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Ph. Clément

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APPROXIMATION BY FINITE ELEMENT FUNCTIONS USING LOCAL REGULARIZATION (*)

par Ph. CLÉMENT (1)

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Abstract — *The aim of this paper is to give an elementary proof of a theorem of approximation of Sobolev spaces $H^q(\Omega)$ by finite elements without to use classical interpolation The construction which we give here allows us in some cases to fit boundary conditions*

1. INTRODUCTION

The mathematical problem of approximation by finite element functions has been first studied by Goël ([2]) and Zlamal ([3]). In [4], Bramble and Zlamal give estimates for the error in terms of Sobolev norms, however their results are based on the existence and continuity of the interpolate; but the interpolate may not exist; for example if $u \in H^1(\Omega)$ where Ω is a two-dimensional domain, by Sobolev's imbedding theorem, the pointwise values of u cannot be defined and consequently no interpolation is possible. In [5] Strang defines an approximation by considering interpolates of regularized functions.

The purpose of this paper is to give an elementary construction of an approximation based on « local » regularization; the results are slightly less restrictive than Strang's ones. As shown by an example, the construction may be modified to fit boundary conditions. Most of the basic tools are known in the literature and their proofs will not be reproduced here; however all details are contained in [1].

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(1) Ecole polytechnique fedérale de Lausanne, Departement de Mathematiques

2. RESULTS

For the sake of concreteness and simplicity, we shall restrict ourselves to the case of triangular finite element subspaces described by the following situation.

Λ is a closed bounded two-dimensional domain with polygonal boundary Γ. One considers a set $\mathfrak{D}$ of decompositions $D = \{ T_1, ..., T_n \}$ of Λ in closed triangles $T_1, T_2, ..., T_n$ called « elements » such that : 1) $\Lambda = \bigcup_{j=1}^{n} T_j$, 2) two triangles T_i and $T_j \in D$ are either disjoint, or have a vertex in common or have a side in common, 3) n depends on D. To each $D = \{ T_1, ..., T_n \} \in \mathfrak{D}$ is associated a set $\{ \varphi_1, \varphi_2, ..., \varphi_m \}$ of independent real functions defined on Λ (m depends on D) and to each φ_i is associated a point $Q_i \in \Lambda$ called « node »; the Q_i's are not necessarily distinct; let $S_1, ..., S_m$ be the supports of $\varphi_1, ..., \varphi_m$. S_i is connected to Q_i by the relation : $S_i = \bigcup_{Q_i \in T_j} T_j$. Furthermore to each φ_i is associated a functional $\gamma_i : C^\infty(\Lambda) \to \mathbf{R}$ of the form $\sum_{|s| = l_i} a_{is} D^s f(Q_i)$; l_i is called the « order » of γ_i.

$V = \left\{ \sum_{i=1}^{m} a_i \varphi_i : a_i \in \mathbf{R} \right\}$ is the finite element space associated to D. For specific examples see [2], [3], [4], [5], [6], [7].

Let $\Phi \subset \Lambda$; $d(\Phi)$ is the diameter of Φ; $\mu(\Phi)$ is the measure of Φ; $H^q(\Phi)$ denotes the Sobolev space of square integrable functions on Φ possessing square integrable derivatives of order $\leqslant q$. For $u, v \in H^q(\Phi)$ and $w \in C^\infty(\Lambda)$ we define the scalar product and seminorms :

$$(u, v)_\Phi = \int_\Phi uv \quad , \quad |u|_{k,\Phi}^2 = \sum_{|s| = k} (D^s u, D^s u)_\Phi \qquad k \leqslant q,$$

$$|w|_{k,\infty,\Phi} = \max_{|s| = k} \sup_{x \in \Phi} |D^s w(x)| ;$$

for a function $u \in H^q(T_j), j = 1, 2, ..., n$ we set $|u|_{k,\Lambda}^2 = \sum_{j=1}^{n} |u|_{k,T_j}^2$, $k \leqslant q$.

In the following c will denote a generic constant independent of $D \in \mathfrak{D}$. We introduce the following hypotheses.

H1. For any $p \in \mathfrak{P}_\rho$ (polynomials of degree $\leqslant \rho$), where ρ is independent of $D \in \mathfrak{D}$, one has for each $T_j \in D$:

$$p(x) = \sum_{Q_i \in T_j} \gamma_i(p) \varphi_i(x) \quad , \quad x \in T_j .$$

The functions φ_i are the basis functions associated with functionals γ_i.

H2. If $f \in C^\infty(\Lambda)$, for any φ_i with $l_i \leqslant \rho$ and any $T_j \in D$, one has

$$|\gamma_i(f)\varphi_i|_{k,T_j} \leqslant c(d(T_j))^{l_i-k}(\mu(T_j))^{\frac{1}{2}}|f|_{l_i,\infty,Q_i}, k \leqslant \rho + 1$$

H3. For any T_j the number of $S_i \supset T_j$ is bounded by c.

H4. For any $D \in \mathfrak{D}$, all the angles of the T_i's are $\geqslant c > 0$.

Now we define the linear mapping $\Pi : H^0(\Lambda) \to V$ by the following construction. Let $u \in H^0(\Lambda)$; to each S_i we associate the polynomial $p_i \in \mathcal{T}_\rho$ which is the best approximation of u with respect to the norm $|\cdot|_{0,S_i}$ i.e. $(u - p_i, p)_{S_i} = 0$ for all $p \in \mathcal{T}_\rho$; we set

$$\Pi u = \sum_{i=1}^{m} \gamma_i(p_i)\varphi_i.$$

Let $h = h(D) = \max_{j=1,2,\cdots,n} d(T_j)$. Under the above situation and hypotheses we shall prove :

Theorem 1

For $u \in H^q(\Lambda)$, $q \leqslant \rho + 1$, one has

$$|u - \Pi u|_{k,\Lambda} \leqslant ch^{q-k}|u|_{q,\Lambda} \quad , \quad k = 0, 1, ..., q; \tag{1}$$

furthermore if $q \leqslant \rho$ one has also

$$\lim_{h\to 0} |u - \Pi u|_{q,\Lambda} = 0. \tag{2}$$

As mentionned in the introduction, Π can be modified in order to fit boundary condition; we restrict ourselves to the case where the function u to approximate takes the value 0 on the boundary Γ of Λ i.e. $u \in \mathring{H}^1(\Lambda)$. We define $\tilde{\Pi} : H^0(\Lambda) \to V$ by

$$\tilde{\Pi} u = \sum_{i=1}^{m}{}' \gamma_i(p_i)\varphi_i$$

where Σ' means that we omit in the sum the terms relative to indices i for which $Q_i \in \Gamma$ and $l_i = 0$. Then under the above hypotheses we shall prove :

Theorem 2

For $u \in \mathring{H}^1(\Lambda) \cap H^q(\Lambda)$, $q \leqslant \rho + 1$ one has

$$|u - \tilde{\Pi} u|_{k,\Lambda} \leqslant ch^{q-k}|u|_{q,\Lambda} \quad , \quad k = 0, 1, ..., q; \tag{3}$$

furthermore if $q \leqslant \rho$ one has also

$$\lim_{h\to 0} |u - \tilde{\Pi} u|_{q,\Lambda} = 0. \tag{4}$$

REMARKS

1. $\tilde{\Pi}u$ will not automatically belong to $\mathring{H}^1(\Lambda)$; however if

a) $\varphi_i \in H^1(\Lambda) i = 1, 2, ..., m$,

b) for all $Q_i \in \Gamma$ one has $l_i = 0$, then specific examples show that $\tilde{\Pi}u \in \mathring{H}^1(\Lambda)$.

2. For one-dimensional finite elements H4 has to be replaced by : if $T_i \cap T_j \neq \emptyset$ then $d(T_i)/d(T_j) \geqslant c > 0$; here the T_i's are segments and $d(T_i)$ is the length of T_i.

3. The local character of the definition of Π clearly implies the following property. Let $u \in H^0(\Lambda)$, $\Phi \subset \Omega \subset \Lambda$ where Φ is closed and Ω open, $u \in H^q(\Omega)$; then for $m \leqslant \rho + 1$ one has $|u - \Pi u|_{k,\Phi} \leqslant ch^{q-k}|u|_{q,\Omega}$ and for $q \leqslant \rho$ one has $\lim_{h\to 0} |u - \Pi u|_{q,\Phi} = 0$.

4. There is a great number of alternative possibilities of defining Π in the same spirit.

5. One can without any difficulty give the same results for $\Lambda \subset \mathbf{R}^n$.

3. PROOFS

In this section we use all the definitions, notations, hypotheses introduced in section 2.

Lemma 1

Let S be any of the supports $S_1, S_2, ..., S_m$, $u \in H^q(S)$, $q \leqslant \rho + 1$, $t \in \mathcal{T}_\rho$ such that $(u - t, p)_S = 0$ for all $p \in \mathcal{T}_\rho$. Then

$$|u - t|_{k,S} \leqslant c(d(S))^{q-k}|u|_{q,S} \quad , \quad 0 \leqslant k \leqslant q.$$

Lemma 1, which supposes H4, has analogue in the literature, see [3], [5], [9]; however, we give below a sketch of the proof.

We restrict ourselves to the case where S is formed by two adjacents triangles; the other cases are treated similarly. Let us consider first the case where $d(S) = 1$. Let Δ_1, Δ_2 be triangles in the (ξ, η) plane with vertices at (0, 0), (0, 1), (1, 0) and (0, 0), (0, − 1), (1, 0) respectively. Let $\Delta = \Delta_1 \cup \Delta_2$. Let in the (x, y) plane S be the domain formed by two triangles T_1 and T_2 having a common side. T_i is the image of Δ_i by the application φ_i :

$$\binom{\xi}{\eta} \mapsto A_i\binom{\xi}{\eta} + b_i = \binom{x}{y} i = 1,2.$$

φ is the application of $\Delta \to S$ such that $\varphi \mid_{\Delta_i} = \varphi_i$. We consider a set $\mathcal{L}$ of domains S of this type satisfying the relations :

$\|A_i\| \leqslant c$ and $\|A_i^{-1}\| \leqslant c$ where c is a generic constant which doesn't depend on $S \in \mathcal{L}$. Let us show that $(u, 1)_S = 0$ implies $|u|_{0,S} \leqslant c\, |u|_{1,S}$ if $u \in H^1(S)$.

Let $v = u \circ \varphi : \Delta \to R$. We verify that v belongs to $H^1(\Delta)$ and if we put

$$\alpha = \frac{(v, 1)_\Delta}{(1, 1)_\Delta},$$

we get $|v - \alpha|_{0,\Delta} \leqslant c|v|_{1,\Delta}$. The hypothesis on the matrices A_i imply that $|u - \alpha|_{0,T_i} \leqslant c|v - \alpha|_{0,\Delta_i}$ and $|v|_{1,\Delta_i} \leqslant c|u|_{1,T_i}$. So

$$|u|_{0,S}^2 \leqslant |u|_{0,S}^2 + |\alpha|_{0,S}^2 = |u - \alpha|_{0,S}^2 = |u - \alpha|_{0,T_1}^2 + |u - \alpha|_{0,T_2}^2$$
$$\leqslant c(|v - \alpha|_{0,\Delta_1}^2 + |v - \alpha|_{0,\Delta_2}^2) = c|v - \alpha|_{0,\Delta}^2 \leqslant |v|_{1,\Delta}^2 \leqslant c|u|_{1,T}^2.$$

This allows us to prove the lemma for $k = 0$. Indeed, let $u \in H^q(S)$, $q \leqslant \rho + 1$, $t \in P_\rho$ such that $(u - t, p)_S = 0$ for all $p \in P_\rho$. Let $\tilde{p}$ the unique polynomial belonging to P_ρ such that $(D^s(u - \tilde{p}), 1)_S = 0$ for $0 \leqslant |s| \leqslant q - 1$; then by applying the preceeding result to $D^s(u - \tilde{p})$ for $|s| = q - 1, q - 2, ..., 0$ we get : $|u - \tilde{p}|_{0,S} \leqslant c|u - \tilde{p}|_{q,S} = c|u|_{q,S}$; then $|u|_{0,S} \leqslant |u - \tilde{p}|_{0,S}$; hence we are done. We obtain the general case from the interpolation formula $|u|_{k,S} \leqslant c(|u|_{0,S} + |u|_{q,S})\ \forall u \in H^q(S)$ and from the fact that $|p|_{k,S} \leqslant c\,|p|_{0,S}$ $\forall p \in P_k$. (These relations can be established by returning to the fundamental domain Δ by the application φ). Indeed, for $\rho = q - 1$, $0 \leqslant k \leqslant q$ we get : $|u - t|_{k,S} \leqslant c(|u - t|_{0,S} + |u - t|_{q,S}) \leqslant c\,|u|_{q,S}$. For $q < \rho + 1$, let $\tilde{t} \in P_{q-1}$ such that $(u - \tilde{t}, p)_S = 0\ \forall p \in P_{q-1}$; then $|u - t|_{k,S} \leqslant |u - \tilde{t}|_{k,S} + |\tilde{t} - t|_{k,S}$; $|u - t|_{k,S} \leqslant c\,|u|_{q,S}$ and $|\tilde{t} - t|_{k,S} \leqslant c\,|\tilde{t} - t|_{0,S} \leqslant c(|u - \tilde{t}|_{0,S} + |u - t|_{0,S}) \leqslant c\,|u|_{q,S}$. We obtain the case $d(S) \neq 1$ by a dilatation.

Lemma 2

Let $T \in D$, $p \in \mathcal{T}_\rho$; then

$$|p|_{k,\infty,T} \leqslant c(\mu(T))^{-\frac{1}{2}}(d(T))^{-k}\,|p|_{0,T} \quad , \quad k = 0, 1, 2, ...$$

Lemma 2, which supposes H4, is an elementary property based on the equivalence of all the norms for a finite dimensional space (see [8]).

Lemma 3

a) If $T_j \subset S_i$ then $d(S_i) \leqslant cd(T_j)$.

b) The number of elements T_j contained in any support S_i is $\leqslant c$.

Lemma 3 is a consequence of H3 and H4.

Proof of theorem 1

Let T be a particular element of D, $Q_1, Q_2, ..., Q_\alpha$ the nodes belonging to T, $\varphi_1, ..., \varphi_\alpha, \gamma_1, ..., \gamma_\alpha, l_1, l_2, ..., l_\alpha, p_1, ..., p_\alpha$ the corresponding basic functions, functionals, orders and polynomials. Let $u \in H^q(\Lambda)$, $q \leqslant \rho + 1$. One has on T :

$$\Pi u = \sum_{i=1}^{\alpha} \gamma_i(p_i)\varphi_i = \sum_{i=1}^{\alpha} \gamma_i(p_1)\varphi_i + \sum_{i=2}^{\alpha} \gamma_i(p_i - p_1)\varphi_i;$$

because of H1, the first sum on the right side is equal to p_1; by lemmas 1 and 3, one gets for $0 \leqslant k \leqslant q$:

$$|\Pi u - u|_{k,T} \leqslant ch^{q-k} |u|_{q,S_1} + \sum_{i=2}^{\alpha} |\gamma_i(p_i - p_1)| \, |\varphi_i|_{k,T} ; \tag{5}$$

in order to estimate the second term of the right side, one remarks that one has by lemmas 1 and 3 :

$$\begin{aligned} |p_i - p_1|_{0,T} &\leqslant |u - p_i|_{0,T} + |u - p_1|_{0,T} \leqslant |u - p_i|_{0,S_i} + |u - p_1|_{0,S_1} \\ &\leqslant c(d(S_i))^q |u|_{q,S_i} + c(d(S_1))^q |u|_{q,S_1} \\ &\leqslant c(d(T))^q(|u|_{q,S_i} + |u|_{q,S_1}); \end{aligned}$$

by lemma 2 and H2, one gets :

$$\begin{aligned} |p_i - p_1|_{l_i,\infty,T} &\leqslant c(\mu(T))^{-\frac{1}{2}}(d(T))^{q-l_i}(|u|_{q,S_i} + |u|_{q,S_1}), \\ |\gamma_i(p_i - p_1)| \, |\varphi_i|_{k,T} &\leqslant ch^{q-k}(|u|_{q,S_i} + |u|_{q,S_1}); \end{aligned}$$

introducing this last inequality in (5) one gets by H3 ($\alpha \leqslant c$) :

$$|\Pi u - u|_{k,T}^2 \leqslant ch^{2(q-k)} \sum_{i=1}^{\alpha} |u|_{q,S_i}^2 ;$$

this relation is valid for any $T_i \in D$; using again H3 and lemma 3, one gets by summing for $i = 1, 2, \ldots, n$ precisely relation (1). Now suppose $q \leqslant \rho$; for any $v \in H^{q+1}(\Lambda)$ one has by (1) :

$$\begin{aligned} |u - \Pi u|_{q,\Lambda} &\leqslant |v - \Pi v|_{q,\Lambda} + |(u - v) - \Pi(u - v)|_{q,\Lambda} \\ &\leqslant ch \, |v|_{q+1,\Lambda} + c \, |u - v|_{q,\Lambda} ; \end{aligned}$$

let $\varepsilon > 0$; one first chooses v such that $c\,|u - v|_{q,\Lambda} < \varepsilon/2$; let $h_0 > 0$ be such that $ch_0 \, |v|_{q+1,\Lambda} < \varepsilon/2$; then for $h < h_0$, $|u - \Pi u|_{q,\Lambda} < \varepsilon$ which proves relation (2).

Lemma 4

Let $T \in D$, τ a side of T, $u \in H_1(T)$; then

$$d(T) \, |u|_{0,\tau}^2 \leqslant c \{ \, |u|_{0,T}^2 + (d(T))^2 \, |u|_{1,T}^2 \, \}.$$

Lemma 4 is a consequence of the trace theorem (see [10]); for a detailed proof, see [1].

Proof of theorem 2

Let $u \in \mathring{H}^1(\Lambda) \cap H^q(\Lambda)$, $1 \leqslant q \leqslant \rho + 1$. Using the notations and arguments of the proof of theorem 1, one remarks (see [6]) that for proving (3) it suffices to show that for $T \in D$ one has

$$|\Pi u - \tilde{\Pi} u|_{k,T}^2 \leqslant ch^{2(q-k)} \sum_{i=1}^{\alpha} |u|_{q,S_i}^2 \quad , \quad 0 \leqslant k \leqslant q. \tag{7}$$

Let $Q_i \in \Gamma$ and $l_i = 0$ for $i = 1, 2, \ldots, \beta$ and for $\beta < i \leqslant \alpha$, $Q_i \notin \Gamma$ or $l_i > 0$; if $\beta = 0$, $\Pi u = \tilde{\Pi} u$ on T so that we can suppose $\beta > 0$; by H2, one has

$$|\Pi u - \tilde{\Pi} u|_{k,T} = \left| \sum_{i=1}^{\beta} \gamma_i(p_i)\varphi_i \right|_{k,T} \leqslant c(d(T))^{-k}(\mu(T))^{\frac{1}{2}} \sum_{i=1}^{\beta} |p_i(Q_i)| ;$$

let $\tau_i \subset \Gamma$ be a side of an element G_i satisfying the relation $Q_i \in \tau_i \subset G_i \subset S_i$, $i = 1, \ldots, \beta$. Since $u \in \overset{\circ}{H}{}^1(\Lambda)$, $|u|_{0,\tau_i} = 0$ and $|p_i|_{0,\tau_i} = |u - p_i|_{0,\tau_i}$; from lemmas 4 and 1, one gets :

$$d(G_i)\,|p_i|^2_{0,\tau_i} \leqslant c\,\{\,|u - p_i|^2_{0,S_i} + (d(S_i))^2\,|u - p_i|^2_{1,S_i}\,\}$$
$$\leqslant c(d(S_i))^{2q}\,|u|^2_{q,S_i}; \qquad (9)$$

a one-dimensional version of lemma 2 allows to write

$$|p_i(Q_i)| \leqslant c(L(\tau_i))^{-\frac{1}{2}}\,|p_i|_{0,\tau_i}, \qquad (10)$$

where $L(\tau)$ is the length of τ; by H4 and lemma 3, (9) and (10) imply :

$$(d(T))^{-2k}\mu(T)(p_i(Q_i))^2 \leqslant c(d(T))^{-2k}\mu(T)(L(\tau_i))^{-1}(d(G_i))^{-1}(d(S_i))^{2q}\,|u|^2_{q,S_i}$$
$$\leqslant ch^{2(q-k)}\,|u|^2_{q,S_i};$$

replacing in (8) and using H3 one obtains (7). It remains to prove (4); let $u \in \overset{\circ}{H}{}^1(\Lambda) \cap H^q(\Lambda)$, $q \leqslant \rho$; by theorem 1, it suffices to verify that $\lim_{h \to 0} |\Pi u - \tilde{\Pi} u|_{q,\Lambda} = 0$; let $J = \{ j : T_j \cap \Gamma \neq \emptyset \}$ and θ be the union of all S_i containing a T_j with $j \in J$; by (7) H3, and lemma 3, one has

$$|\Pi u - \tilde{\Pi} u|^2_{q,\Lambda} = \sum_{j \in J} |\Pi u - \tilde{\Pi} u|_{q,T_j} \leqslant c\,|u|^2_{q,\theta};$$

since $\lim_{h \to 0} \mu(\theta) = 0$, one also has $\lim_{h \to 0} |\Pi u - \tilde{\Pi} u|_{q,\Lambda} = 0$.

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