L^2(\Gamma)$ *and* $p_2 \in L^2(\Omega)$ *such that* $(u; p_1, p_2)$ *is the saddle point on* $L^2(\Gamma) \times (L^2(\Gamma) \times L^2(\Omega))$ *of the Lagrangian* $\mathscr{L}$ *defined by*

$$\mathscr{L}(v; q_1, q_2) = \frac{\nu}{2} \| v \|^2_{L^2(\Gamma)} - \| q_1 \|_{L^1(\Gamma)} - \frac{1}{2} \| q_2 \|^2_{L^2(\Omega)}$$
$$- (Lv - y_d, q_2)_{L^2(\Omega)} - (v, q_1)_{L^2(\Gamma)}, \tag{2.1}$$

i.e.:
$$\mathscr{L}(u; q_1, q_2) \leqq \mathscr{L}(u; p_1, p_2) \leqq \mathscr{L}(v; p_1, p_2) \tag{2.2}$$

for each $v \in L^2(\Gamma)$, $q_1 \in L^2(\Gamma)$, $q_2 \in L^2(\Omega)$.

The saddle point characterization leads to the relations

$$\nu u - p_1 - L^* p_2 = 0, \tag{2.3}$$

$$p_2 + L u - y_d = 0, \tag{2.4}$$

$$(u, p_1)_{L^2(\Gamma)} + \| p_1 \|_{L^1(\Gamma)} = \inf_{q_1 \in L^2(\Gamma)} \{ (u, q_1)_{L^2(\Gamma)} + \| q_1 \|_{L^1(\Gamma)} \}. \tag{2.5}$$

Proof: We appeal to duality theory as discussed in Ekeland and Temam ([3], chap. III) and use similar notation.

Problem (P) can be expressed as

$$\text{(P)} \qquad \operatorname*{Inf}_{v \in L^2(\Gamma)} [J(v) + \chi_K(v)],$$

where χ_K is the indicator function of K:

$$\chi_K(v) = \begin{cases} 0 & \text{if} \quad v \in K, \\ +\infty & \text{if} \quad v \notin K. \end{cases}$$

We consider the perturbation functional

$$\Phi(v; q_1, q_2) = \frac{\nu}{2} \| v \|^2_{L^2(\Gamma)} + \frac{1}{2} \| L v - q_2 - y_d \|^2_{L^2(\Omega)} + \chi_K(v - q_1), \tag{2.6}$$

where $v \in L^2(\Gamma)$, $q_1 \in L^2(\Gamma)$ and $q_2 \in L^2(\Omega)$. Notice that

$$\Phi(v; 0, 0) = J(v) + \chi_K(v), \qquad \forall v \in L^2(\Gamma)$$

and that $\Phi \in \Gamma_0(L^2(\Gamma) \times L^2(\Gamma) \times L^2(\Omega))$.

We identify $(L^2(\Gamma))^*$ with $L^2(\Gamma)$ and $(L^2(\Omega))^*$ with $L^2(\Omega)$. The Lagrangian corresponding to Φ is thus defined on $L^2(\Gamma) \times L^2(\Gamma) \times L^2(\Omega)$ and

$$-\mathscr{L}(v; q_1^*, q_2^*) = \operatorname*{Sup}_{(q_1, q_2) \in L^2(\Gamma) \times L^2(\Omega)} \{ (q_1, q_1^*)_{L^2(\Gamma)} + (q_2, q_2^*)_{L^2(\Omega)} - \Phi(v; q_1, q_2) \}.$$

Therefore

$$\begin{aligned} -\mathscr{L}(v; q_1^*, q_2^*) = & -\frac{\nu}{2} \| v \|^2_{L^2(\Gamma)} + \sup_{q_1 \in L^2(\Gamma)} \{ (q_1, q_1^*)_{L^2(\Gamma)} - \chi_K(v - q_1) \} \\ & + \sup_{q_2 \in L^2(\Gamma)} \{ (q_2, q_2^*)_{L^2(\Omega)} - \frac{1}{2} \| L v - y_d - q_2 \|_{L^2(\Omega)} \} \\ = & -\frac{\nu}{2} \| v \|^2_{L^2(\Gamma)} + (v, q_1^*)_{L^2(\Gamma)} + (L v - y_d, q_2^*)_{L^2(\Omega)} \\ & + \sup_{\substack{|q_1(x)| \leq 1 \\ \text{a.e. on } \Gamma}} (q_1, q_1^*)_{L^2(\Gamma)} + \sup_{q_2 \in L^2(\Omega)} \{ (q_2, q_2^*)_{L^2(\Omega)} - \frac{1}{2} \| q_2 \|^2_{L^2(\Omega)} \} \\ = & -\frac{\nu}{2} \| v \|^2_{L^2(\Gamma)} + (v, q_1^*)_{L^2(\Gamma)} + (L v - y_d, q_2^*)_{L^2(\Omega)} + \| q_1^* \|_{L^1(\Gamma)} + \frac{1}{2} \| q_2^* \|^2_{L^2(\Omega)}. \end{aligned}$$

Thus

$$\mathscr{L}(v; q_1, q_2)=\frac{\nu}{2}\|v\|^2_{L^2(\Gamma)}-\|q_1\|_{L^1(\Gamma)}-\frac{1}{2}\|q_2\|^2_{L^2(\Omega)}$$
$$-(Lv-y_d, q_2)_{L^2(\Omega)}-(v, q_1)_{L^2(\Gamma)}. \quad (2.7)$$

Since there are no state constraints, the general discussion of Mossino ([12], pp. 232-233) is valid (the duality discussed in [12] corresponds to the perturbation scheme that we are using); the primal and dual problems have the same value and each has a unique solution. Therefore, u being the solution of the primal problem and (p_1, p_2) being the solution of the dual problem, $(u; p_1, p_2)$ is the unique saddle point of $\mathscr{L}$ on $L^2(\Gamma)\times(L^2(\Gamma)\times L^2(\Omega))$ (*see* [2], p. 57);

$$\mathscr{L}(u; q_1, q_2)\leqq\mathscr{L}(u; p_1, p_2)\leqq\mathscr{L}(v; p_1, p_2),$$
$$\forall v\in L^2(\Gamma), \quad q_1\in L^2(\Gamma), \quad q_2\in L^2(\Omega).$$

Thus

$$\left\langle \frac{\partial\mathscr{L}}{\partial v}(u; p_1, p_2), \delta v\right\rangle=0, \qquad \forall\,\delta v\in L^2(\Gamma), \quad (2.8)$$

$$\left\langle \frac{\partial\mathscr{L}}{\partial q_2}(u; p_1, p_2), \delta q_2\right\rangle=0, \qquad \forall\,\delta q_2\in L^2(\Omega), \quad (2.9)$$

and

$$-(u, p_1)_{L^2(\Gamma)}-\int_\Gamma |p_1|\,d\Gamma\geqq-(u, q_1)-\int_\Gamma |q_1|\,d\Gamma, \qquad \forall q_1\in L^2(\Gamma).$$

(2.8) means

$$(\nu u-p_1, \delta v)_{L^2(\Gamma)}-(L(\delta v), p_2)_{L^2(\Omega)}=0,$$

and therefore

$$(\nu u-p_1-L^* p_2, \delta v)_{L^2(\Gamma)}=0, \qquad \forall\,\delta v\in L^2(\Gamma).$$

Thus (2.3) is valid.

Equation (2.9) immediately yields (2.4), and the proof of the lemma is complete.

REMARK: The form of the Lagrangian (2.1) resulted from the perturbation scheme that we chose (2.6). It leads more easily to error estimates than the Lagrangian:

$$\tilde{\mathscr{L}}(y, v; q_1, q_2)=\frac{1}{2}\|y-y_d\|^2_{L^2(\Omega)}+\frac{\nu}{2}\|v\|^2_{L^2(\Gamma)}-\|q_1\|_{L^1(\Gamma)}$$
$$-(v, q_1)_{L^2(\Gamma)}-(q_2, y-Lv)_{L^2(\Omega)},$$

which results from the pertubation functional

$$\tilde{\Phi}(y, v; q_1, q_2)=\frac{1}{2}\|y-y_d\|^2_{L^2(\Omega)}+\frac{\nu}{2}\|v\|^2_{L^2(\Gamma)}+\chi_K(v-q_1)+\chi_{\{0\}}(y-Lv).$$

The next lemma expresses the corresponding characterization of the solution of the approximating problem (P_h).

LEMMA 2.2: *If u_h is the solution of (P_h), there exist $p_{1h} \in L_h^2(\Gamma)$ and $p_{2h} \in L_h^2(\Omega)$ such that $(u_h; p_{1h}, p_{2h})$ is the saddle point on $L_h^2(\Gamma) \times (L_h^2(\Gamma) \times L_h^2(\Omega))$ of the Lagrangian $\mathscr{L}_h$ defined by*

$$\mathscr{L}_h(v_h; q_{1h}, q_{2h}) = \frac{\nu}{2}\| v_h \|^2_{L^2(\Gamma)} - \| q_{1h} \|_{L^1(\Gamma)} - \frac{1}{2}\| q_{2h} \|^2_{L^2(\Omega)} - (L_h v_h - s_h y_d, q_{2h})_{L^2(\Omega)} - (v_h, q_{1h})_{L^2(\Gamma)}, \quad (2.10)$$

i.e.:

$$\left.\begin{array}{l} \mathscr{L}_h(u_h; q_{1h}, q_{2h}) \leqq \mathscr{L}_h(u_h; p_{1h}, p_{2h}) \leqq \mathscr{L}_h(v_h; p_{1h}, p_{2h}), \\ \forall v_h \in L_h^2(\Gamma), \quad q_{1h} \in L_h^2(\Gamma), \quad q_{2h} \in L_h^2(\Omega), \end{array}\right\} \quad (2.11)$$

and we have

$$u_h - p_{1h} - t_h L_h^* p_{2h} = 0, \quad (2.12)$$

$$p_{2h} + s_h L_h u_h - s_h y_d = 0, \quad (2.13)$$

$$(u_h, p_{1h})_{L^2(\Gamma)} + \int_\Gamma |p_{1h}| \, d\Gamma = \inf_{q_{1h} \in L_h^2(\Gamma)} \left\{ (u_h, q_{1h})_{L^2(\Gamma)} + \int_\Gamma |q_{1h}| \, d\Gamma \right\}. \quad (2.14)$$

Proof: The proof is almost identical to that of lemma (2.1), and we shall only indicate the necessary modifications.

(P_h) is expressed as

$$\operatorname*{Inf}_{v_h \in L_h^2(\Gamma)} [J_h(v_h) + \chi_{K_h}(v_h)],$$

where χ_{K_h} is the indicator function of $K_h = \{ v_h \in L_h^2(\Gamma) : |v_h(x)| \leqq 1, \forall x \in \Gamma \}$. The perturbation functional is

$$\Phi_h(v_h; q_{1h}, q_{2h}) = \frac{\nu}{2}\| v_h \|^2_{L^2(\Gamma)} + \frac{1}{2}\| s_h L_h v_h - q_{2h} - s_h y_d \|^2_{L^2(\Omega)} + \chi_{K_h}(v_h - q_{1h})$$

for $(v_h, q_{1h}, q_{2h}) \in L_h^2(\Gamma) \times L_h^2(\Gamma) \times L_h^2(\Omega)$, so that the corresponding Lagrangian is computed to be

$$\mathscr{L}_h(v_h; q_{1h}, q_{2h}) = \frac{\nu}{2}\| v_h \|^2_{L^2(\Gamma)} - \| q_{1h} \|_{L^1(\Gamma)} - \frac{1}{2}\| q_{2h} \|^2_{L^2(\Omega)} - (s_h L_h v_h - s_h y_d, q_{2h})_{L^2(\Omega)} - (v_h, q_{1h})_{L^2(\Gamma)},$$

which is equivalent to (2.10) since s_h is a projection. (2.12) follows from

$$\left\langle \frac{\partial \mathscr{L}_h}{\partial v_h}(v_h; p_{1h}, p_{2h}), \delta v_h \right\rangle = 0, \qquad \forall \delta v_h \in L_h^2(\Gamma),$$

i.e.:

$$(\nu u_h - p_{1h},\ \delta v_h)_{L^2(\Gamma)} - (L_h(\delta v_h),\ p_{2h})_{L^2(\Omega)} = 0,$$

which yields

$$(\nu u_h - p_{1h} - L_h^* p_{2h},\ \delta v_h)_{L^2(\Gamma)} = 0, \qquad \forall\, \delta v_h \in L_h^2(\Gamma).$$

Since in general $L_h^* p_{2h} \notin L_h^2(\Gamma)$, we cannot assert that $\nu u_h - p_{1h} - L_h^* p_{2h} = 0$, but we can assert that

$$\nu u_h - p_{1h} - t_h L_h^* p_{2h} = 0,$$

where $t_h : L^2(\Gamma) \to L_h^2(\Gamma)$ *is the projection introduced in section* 1. (2.13) *follows in similar manner.*

Q.E.D.

The regularity result about the optimal control can be obtained as in Lions [11], pp. 53 and 58.

LEMMA 2.3: *The solution u of* (P) *is in* $H^1(\Gamma)$, *the corresponding* $p_1 \in H^1(\Gamma)$ *and* $p_2 \in H^1(\Omega)$.

Proof: Owing to (2.3), (2.4), the definition of L (1.11) and lemma 1.1, letting $y(u) = y$, $\tilde{y}(y - y_d) = \tilde{y}$ (1.7), we have

$$\left.\begin{aligned} -\Delta y + y = 0 \quad &\text{in } \Omega, \\ \frac{\partial y}{\partial n} = u \quad &\text{on } \Gamma, \end{aligned}\right\} \tag{2.15}$$

$$\left.\begin{aligned} -\Delta \tilde{y} + \tilde{y} = y - y_d \quad &\text{in } \Omega, \\ \frac{\partial \tilde{y}}{\partial n} = 0 \quad &\text{on } \Gamma, \end{aligned}\right\} \tag{2.16}$$

$$p_2 = -(y - y_d), \tag{2.17}$$

$$p_1 = \nu u + \gamma_0 \tilde{y}, \tag{2.18}$$

and

$$\left.\begin{aligned} (\gamma_0 \tilde{y} + \nu u,\ v - u)_{L^2(\Gamma)} \geqq 0, \\ \forall\, v \in K = \{ v \in L^2(\Gamma) : |v(x)| \leqq 1 \text{ a.e. on } \Gamma \}. \end{aligned}\right\} \tag{2.19}$$

By (2.19)

$$u = P\left(-\frac{\gamma_0 \tilde{y}}{\nu} \right), \tag{2.20}$$

where $P = L^2(\Gamma) \to L^2(\Gamma)$ is the projection into K.

Now, $\tilde{y} \in H^2(\Omega)$ since y, $y_d \in L^2(\Omega)$ (1.8), so that $\gamma_0 \tilde{y} \in H^{3/2}(\Gamma)$. *A fortiori*, $\gamma_0 \tilde{y} \in H^1(\Gamma)$. P leaves $H^1(\Gamma)$ invariant (*see* [7], p. 50 and [10]) so that $u \in H^1(\Gamma)$ by (2.20).

Since $y_d \in H^1(\Omega)$, $p_2 \in H^1(\Omega)$ by (2.17), and $p_1 \in H^1(\Gamma)$ by (2.18).

Q.E.D.

3. THE RATE OF CONVERGENCE OF APPROXIMATE SOLUTIONS OF (P)

On the basis of the previous preliminary sections we are now ready to prove the convergence result.

THEOREM 3.1: *If u is the solution of* (P) *and u_h is the solution of* (P_h), *the estimates*

$$\| u - u_h \|_{L^2(\Gamma)} \leqq C h \tag{3.1}$$

and

$$\| y(u) - y_h(u_h) \|_{H^1(\Omega)} \leqq C h \tag{3.2}$$

are valid.

Proof: By lemma 2.2 (2.11):

$$\mathscr{L}_h(u_h;\, p_{1h},\, p_{2h}) \geqq \mathscr{L}_h(u_h;\, q_{1h},\, q_{2h}), \qquad \forall\, q_{1h} \in L_h^2(\Gamma), \quad \forall\, q_{2h} \in L_h^2(\Omega),$$

and in particular

$$\mathscr{L}_h(u_h;\, p_{1h},\, p_{2h}) \geqq \mathscr{L}_h(u_h;\, t_h p_1,\, s_h p_2), \tag{3.3}$$

where s_h and t_h are the projections introduced in section 1.

It is readily seen that

$$\int_\Gamma | t_h p_1 | \, d\Gamma \leqq \int_\Gamma | p_1 | \, d\Gamma,$$

and we have

$$(u_h,\, p_1)_{L^2(\Gamma)} = (u_h,\, t_h p_1)_{L^2(\Gamma)},$$

so that

$$\mathscr{L}_h(u_h;\, t_h p_1,\, s_h p_2) \geqq \mathscr{L}_h(u_h;\, p_1,\, s_h p_2). \tag{3.4}$$

By (3.3) and (3.4),

$$\mathscr{L}_h(u_h;\, p_{1h},\, p_{2h}) \geqq \mathscr{L}_h(u_h;\, p_1,\, s_h p_2). \tag{3.5}$$

Now

$$\begin{aligned} &\mathscr{L}_h(u_h;\, p_1,\, s_h p_2) - \mathscr{L}_h(t_h u;\, p_1,\, s_h p_2) \\ &\quad = \frac{\nu}{2} \| u_h \|_{L^2(\Gamma)}^2 - \frac{\nu}{2} \| t_h u \|_{L^2(\Gamma)}^2 - (u_h,\, p_1)_{L^2(\Gamma)} + (t_h u,\, p_1)_{L^2(\Gamma)} \\ &\qquad - (L_h u_h,\, s_h p_2)_{L^2(\Omega)} + (L_h t_h u,\, s_h p_2)_{L^2(\Omega)} \\ &\quad = \nu (u_h - t_h u,\, t_h u)_{L^2(\Gamma)} + \frac{\nu}{2} \| u_h - t_h u \|_{L^2(\Gamma)}^2 \end{aligned}$$

$$-(u_h-t_h u,\ p_1)_{L^2(\Gamma)}-(L_h(u_h-t_h u),\ s_h p_2)_{L^2(\Omega)}$$

$$=\nu(u_h-t_h u,\ t_h u)_{L^2(\Gamma)}+\frac{\nu}{2}\|u_h-t_h u\|^2_{L^2(\Gamma)}$$

$$-(u_h-t_h u,\ p_1)_{L^2(\Gamma)}-(u_h-t_h u,\ L_h^* s_h p_2)_{L^2(\Gamma)}$$

$$=\frac{\nu}{2}\|u_h-t_h u\|^2_{L^2(\Gamma)}+(u_h-t_h u,\ \nu\, t_h u-p_1-L_h^* s_h p_2)_{L^2(\Gamma)}. \quad (3.6)$$

From (3.5) and (3.6):

$$\frac{\nu}{2}\|u_h-t_h u\|^2_{L^2(\Gamma)}\leqq \mathscr{L}_h(u_h;\ p_{1h},\ p_{2h})-\mathscr{L}_h(t_h u;\ p_1,\ s_h p_2)$$

$$+(u_h-t_h u,\ -\nu\, t_h u+p_1+L_h^* s_h p_2)_{L^2(\Gamma)}$$

$$\leqq \mathscr{L}_h(t_h u;\ p_{1h},\ p_{2h})-\mathscr{L}_h(t_h u;\ p_1,\ s_h p_2)$$

$$+(u_h-t_h u,\ -\nu\, t_h u+p_1+L_h^* s_h p_2)_{L^2(\Gamma)} \quad (3.7)$$

by lemma 2.2 (2.11). Now

$$\mathscr{L}_h(t_h u;\ p_{1h},\ p_{2h})-\mathscr{L}_h(t_h u;\ p_1,\ s_h p_2)$$

$$=-(t_h u,\ p_{1h}-p_1)_{L^2(\Gamma)}-\frac{1}{2}\|p_{2h}\|^2_{L^2(\Omega)}+\frac{1}{2}\|s_h p\|^2_{L^2(\Omega)}$$

$$-\int_\Gamma |p_{1h}|\,d\Gamma+\int_\Gamma |p_1|\,d\Gamma-(L_h(t_h u)-s_h y_d,\ p_{2h}-s_h p_2)_{L^2(\Omega)}$$

$$=-(t_h u,\ p_{1h}-p_1)_{L^2(\Gamma)}-\frac{1}{2}\|p_{2h}-s_h p_2\|^2_{L^2(\Omega)}$$

$$+(-L_h(t_h u)+s_h y_d-s_h p_2,\ p_{2h}-s_h p_2)_{L^2(\Omega)}-\int_\Gamma |p_{1h}|\,d\Gamma+\int_\Gamma |p_1|\,d\Gamma. \quad (3.8)$$

By (3.7) and (3.8),

$$\frac{\nu}{2}\|u_h-t_h u\|^2_{L^2(\Gamma)}\leqq(-L_h(t_h u)+s_h y_d-s_h p_2,\ p_{2h}-s_h p_2)_{L^2(\Omega)}$$

$$+(-\nu\, t_h u+p_1+L_h^*(s_h p_2),\ u_h-t_h u)_{L^2(\Gamma)}$$

$$-(t_h u,\ p_{1h}-p_1)_{L^2(\Gamma)}-\int_\Gamma |p_{1h}|\,d\Gamma+\int_\Gamma |p_1|\,d\Gamma. \quad (3.9)$$

By lemma 2.1 (2.5):

$$(u,\ p_{1h}-p_1)_{L^2(\Gamma)}-\int_\Gamma |p_1|\,d\Gamma+\int_\Gamma |p_{1h}|\,d\Gamma\geqq 0. \quad (3.10)$$

(3.10), and the fact that s_h and t_h are projections, enable us to conclude from (3.9):

$$\frac{\nu}{2}\| u_h - t_h u \|^2_{L^2(\Gamma)} \leqq (-L_h(t_h u) + y_d - p_2, p_2 h - s_h p_2)_{L^2(\Omega)}$$
$$+(-\nu u + p_1 + L_h^*(s_h p_2), u_h - t_h u)_{L^2(\Gamma)} + (u - t_h u, p_{1h} - p_1)_{L^2(\Gamma)}. \quad (3.11)$$

We'shall now estimate each term on the right-hand side of (3.11) separately.

By lemma 2.1 (2.4),

$$(-L_h(t_h u) + y_d - p_2, p_{2h} - s_h p_2)_{L^2(\Omega)} = (-L_h(t_h u) + Lu, p_{2h} - s_h p_2)_{L^2(\Omega)}, \quad (3.12)$$

and

$$(-L_h(t_h u) + Lu, p_{2h} - s_h p_2)_{L^2(\Omega)}$$
$$= (-L_h(t_h u) + Lu, p_{2h} - p_2)_{L^2(\Omega)} + (-L_h(t_h u) + Lu, p_2 - s_h p_2)_{L^2(\Omega)}$$
$$= (-L_h(t_h u) + Lu, -s_h L_h(u_h) + s_h y_d + Lu - y_d)_{L^2(\Omega)}$$
$$+ (-L_h(t_h u) + Lu, p_2 - s_h p_2)_{L^2(\Omega)}, \quad (3.13)$$

by lemma 2.1 (2.4) and lemma 2.2 (2.13).

By (3.12) and (3.13),

$$(-L_h(t_h u) + y_d - p_2, p_{2h} - s_h p_2)_{L^2(\Omega)}$$
$$\leqq \| Lu - L_h(t_h u) \|_{L^2(\Omega)} \cdot \| Lu - s_h L_h(u_h) \|_{L^2(\Omega)}$$
$$+ \| Lu - L_h(t_h u) \|_{L^2(\Omega)} \cdot \| y_d - s_h y_d \|_{L^2(\Omega)}$$
$$+ \| Lu - L_h(t_h u) \|_{L^2(\Omega)} \cdot \| p_2 - s_h p_2 \|_{L^2(\Omega)}. \quad (3.14)$$

By lemma 1.1,

$$\| Lu - L_h(t_h u) \|_{L^2(\Omega)} \leqq C(\| u - t_h u \|_{L^2(\Gamma)} + \| u \|_{H^{1/2}(\Gamma)} \cdot h), \quad (3.15)$$

and by lemma 2.3, $u \in H^1(\Gamma)$, so that

$$\| u - t_h u \|_{L^2(\Gamma)} \leqq C \| u \|_{H^1(\Gamma)} \cdot h, \quad (3.16)$$

by (1.15). Therefore

$$\| Lu - L_h(t_h u) \|_{L^2(\Omega)} \leqq Ch. \quad (3.17)$$

By lemma 2.3, $p_2 \in H^1(\Omega)$, and it is given that $y_d \in H^1(\Omega)$, so that by (1.14):

$$\| y_d - s_h y_d \|_{L^2(\Omega)} \leqq Ch, \quad (3.18)$$
$$\| p_2 - s_h p_2 \|_{L^2(\Omega)} \leqq Ch. \quad (3.19)$$

Combining (3.14), (3.17), (3.18) and (3.19) we obtain

$$(-L_h(t_h u) + y_d - p_2, p_{2h} - s_h p_2)_{L^2(\Omega)} \leqq C(\| Lu - s_h L_h(u_h) \|_{L^2(\Omega)} \cdot h + h^2). \quad (3.20)$$

On the other hand,

$$\|Lu-s_h L_h(u_h)\|_{L^2(\Omega)} \leqq \|Lu-s_h Lu\|_{L^2(\Omega)}$$
$$+\|s_h Lu-s_h L_h(u_h)\|_{L^2(\Omega)} \leqq C\|Lu\|_{H^1(\Omega)}.h+\|Lu-L_h u_h\|_{L^2(\Omega)} \quad (3.21)$$

by (1.14) and the definition of s_h.

From the definition of L

$$\|Lu\|_{H^1(\Omega)} \leqq C\|u\|_{L^2(\Gamma)},$$

and we obtain from (3.21),

$$\|Lu-s_h L_h(u_h)\|_{L^2(\Omega)} \leqq Ch+\|Lu-L_h u_h\|_{L^2(\Omega)}. \quad (3.22)$$

Now

$$\|Lu-L_h u_h\|_{L^2(\Omega)} \leqq C(\|u-u_h\|_{L^2(\Gamma)}+\|u\|_{H^{1/2}(\Gamma)}.h)$$
$$\leqq C(\|u-t_h u\|_{L^2(\Gamma)}+\|t_h u-u_h\|_{L^2(\Gamma)}+\|u\|_{H^{1/2}(\Gamma)}.h)$$
$$\leqq C(h+\|t_h u-u_h\|_{L^2(\Gamma)}), \quad (3.23)$$

by lemma 1.1 and by (3.16). (3.22) and (3.23) yield

$$\|Lu-s_h L_h(u_h)\|_{L^2(\Omega)} \leqq C(h+\|t_h u-u_h\|_{L^2(\Gamma)}). \quad (3.24)$$

From (3.20) and (3.24) we obtain

$$(-L_h(t_h u)+y_d-p_2, \dot{p}_{2h}-s_h p_2)_{L^2(\Omega)} \leqq C(h^2+\|u_h-t_h u\|_{L^2(\Gamma)}.h). \quad (3.25)$$

We now estimate the second term on the r.h.s. of (3.11):

$$(-\nu u+p_1+L_h^*(s_h p_2), u_h-t_h u)_{L^2(\Gamma)}=(L_h^*(s_h p_2)-L^* p_2, u_h-t_h u)_{L^2(\Gamma)} \quad (3.26)$$

by lemma 2.1 (2.3).

$$(L_h^*(s_h p_2)-L^* p_2, u_h-t_h u)_{L^2(\Gamma)} \leqq \|L_h^*(s_h p_2)-L^* p_2\|_{L^2(\Gamma)}.\|u_h-t_h u\|_{L^2(\Gamma)}$$
$$\leqq C(\|s_h p_2-p_2\|_{L^2(\Omega)}+\|p_2\|_{L^2(\Omega)}.h)\|u_h-t_h u\|_{L^2(\Gamma)}, \quad (3.27)$$

by lemma 1.1, and

$$\|s_h p_2-p_2\|_{L^2(\Omega)} \leqq Ch$$

by (3.19), so that we obtain

$$(L_h^*(s_h p_2)-L^* p_2, u_h-t_h u)_{L^2(\Gamma)} \leqq C\|u_h-t_h u\|_{L^2(\Gamma)}.h,$$

and

$$(-\nu u+p_1+L_h^*(s_h p_2), u_h-t_h u)_{L^2(\Gamma)} \leqq C\|u_h-t_h u\|_{L^2(\Gamma)}.h. \quad (3.28)$$

The last term on the r.h.s. of (3.11) is

$$(u-t_h u, p_{1h}-p_1)_{L^2(\Gamma)}=(u-t_h u, \nu u_h-t_h L_h^*(p_{2h})-\nu u+L^* p_2)_{L^2(\Gamma)}$$

by lemma 2.1 (2.3) and lemma 2.2 (2.12).

Thus

$$(u-t_h u,\ p_{1h}-p_1)_{L^2(\Gamma)}=\nu(u-t_h u,\ u_h-t_h u)_{L^2(\Gamma)}$$
$$+\nu(u-t_h u,\ t_h u-u)_{L^2(\Gamma)}+(u-t_h u,\ L^* p_2-t_h L_h^*(p_{2h}))_{L^2(\Gamma)}$$
$$\leqq \nu(u-t_h u,\ u_h-t_h u)_{L^2(\Gamma)}+(u-t_h u,\ L^* p_2-t_h L_h^*(p_{2h}))_{L^2(\Gamma)}$$
$$\leqq C\|u\|_{H^1(\Gamma)}\cdot h(\|u_h-t_h u\|_{L^2(\Gamma)}+\|L^* p_2-t_h L_h^*(p_{2h})\|_{L^2(\Gamma)}), \quad (3.29)$$

by (3.16). Now,

$$\|L^* p_2-t_h L_h^*(p_{2h})\|_{L^2(\Gamma)}\leqq \|L^* p_2-t_h L^* p_2\|_{L^2(\Gamma)}+\|t_h(L^* p_2-L_h^* p_{2h})\|_{L^2(\Gamma)}$$
$$\leqq C\|L^* p_2\|_{H^1(\Gamma)}\cdot h+\|L^* p_2-L_h^* p_{2h}\|_{L^2(\Gamma)} \quad (3.30)$$

by the definition of t_h and (1.15).

By lemma 1.1 and (1.8), (3.30) yields

$$\|L^* p_2-t_h L_h^*(p_{2h})\|_{L^2(\Gamma)}\leqq C(\|p_2-p_{2h}\|_{L^2(\Omega)}+\|p_2\|_{L^2(\Omega)}\cdot h). \quad (3.31)$$

From lemma 2.1 (2.4) and lemma 2.2 (2.13),

$$p_2-p_{2h}=(Lu-s_h L_h u_h)+(y_d-s_h y_d),$$

so that

$$\|p_2-p_{2h}\|_{L^2(\Omega)}\leqq \|Lu-s_h L_h u_h\|_{L^2(\Omega)}+\|y_d-s_h y_d\|_{L^2(\Omega)}$$
$$\leqq C(h+\|u_h-t_h u\|_{L^2(\Gamma)})+C\|y_d\|_{H^1(\Omega)}\cdot h, \quad (3.32)$$

by (3.24) and (3.18).

From (3.31) and (3.32) we obtain

$$\|L^* p_2-t_h L_h^*(p_{2h})\|_{L^2(\Gamma)}\leqq C(h+\|u_h-t_h u\|_{L^2(\Gamma)}), \quad (3.33)$$

and from (3.29) and (3.33),

$$(u-t_h u,\ p_{1h}-p_1)_{L^2(\Gamma)}\leqq C(h^2+\|u_h-t_h u\|_{L^2(\Gamma)}\cdot h). \quad (3.34)$$

This completes the estimation of the three terms on the r.h.s. of (3.11). Inserting the estimates (3.25), (3.28) and (3.34) in (3.11), we finally obtain

$$\|u_h-t_h u\|^2_{L^2(\Gamma)}\leqq C(h^2+\|u_h-t_h u\|_{L^2(\Gamma)}\cdot h),$$

so that

$$\|u_h-t_h u\|_{L^2(\Gamma)}\leqq Ch$$

and

$$\|u-u_h\|_{L^2(\Gamma)}\leqq \|u-t_h u\|_{L^2(\Gamma)}+\|t_h u-u_h\|_{L^2(\Gamma)}\leqq C(\|u\|_{H^1(\Gamma)}\cdot h+h),$$

and statement (3.1) of the theorem has been established. (3.2) follows from (3.1) by lemma 1.1, and this completes the proof of theorem 3.1.

CONCLUSION

In this paper we singled out a specific problem to illustrate the use of Fenchel-Rockafellar duality theory in obtaining approximation results related to optimal control problems. The technique is quite general and may be applied to the study of control problems governed by linear equations, whether higher-order elliptic (self-adjoint or not), parabolic of hyperbolic (on cylindrical domains).

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