P. Lesaint

Finite element methods for the transport equation


<http://www.numdam.org/item?id=M2AN_1974__8_2_67_0>

© AFCET, 1974, tous droits réservés.

L’accès aux archives de la revue « Revue française d’automatique, informatique, recherche opérationnelle. Analyse numérique » implique l’accord avec les conditions générales d’utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d’une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.
FINITE ELEMENT METHODS
FOR THE TRANSPORT EQUATION

by P. LESAIN'T (1)

Communiqué par P.-A. RAVIART.

Abstract. — Finite element methods for solving the two dimensional $x-y$ transport equation are considered and some practical numerical schemes are defined. A particular emphasis is put upon the bounds for the errors due to the spatial discretization.

I. INTRODUCTION AND POSITION OF THE PROBLEM

The neutron transport equation in plane $x-y$ geometry corresponds to the following first order problem :

\begin{equation}
A\varphi = \mu \frac{\partial \varphi}{\partial x} + \nu \frac{\partial \varphi}{\partial y} + \sigma \varphi = f \quad \text{for} \quad (x, y) \times (\mu, \nu) \in \Omega \times Q,
\end{equation}

\begin{equation}
\varphi(x, y, \mu, \nu) = 0 \quad \text{for} \quad (x, y) \times (\mu, \nu) \in \Gamma \times Q \quad \text{if} \quad B = \mu n_x + \nu n_y < 0,
\end{equation}

where $\Omega$ is the open square $]0, 1[ \times ]0, 1[$, $\Gamma$ is the boundary of $\Omega$, $n_x$ and $n_y$ denote the components of the outer normal on $\Gamma$, and $Q$ is the unit disk $\mu^2 + \nu^2 \leqslant 1$. The function $\varphi(x, y, \mu, \nu)$ represents the flux of neutrons at the point $(x, y)$ in the angular direction $(\mu, \nu)$. The quantity $\sigma$ denotes the cross section and $f$ takes into account the scattering, fission and inhomogeneous sources. The boundary conditions (1.2) simply mean that the flux of neutrons entering into the system is equal to zero. Let $M$ be defined by $M = (B^2)^{1/2}$. The boundary condition (1.2) can be written as follows

\begin{equation}
(B - M) \varphi = 0 \quad \text{for} \quad (x, y) \times (\mu, \nu) \in \Gamma \times Q.
\end{equation}

(1) Centre d'Etudes de Limeil, Villeneuve-Saint-Georges, France.

Revue Française d'Automatique, Informatique et Recherche Opérationnelle n° août 1974, R-2.
Let \((\cdot, \cdot)_{L^2(\Omega \times Q)}\) denote the usual inner product in the space \(L^2(\Omega \times Q)\) and let \(\|\cdot\|_{L^2(\Omega \times Q)}\) be the corresponding norm. Problem (1.1), (1.3) can be considered as a symmetric positive Friedrichs' system ([5]) and one has the following result ([5], [16]):

**Theorem 1.1**: Assume that \(f \in L^2(\Omega \times Q)\) and that \(\sigma \in L^\infty(\Omega \times Q)\). Then problem (1.1), (1.3) has a unique strong solution \(\varphi\), in the following sense:

There exists a sequence \(\{\varphi_j\}\) with \(\varphi_j \in H^1(\Omega \times Q)\) and such that \(\varphi_j\) satisfies the boundary conditions (1.3), with the property that

\[
\lim_{j \to +\infty} \left\{ \|A\varphi_j - f\|_{L^2(\Omega \times Q)} + \|\varphi_j - \varphi\|_{L^2(\Omega \times Q)} \right\} = 0.
\]

In what follows, we always assume that problem (1.1), (1.2) has a unique strong smooth solution \(\varphi\) (at least \(\varphi \in H^1(\Omega \times Q)\)) and that \(f \in C^0(\Omega \times Q)\). To solve problem (1.1), (1.3) by a Galerkin type method, we consider as in [11] the following formulation: If \(\varphi \in H^1(\Omega \times Q)\), one may write

\[
(A\varphi, \psi)_{L^2(\Omega \times Q)} - \left( \frac{B - M}{2} \varphi, \psi \right)_{L^2(\Omega \times Q)} = (f, \psi)_{L^2(\Omega \times Q)},
\]

for all \(\psi \in H^1(\Omega \times Q)\).

As is usually done, we shall consider separately the discretizations in the angular variables \((\mu, \nu)\) and in the spatial variables \((x, y)\).

**Angular discretization.** Let us consider a triangulation \(\Omega\) of the domain \(Q\) in triangles \(T_l\), \(1 \leq l \leq L\), the boundary of \(Q\) being approximated by a polygonal line. Let \(Q_{\mu}\) be the reunion of all triangles \(T_l\), \(1 \leq l \leq L\). We define the following geometrical parameter for each triangle \(T_l\):

\[
h(T_l) = \text{diameter of } T_l,
\]

\[
\rho(T_l) = \text{diameter of the inscribed circle in } T_l.
\]

We assume that the triangulation \(\Omega\) is a *regular* family, i.e. there exists a constant \(\alpha > 0\) independent of the triangulation such that

\[
(1.5) \quad \frac{h(T_l)}{\rho(T_l)} \leq \alpha, \quad \text{for } 1 \leq l \leq L.
\]

Let \(\mathcal{U}_{\mu}\) denote the space of functions whose restriction to each triangle is a polynomial of degree \(\leq k\) in \(\mu\) and \(\nu\), the dimension \(N\) of the space \(\mathcal{U}_{\mu}\) being then equal to \(\frac{k(k + 1)}{2} L\). We shall consider the following problem:

we want to find \(\varphi_{\mu} \in H^1(\Omega) \times \mathcal{U}_{\mu}\) which satisfies:

\[
(1.6) \quad (A\varphi_{\mu}, \psi)_{L^2(\Omega \times Q_{\mu})} - \left( \frac{B - M}{2} \varphi_{\mu}, \psi \right)_{L^2(\Omega \times Q_{\mu})} = (f, \psi)_{L^2(\Omega \times Q_{\mu})},
\]

for all \(\psi \in H^1(\Omega) \times \mathcal{U}_{\mu}\).
Let \( \{ \psi_m^\mu(\mu, \nu) \} \), \( 1 \leq m \leq \frac{k(k + 1)}{2} \) and \( 1 \leq l \leq L \), be a basis of the space \( \mathcal{Y}_\mu \). The functions \( \varphi_\mu \) and \( \psi \) can be written as follows:

\[
\varphi_\mu = \sum_{i,m} u_i^\mu(x, y) \psi_i^\mu(\mu, \nu), \quad \psi = \sum_{i,m} v_i^\mu(x, y) \psi_i^\mu(\mu, \nu)
\]

What we need to do now is to replace \( \psi \) in expression (1.6) by all the functions \( v_i^\mu(x, y) \psi_i^\mu(\mu, \nu), 1 \leq m \leq \frac{k(k + 1)}{2} \), \( 1 \leq l \leq L \) and to calculate the following integrals:

\[
\int_{T_l} \mu \psi_i^\mu \psi_i^\mu d\mu d\nu, \quad \int_{T_l} v_i^\mu \psi_i^\mu d\mu d\nu, \quad \int_{T_l} \psi_i^\mu \psi_i^\mu d\mu d\nu \text{ and }
\int_{T_l} f^\mu d\mu d\nu, \quad \text{for } 1 \leq m, n \leq \frac{k(k + 1)}{2}, \quad 1 \leq l \leq L.
\]

In what follows, we shall restrict our attention to the case where \( \mathcal{Y}_\mu \) is the space of functions which are constant on each triangle. Let \( \varphi_i \) (resp. \( \mu_i \), \( \nu_i \) and \( f_i \)) denote the value of the flux \( \varphi_\mu \) (resp. \( \mu \), \( \nu \) and \( f \)) at the centroid of \( T_l \). And let us consider the following quadrature formula:

\[(1.7) \quad \int_{T_l} g(\mu, \nu) d\mu d\nu \sim \text{area}(T_l)g_i.\]

If we use formula (1.7) to calculate the integrals arising in expression (1.6), we get the following family of problems: to find \( \varphi_i \in H^1(\Omega) \) such that:

\[(1.8) \quad \int_\Omega \left( \mu \frac{\partial \varphi_i}{\partial x} + \nu \frac{\partial \varphi_i}{\partial y} + \sigma \varphi_i - f_i \right) v \, dx \, dy - \int_\Gamma \frac{B_l - M_l}{2} \varphi_i \nu \, d\gamma = 0,
\]

for all \( v \in H^1(\Omega) \), where \( B_l = \mu_i n_x + \nu_i n_y \) and \( M_l = (B_i^2)^{1/2} \), for \( 1 \leq l \leq L \).

Let \( \| \cdot \|_\mu \) be the discrete norm defined by

\[(1.9) \quad \| \varphi \|_\mu^2 = \sum_{l=1}^L \text{area}(T_l) \int_\Omega (\varphi(x, y, \mu_l, \nu_l))^2 \, dx \, dy.
\]

We have the following classical error estimate:

**Theorem 1.2**: We assume that problem (1.1), (1.3) has a smooth strong solution \( \varphi \). Let \( \varphi_\mu \) be the solution of problem (1.6), the integrals being calculated using formula (1.7). Then we have:

\[(1.10) \quad \| \varphi - \varphi_\mu \|_\mu = o(\Delta \mu^2),\]

where \( \Delta \mu \) denotes the supremum of the diameters of the triangles \( T_l \), \( 1 \leq l \leq L \).

\( \text{n}^o \text{août 1974, R-2.} \)
**Remark 1.1**: The use of polynomials of degree zero leads us to a discrete ordinate method [10]. The method described above can be successfully applied when we use polynomials of higher degree on triangles [15] or on quadrilateral elements. We can then expect a better accuracy for the numerical results and we obtain some coupling between the angular directions.

**Remark 1.2**: The method described above is an example of the application of a discontinuous method ([13], [17]) to angular variables.

Let \((\cdot, \cdot)_{L^2(\Omega)}\) denote the usual inner product in \(L^2(\Omega)\) and let \(|\cdot|_{L^2(\Omega)}\) denote the corresponding norm (for \(L^2(\Gamma)\), we shall use the same notations, with \(\Omega\) replaced by \(\Gamma\)). We have now to consider the following problem for the spatial variables, the angular variables \(\mu\) and \(\nu\) being considered as parameters: we want to find \(u\) such that:

\[
(1.11) \quad Au = \mu \frac{\partial u}{\partial x} + \nu \frac{\partial u}{\partial y} + \sigma u = f \quad \text{in} \quad \Omega,
\]

\[
(1.12) \quad (B - M)u = 0 \quad \text{on} \quad \Gamma.
\]

When \(u \in H^1(\Omega)\), this problem can be written as follows [11]:

\[
(1.13) \quad (Au, v)_{L^2(\Omega)} - \left( \frac{B - M}{2} u, v \right)_{L^2(\Gamma)} = (f, v)_{L^2(\Omega)} \quad \text{for all} \quad v \in H^1(\Omega).\]

We shall use the following result:

**Lemma 1.1**: For all \(v \in H^1(\Omega)\), we have:

\[
(Au, v)_{L^2(\Omega)} - \left( \frac{B - M}{2} u, v \right)_{L^2(\Gamma)} \geq \sigma |v|_{L^2(\Omega)}^2 + \left( \frac{M}{2} \right)^{1/2} |v|_{L^2(\Gamma)}^2.
\]

In what follows, we always assume that \(\mu\) and \(\nu\) are positive. If we want to consider the case where \(\mu\) and (or) \(\nu\) are negative, we just exchange \(x\) in \(-x\) and (or) \(y\) in \(-y\). Define \(\Gamma_i, 0 \leq i \leq 3\), by:

\[
\Gamma_0 = \Gamma \cap \{ x = 0 \}, \quad \Gamma_1 = \Gamma \cap \{ y = 0 \}, \quad \Gamma_2 = \Gamma \cap \{ x = 1 \}
\]

and \(\Gamma_3 = \Gamma \cap \{ y = 1 \}\)

To solve problem (1.11), (1.12), we shall use a finite element method. This approach of the problem has already been considered by several authors ([6], [14], [15], [17], ...) and gives good results [8]. To define numerical schemes, we shall use formulation (1.13) along with finite dimensional spaces of test functions constructed with four nodes isoparametric quadrilateral elements [19] whose diameter are smaller or equal to \(h\) (§ II). Existence of the approximate solutions can be shown (§ III) by using results similar to lemma 1.1. Then, generalizing results of [4], we get an error of order \(h^2\) when the elements are equal rectangles (§ IV). When we use numerical quadrature formulas, we
solve a local problem on each element, which is precisely a collocation method. Thus we get quasi-explicit numerical schemes (§ II) which are conditionally stable and accurate to the order $h^2$ if the quadrilaterals are not too distorted, and which are generalizations of classical schemes (D.S.N. [7]).

II. NUMERICAL SCHEMES FOR SOLVING PROBLEM (1.13)

Consider a triangulation $\mathcal{T}_h$ of $\Omega$ made up of convex quadrilateral elements $Z$, such that $UK = \bar{\Omega}$, and such that from any vertex belonging to the interior of $\Omega$ start four edges (such a triangulation may arise from the deformation of a regular grid). With each element $K \in \mathcal{T}_h$, we associate the geometrical parameters:

$h(K) = \text{diameter of } K,$

$\rho(K) = \sup \{ \text{diameters of the spheres contained in } K \}$

$\theta_i(K) = \text{angle of the quadrilateral } K, \text{ for } 1 \leq i \leq 4.$

Let $h$ be defined by $h = \sup \{ h(K); K \in \mathcal{T}_h \}$.

We assume that the triangulation $\mathcal{T}_h$ is a regular family of elements [3], in the following sense: we have:

$h(K) \leq \beta \rho(K), \text{ for all } K \in \mathcal{T}_h,$

$\max \{ |\cos \theta_i(K)|, 1 \leq i \leq 4 \} \leq \gamma, \text{ for all } K \in \mathcal{T}_h$

where $\beta$ and $\gamma$ are two constants independent of the triangulation and such that $\beta > 0$ and $0 < \gamma < 1$.

Let $I$ (resp. $J$) be the number of quadrilaterals with an edge belonging to $\Gamma_1$ (resp. $\Gamma_0$). The number of quadrilaterals included in $\bar{\Omega}$ is then equal to $IJ$ and the number of vertices in $\bar{\Omega}$ is equal to $(I + 1)(J + 1)$. We shall numerotate the quadrilaterals from the left to the right and from the bottom to the top so that $\bar{\Omega} = \bigcup K_{i,j}, 0 \leq i \leq I - 1, 0 \leq j \leq J - 1$. Consider now the quadrilateral $K$ with vertices $A_i = (x_i, y_i), 1 \leq i \leq 4$ (fig. 2.1). There exists a unique invertible bilinear mapping $F_K$ such that $K$ is the image by $F_K$ of the square $\hat{K} = [-1, +1] \times [-1, +1]$. This mapping is defined as follows:

\begin{align*}
(2.1) \quad x &= \frac{(1 + \xi)(1 + \eta)}{4} x_1 + \frac{(1 - \xi)(1 + \eta)}{4} x_2 \\
&\quad + \frac{(1 - \xi)(1 - \eta)}{4} x_3 + \frac{(1 + \xi)(1 - \eta)}{4} x_4 ;
\end{align*}

\begin{align*}
(2.2) \quad y &= \frac{(1 + \xi)(1 + \eta)}{4} y_1 + \frac{(1 - \xi)(1 + \eta)}{4} y_2 \\
&\quad + \frac{(1 - \xi)(1 - \eta)}{4} y_3 + \frac{(1 + \xi)(1 - \eta)}{4} y_4 .
\end{align*}
To construct a finite dimensional space in which we shall look for an approximate solution $u_h$, we shall use either conforming or non-conforming elements.

Figure 2.1
The isoparametric mapping $F_K$

**Conforming case, definition of the space $V_h$**

Let $\hat{Q}(1)$ be the space of polynomials defined by $\hat{Q}(1) = \{ \hat{q} : K \to R; \hat{q} = a + b\xi + c\eta + d\xi\eta \}$. There exists a unique polynomial of $\hat{Q}(1)$ which takes given values at the vertices $A_i$, $1 \leq i \leq 4$. The space $P_K$ of the shape functions over the element $K$ is defined by:

\begin{equation}
P_K = \{ p : p = \hat{p}F_K^{-1} , \quad \hat{p} \in \hat{Q}(1) \}
\end{equation}

We shall define $V_h$ as the space of functions defined and continuous over $\Omega$ and whose restriction to each element $K$ belongs to $P_K$. The dimension of $V_h$ is equal to $(I + 1)(J + 1)$ and the degrees of freedom of $V_h$ can be chosen as the values of the functions of $V_h$ at the vertices belonging to $\Gamma_0 \cup \Gamma_1$ and at the centroids of the quadrilateral elements $K$. Let $v_h$ be a function of $V_h$ taking the values $v_i$ at the vertices $A_i$, $1 \leq i \leq 4$ of quadrilateral $K$; then the restriction of $v_h$ to the quadrilateral $K$ can be expressed in local coordinates $\xi$, $\eta$ as:

\begin{equation}
\hat{v}_h(\xi, \eta) = \frac{(1 + \xi)(1 + \eta)}{4} v_1 + \frac{(1 - \xi)(1 + \eta)}{4} v_2 + \frac{(1 - \xi)(1 - \eta)}{4} v_3 + \frac{(1 + \xi)(1 - \eta)}{4} v_4
\end{equation}

Given a function $u$ defined and continuous over $\Omega$, its interpolate $r_hu$ will be the unique function of $V_h$ taking the same values as $u$ at the vertices of the quadrilaterals $K \in C_h$. Let $V_h^0$ be the subspace of $V_h$ spanned by the functions of $V_h$ which are equal to zero at the vertices belonging to $\Gamma$. Any function $v_h$ of $V_h$ can be written as:

\begin{equation}
v_h = v_h^0 + v_h^b , \quad v_h^0 \in V_h^0 \text{ and } v_h^b = v_h \text{ at all the vertices}
\end{equation}
belonging to the interior of Ω. Such a decomposition is unique and the function \( v_h \) depends only on the values of \( v_h \) at the vertices belonging to \( Γ \).

**Non-conforming case, definition of the space \( W_h \):** Let \( \hat{P}(1) \) be the space of polynomials defined by \( \hat{P}(1) = \{ \hat{p} : \hat{K} \rightarrow R; \hat{p} = a + b\xi + c\eta \} \). There exists a unique polynomial of \( \hat{P}(1) \) which takes given values \( \hat{p}_i \) at the mid-points \( a_i, 1 \leq i \leq 4 \), of the edges of \( K \) if we have:

\[
\hat{p}_1 + \hat{p}_3 = \hat{p}_2 + \hat{p}_4
\]

The space \( P_K \) of the shape functions over the element \( K \) is defined by:

\[
P_K = \{ p ; p = \hat{p}_0 F_K^{-1}, \hat{p} \in \hat{P}(1) \}
\]

We shall define \( W_h \) as the space of functions continuous at the mid-points of the edges of the quadrilaterals and whose restriction to each quadrilateral \( K \) belongs to \( P_K \). The dimension of \( W_h \) is equal to \( IJ + I + J \) and the degrees of freedom of \( W_h \) can be chosen as the values of the functions of \( W_h \) at the mid-points of the edges included in \( Γ_0 \cup Γ_1 \) and at the centroids of the elements \( K \in G_h \). The function \( w_h \) of \( W_h \) taking the values \( w_i \) at the mid-points \( a_i \), \( 1 \leq i \leq 4 \), of the edges of \( K \) and the value \( w_0 \) at the centroid of \( K \), can be expressed in local coordinates \( \xi, \eta \) as:

\[
(2.7) \quad \hat{w}_h(\xi, \eta) = w_0 + \frac{w_4 - w_2}{2} \xi + \frac{w_4 - w_3}{2} \eta,
\]

with \( 2w_0 = w_1 + w_3 = w_2 + w_4 \)

Given a function \( u \) defined and continuous over \( \tilde{Ω} \), its interpolate \( r_h u \) will be the unique function of \( W_h \) such that the value of \( r_h u \) at the mid-point of any edge of the quadrilateral is equal to the average of the values of \( u \) at the endpoints of this edge. It is still possible to write any function \( w_h \) of \( W_h \) as \( w_h = w_h^0 + w_h^\circ \), with \( w_h^\circ = w_h^0 \) at the mid-points of the edges which have no point in common with \( Γ \) and \( w_h^\circ = 0 \) at the mid-points of the edges included in \( Γ \). Such a decomposition is not unique and the function \( w_h^\circ \) does not depend only on the values of \( w_h \) on the boundary \( Γ \).

We shall give now some numerical quadrature formulas, which will be useful to evaluate the integrals arising in the inner products. Consider the following formula on the square \( \hat{K} \):

\[
(2.8) \quad \int_{\hat{K}} \hat{f}(\xi, \eta) \, d\xi \, d\eta \sim 4\hat{f}(0, 0),
\]

which induces on the quadrilateral \( K \) the following formula:

\[
(2.9) \quad \int_K f(x, y) \, dx \, dy \sim \text{area}(K) \cdot f(G_K),
\]

where \( G_K \) is the centroid of the quadrilateral \( K \).
We define the two following formulas on any edge \( A_i A_j \):

\[
(2.10) \quad \int_{A_i}^{A_j} f \, dt \sim \frac{A_i A_j}{2} \left( f(A_i) + f(A_j) \right)
\]

\[
(2.11) \quad \int_{A_i}^{A_j} f \, dt \sim A_i A_j f(a_{ij})
\]

where \( t \) is a curvilinear abscissa along \( A_i A_j \), and \( a_{ij} \) is the mid-point of \( A_i A_j \). With those formulas, we can define discrete inner products between functions of \( V_h \) or \( W_h \) as follows: let \( v_h, w_h \in V_h \) (or \( W_h \)), we define \((v_h, w_h)_h\) by:

\[
(2.12) \quad (v_h, w_h)_h = \sum_{K \in T_h} \text{area}(K)v_h(G_K)w_h(G_K).
\]

Let \(| \cdot |_h\) denote the corresponding semi-norm.

Let \( s \) denote any edge belonging to \( \Gamma \), for any \( v_h, w_h \in V_h \), we define \( \langle v_h, w_h \rangle_h \) by:

\[
(2.13) \quad \langle v_h, w_h \rangle_h = \sum_{s \in \Gamma_0 \cup \Gamma_1} \frac{A_s B_s}{2} \left( (v_h w_h)(A_s) + (v_h w_h)(B_s) \right) + \sum_{s \in \Gamma_2 \cup \Gamma_3} (A_s B_s)(v_h w_h)(G_s)
\]

where \( A_s \) and \( B_s \) denote the end-point of \( s \) and where \( G_s \) denote the mid-point of \( s \). The corresponding norm will be denoted by \( \langle \cdot, \cdot \rangle_h \). For any \( v_h, w_h \in V_h \) (or \( W_h \)) we define \([\cdot, \cdot]_h\) by:

\[
[v_h, w_h]_h = \sum_{s \in \Gamma} (A_s B_s)(v_h w_h)(G_s).
\]

The corresponding semi-norm will be denoted by \([\cdot]_h\).

Let \( (\cdot, \cdot)_{L^2(\Omega)} \) (resp \( (\cdot, \cdot)_{L^2(\Gamma)} \)) denote the inner product in \( V_h \) or \( W_h \) induced by the inner product in \( L^2(\Omega) \) (resp \( L^2(\Gamma) \)). In the non-conforming case, we shall use the following notation:

\[
(u_h, v_h)^* = \sum_{K \in T_h} (u_h, v_h)_{L^2(K)}, \quad \text{for} \quad u_h, v_h \in W_h.
\]

**Remark 2.1**: We have the following almost classical inequalities, where \( c \) is a constant independent of \( h \):

\[
[v_h]_h \leq \langle v_h \rangle_h \leq c |v_h|_{L^2(\Gamma)} \leq ch^{-1} \langle v_h \rangle_h \quad \text{for all} \quad v_h \in V_h,
\]

\[
[w_h]_h \leq c |w_h|_{L^2(\Gamma)} \quad \text{for all} \quad w_h \in W_h,
\]

\[
|v_h|_h \leq |v_h|_{L^2(\Omega)} \leq ch^{-2} |v_h|_h \quad \text{for all} \quad v_h \in V_h.
\]
Now we can define the following problems:

**Scheme 1**: To find $u_h \in V_h$ such that:

$$(Au_h, v_h)_{L^2(\Omega)} - \left(\frac{B-M}{2}u_h, v_h\right)_{L^2(\Gamma)} = (f, v_h)_{L^2(\Omega)} \text{ for all } v_h \in V_h.$$ 

**Scheme 2**: To find $u_h \in W_h$ such that:

$$(Au_h, w_h) = \left(\frac{B-M}{2}u_h, w_h\right)_{L^2(\Omega)} = (f, w_h)_{L^2(\Omega)} \text{ for all } w_h \in W_h.$$ 

**Scheme 3**: To find $V_h \in V_h$ such that:

$$(Au_h, v_h) - \left(\frac{B-M}{2}u_h, v_h\right)_{h} = (f, v_h)_h \text{ for all } v_h \in V_h.$$ 

**Scheme 4**: To find $u_h \in W_h$ such that:

$$(Au_h, w_h)_h = \left(\frac{B-M}{2}u_h, w_h\right)_{h} = (f, w_h)_h \text{ for all } w_h \in W_h.$$ 

When one wants to give a numerical solution for scheme 1, one has to invert a nine-diagonal matrix with a total bandwidth equal to $2I + 1$ (resp. $2J + 1$) if one numerates the vertices from the left to the right and then from the bottom to the top (resp. from the bottom to the top and then from the left to the right). The situation is still more complicated for scheme 2. But we shall see that for schemes 3 and 4, one can get a quasi-explicit resolution.

**Lemma 2.1**: Scheme 3 can be written as follows, on each quadrilateral $K$ with vertices $A_i(x_i, y_i)$, where $u_i = u_{\delta}(A_i)$, $1 \leq i \leq 4$:

$$\begin{align*}
(2.14) \quad & (u_1 - u_3)(\mu(y_2 - y_4) - \nu(x_2 - x_4)) \\
& + (u_2 - u_4)(-\mu(y_1 - y_3) + \nu(x_1 - x_3)) + ((y_1 - y_3)(x_4 - x_2) \\
& + (y_2 - y_4)(x_1 - x_3))(\sigma(G_K) \frac{u_1 + u_2 + u_3 + u_4}{4} - f(G_K)) = 0
\end{align*}$$

where $G_K$ is the centroid of $K$, $u_h = 0$ at the vertices belonging to $\Gamma_0 \cup \Gamma_1$.

**Proof**: According to the definition of scheme 3, one may write:

$$\sum_{K \in \mathcal{G}_h} \text{area}(K)(Au_h \cdot v_h)(G_K) - \sum_{s \in \Gamma_0 \cup \Gamma_1} A_s B_s \left\{ \left(\frac{B-M}{2}u_h \cdot v_h\right)(A_s) \\
+ \left((B-M)u_h \cdot v_h\right)(B_s) \right\} = \sum_{K \in \mathcal{G}_h} \text{area}(K)(f \cdot v_h)(G_K).$$

where $A_s$ and $B_s$ are the end-points of the edge $s$ and where $A_s B_s$ is the distance between $A_s$ and $B_s$.

n° août 1974, R-2.
The values of $v_h$ at the centroid of the elements $K$ and at the vertices belonging to $\Gamma_0 \cup \Gamma_1$ are degrees of freedom of the space $V_h$. So one may write :

$$ (Au_h)(G_K) = f(G_K) \quad \text{for all} \quad K \in \mathcal{C}_h,$$

$u_h = 0$ at the vertices belonging to $\Gamma_0 \cup \Gamma_1$.

Now let $J_F$ be the jacobian of the isoparametric transformation $F_K$ defined by (2.1) and (2.2). One has the following classical formulas :

$$ J_F(\xi, \eta) \frac{\partial u_h}{\partial x}(\xi, \eta) = \frac{\partial u_h}{\partial \xi} \frac{\partial y}{\partial \eta} - \frac{\partial u_h}{\partial \eta} \frac{\partial y}{\partial \xi}, $$

$$ J_F(\xi, \eta) \frac{\partial u_h}{\partial y}(\xi, \eta) = - \frac{\partial u_h}{\partial \xi} \frac{\partial x}{\partial \eta} + \frac{\partial u_h}{\partial \eta} \frac{\partial x}{\partial \xi}. $$

Combining equalities (2.15), (2.16) and (2.17) one easily gets equality (2.14).

In the same manner, one can prove the following result :

Lemma 2.2 : Scheme 4 can be written as follows, on each quadrilateral $K$ with vertices $A_i(x_i, y_i)$, where $u_i = u_h(a_i)$, $1 \leq i \leq 4$ :

$$ (u_4 - u_2)(\mu(y_1 + y_2 - y_3 - y_4) - \nu(x_1 + x_2 - x_3 - x_4)) + $$

$$ + (u_1 - u_3)(- \mu(y_1 - y_2 + y_3 + y_4) + \nu(x_1 - x_2 + x_3 + x_4)) + $$

$$ + ((y_1 - y_3)(x_4 - x_2) + (y_2 - y_4)(x_1 - x_3)) $$

$$ + (\sigma(G_K)u(G_K) - f(G_K)) = 0 $$

(2.19)

$$ 2u(G_K) = u_1 + u_3 = u_2 + u_4, $$

$u_h = 0$ at the mid-points of the edges included in $\Gamma_0 \cup \Gamma_1$.

Remark 2.2. : Both schemes 3 and 4 are quasi-explicit if one solves the system by starting from the elements adjacent to $\Gamma_0 \cap \Gamma_1$.

Remark 2.3. : In the conforming case (scheme 3), if one wants to calculate the value $u_1$ as a function of $u_2$, $u_3$ and $u_4$, or in the non-conforming case (scheme 4), if one wants to calculate $u_1$ and $u_4$ as a function of $u_2$ and $u_3$, a practical necessary condition of resolution seems to be the following :

$$ \mu(y_2 - y_4) - \nu(x_2 - x_4) > 0. $$

This condition means that the characteristic direction $(\mu, \nu)$ makes a positive angle with the diagonal $A_4A_2$ of the quadrilateral $K$. We shall see later on that this condition is not sufficient for stability.

Remark 2.4. : Let us assume now that the domain $\Omega$ is divided into equal rectangles with edges parallel to the axes and equal respectively to $\Delta x = \frac{1}{J}$ and $\Delta y = \frac{1}{J}$. In the conforming case, we shall write $u_{i,j}$ for $u_h(x_i, y_j)$, for $0 \leq i \leq I$, $0 \leq j \leq J$. 

Revue Française d'Automatique, Informatique et Recherche Opérationnelle
0 ≤ j ≤ J, and in both cases, we shall write $u_{i+1/2,j}$ for $u_h\left(\frac{x_i + x_{i+1}}{2}, y_j\right)$, for $0 ≤ i ≤ I - 1$ and $0 ≤ j ≤ J$, and $u_{i,j+1/2}$ for $u_h\left(x_i, \frac{y_j + y_{j+1}}{2}\right)$, $0 ≤ i ≤ I$ and $0 ≤ j ≤ J - 1$. Scheme 3 can then be written as follows:

$$
\left[ \frac{u_{i+1/2,j+1} + u_{i+1,j}}{2 \Delta x} \right] + \left[ \frac{v_{i+1/2,j+1} + v_{i,j}}{2 \Delta y} \right] + \frac{\sigma_{i+1/2,j+1} u_{i+1,j+1} + u_{i+1,j} + u_{i,j+1} + u_{i,j}}{4} = f_{i+1/2,j+1/2}
$$

for $0 ≤ i ≤ I - 1$, $0 ≤ j ≤ J - 1$,

$$
u_{0,j} = u_{i,0} = 0 \text{ for } 0 ≤ i ≤ I, 0 ≤ j ≤ J.
$$

Scheme 4 can be written as follows (classical D.S.N. scheme [7]):

$$
\mu \frac{u_{i+1/2,j+1/2} - u_{i,j+1/2}}{\Delta x} + \nu \frac{u_{i+1/2,j+1} - u_{i+1/2,j}}{\Delta y} + \sigma_{i+1/2,j+1/2} u_{i+1/2,j+1/2} = f_{i+1/2,j+1/2},
$$

$$
2u_{i+1/2,j+1/2} = u_{i+1,j+1/2} + u_{i,j+1/2} = u_{i+1/2,j+1} + u_{i+1/2,j},
$$

for $0 ≤ i ≤ I - 1$ and $0 ≤ j ≤ J - 1$;

$$
u_{0,j+1/2} = u_{i+1/2,0} \text{ for } 0 ≤ j ≤ J - 1, 0 ≤ i ≤ I - 1.
$$

III. EXISTENCE AND STABILITY OF THE APPROXIMATE SOLUTIONS

We can already give the following result for scheme 1.

**Lemma 3.1:** For any $v_h ∈ V_h$, we have:

$$
(A v_h, v_h)_{L^2(Ω)} - \left( \frac{B - M}{2} v_h, v_h \right)_{L^2(Γ)} \geq \sigma |v_h|_{L^2(Ω)}^2 + \left( \frac{M}{2} \right)^{1/2} |v_h|_{L^2(Ω)}^2
$$

n° août 1974, R-2.
Scheme 1 has a unique solution $u_h \in V_h$ and satisfying:

\[
(3.2) \quad |u_h|^2_{L^2(\Omega)} + \left( \frac{M}{2} \right)^{1/2} |u_h|^2_{L^2(\Gamma)} \leq c |f|_{L^2(\Omega)},
\]

where $c$ is a constant independent of $h$.

**Proof**: Since $V_h$ is a subspace of $H^1(\Omega)$, one can apply lemma 1.1 for any $v_h \in V_h$ and we get inequality (3.1). Inequality (3.2) is a consequence of inequality (3.1).

For the other schemes, we shall need the following hypotheses:

**Hypothesis 3.1**: The distance $z(K)$ between the mid-points of the diagonals of quadrilateral $K$ satisfies the following inequality $z(K) \leq \lambda h(K)^2$ for all $K \in \mathcal{T}_h$, where $\lambda$ is a constant independent of the triangulation.

**Hypothesis 3.2**: Let $A_i(x_i, y_i)$, $1 \leq i \leq 4$, be the vertices of quadrilateral $K$; we have:

\[
|\mu(y_1 - y_2 + y_3 - y_4) - \nu(x_1 - x_2 + x_3 - x_4)| \leq c_0 h(K) |\mu(y_1 - y_4 + y_2 - y_3) - \nu(x_1 - x_4 + x_2 - x_3)|,
\]

\[
|\mu(y_1 - y_2 + y_3 - y_4) - \nu(x_1 - x_2 + x_3 - x_4)| \leq c_0 h(K) |\mu(y_1 - y_2 + y_4 - y_3) - \nu(x_1 - x_2 + x_4 - x_3)|,
\]

for all $v \in \mathcal{T}_h$, where $c_0$ is a constant independent of $h$.

**Remark 3.1**: Let $h_0$ be a real positive number such that $0 < h_0 < \frac{1}{3c_0}$.

Hypothesis 3.2 implies that, if $h \leq h_0$, the angles done by the characteristic direction $(\mu, \nu)$ with two opposite edges $A_iA_{i+1}$ and $A_{i+3}A_{i+2}$, $i = 1, 2$, (with $A_5 = A_1$) have either the same sign or they are both equal to zero. When these angles are not equal to zero, any quadrilateral $K$ has always two adjacent edges through which the flux of neutrons enters into the quadrilateral and two adjacent edges (opposite to the others) through which the flux goes outside the quadrilateral.

We have the following result for scheme 2:

**Lemma 3.2**: We assume that hypothesis 3.1 is satisfied and that $\sigma$ is greater than a constant depending only on $\nu$, $\nu$ and $\lambda$ (in the case where this last condition is not satisfied, one can always consider a new function $v$ defined by $v = u \exp \left(-D \left(\frac{x}{\mu} + \frac{y}{\nu}\right)\right)$ where $D$ is a positive constant suitably chosen; we then have:

\[
(3.3) \quad (Av_h, v_h)_{L^2(\Omega)} - \left(\frac{B - M}{2} v_h, v_h\right)_{L^2(\Gamma)} \geq c |v_h|^2_{L^2(\Omega)} + \left|(M)^{1/2} v_h\right|^2_{L^2(\Gamma \cup \Gamma_1)} + [M^{1/2} v_h]^2,
\]

for all $v_h \in W_h$, where $c$ is a constant independent of $h$. 

*Revue Française d'Automatique, Informatique et Recherche Opérationnelle*
Scheme 2 has a unique solution \( u_h \in W_h \) and satisfying:

\[
|u_h|_{L^2(\Omega)}^2 + [(M)^{1/2}u_h]_{L^2(\Gamma_0 U \Gamma_1)}^2 + [[(M)^{1/2}u_h]]_h^2 \leq c |f|_{L^2(\Omega)}^2.
\]

**Proof:** Let \( K \) be the quadrilateral with vertices \( A_i(x_i, y_i), 1 \leq i \leq 4 \). We have:

\[
\int_K \left( \mu \frac{\partial v_h}{\partial x} + \nu \frac{\partial v_h}{\partial y} \right) v_h \, dx \, dy = \frac{1}{2} \sum_{i=1}^4 \left( \mu(y_{i+1} - y_i) - \nu(x_{i+1} - x_i) \right)
+ \frac{1}{12} (v_1^2 + v_3^2 - v_2^2 - v_4^2)(\mu(y_1 - y_2 + y_3 - y_4) - \nu(x_1 - x_2 + x_3 - x_4)),
\]

with \( x_5 = x_1 \) and \( y_5 = y_1 \).

It is easy to see that:

\[
\int_K \sigma v_h^2 \, dx \, dy \geq c(h(K))^2(v_1^2 + v_2^2 + v_3^2 + v_4^2)
\]

where the constant \( c \) is independent of \( h \).

Assume now that the quadrilateral \( K \) has an edge (for example \( A_3A_4 \)) belonging to \( \Gamma_1 \). We have:

\[
-\int_{A_3A_4} \frac{B - M}{2} v_h^2 \, dx = \nu(x_4 - x_3) \left( v_3^2 + \frac{1}{12} (v_4 - v_2)^2 \right)
\]

We can get the same type of equality for the edges belonging to \( \Gamma_0 \). Combining equalities (3.5) and (3.7), inequality (3.6) and hypothesis 3.1, we easily get inequality (3.3).

We shall now give some results for schemes 3 and 4.

**Lemma 3.3:** Let us assume that hypothesis 3.2 holds and that \( h \leq h_0 \). Then one can always numerotate the vertices \( A_i, 1 \leq i \leq 4 \) of any quadrilateral \( K \in C_h \) in such a way that we have:

\[\mu(y_1 - y_4) - \nu(x_1 - x_4) > 0, \mu(y_2 - y_1) - \nu(x_2 - x_1) > 0,\]

\[\mu(y_3 - y_2) - \nu(x_3 - x_2) < 0 \text{ and } \mu(y_4 - y_3) - \nu(x_4 - x_3) < 0.\]
Then, for scheme 3, we have:

\[
\text{area}(K) \left( \mu \frac{\partial v_h}{\partial x} + \nu \frac{\partial v_h}{\partial y} \right) v_h(G_K) \\
\geq \frac{1 - 2c_0 h(K)}{1 - c_0 h(K)} \left( \mu \frac{y_1 - y_4}{2} - \nu \frac{x_1 - x_4}{2} \right) \left( \frac{v_1 + v_4}{2} \right)^2 \\
+ \frac{1 - 2c_0 h(K)}{1 - c_0 h(K)} \left( \mu \frac{y_2 - y_1}{2} - \nu \frac{x_2 - x_1}{2} \right) \left( \frac{v_1 + v_2}{2} \right)^2 \\
+ \frac{1}{1 - c_0 h(K)} \left( \mu \frac{y_3 - y_2}{2} - \nu \frac{x_3 - x_2}{2} \right) \left( \frac{v_2 + v_3}{2} \right)^2 \\
+ \frac{1}{1 - c_0 h(K)} \left( \mu \frac{y_4 - y_3}{2} - \nu \frac{x_4 - x_3}{2} \right) \left( \frac{v_3 + v_4}{2} \right)^2
\]

In the same way, for scheme 4, we have, for all \( v_h \in W_h \):

\[
\text{area}(K) \left( \mu \frac{\partial v_h}{\partial x} + \nu \frac{\partial v_h}{\partial y} \right) v_h(G_K) \geq \\
\geq \frac{1 - 3c_0 h(K)}{1 - c_0 h(K)} \left( \mu \frac{y_1 - y_4}{2} - \nu \frac{x_1 - x_4}{2} \right) v_4^2 \\
+ \frac{1 - 3c_0 h(K)}{1 - c_0 h(K)} \left( \mu \frac{y_2 - y_1}{2} - \nu \frac{x_2 - x_1}{2} \right) v_1^2 \\
+ \frac{1 + c_0 h(K)}{1 - c_0 h(K)} \left( \mu \frac{y_3 - y_2}{2} - \nu \frac{x_3 - x_2}{2} \right) v_2^2 \\
+ \frac{1 + c_0 h(K)}{1 - c_0 h(K)} \left( \mu \frac{y_4 - y_3}{2} - \nu \frac{x_4 - x_3}{2} \right) v_3^2
\]

**Proof:** In the conforming case, we have for all \( v_h \in V_h \):

\[
\text{area}(K) \left( \mu \frac{\partial v_h}{\partial x} + \nu \frac{\partial v_h}{\partial y} \right) v_h(G_K) = \sum_{i=1}^3 \left( \mu \frac{y_{i+1} - y_i}{2} - \nu \frac{x_{i+1} - x_i}{2} \right) \left( \frac{v_i + v_{i+1}}{2} \right)^2 \\
+ \frac{1}{8} \left( \mu (y_1 - y_2 + y_3 - y_4) - \nu (x_1 - x_2 + x_3 - x_4) \right) \\
\left[ \left( \frac{v_1 + v_4}{2} - \frac{v_2 + v_3}{2} \right)^2 - \left( \frac{v_1 + v_2}{2} - \frac{v_3 + v_4}{2} \right)^2 \right]
\]

with \( x_5 = x_1, y_5 = y_1 \) and \( v_5 = v_1 \). If we combine this identity with hypothesis (3.2), we get inequality (3.8).

*Revue Française d'Automatique, Informatique et Recherche Opérationnelle*
The proof is the same in the non-conforming case if we start from the following identity, for all $v_h \in W_h$:

$$\text{area}(K) \left( \mu \frac{\partial v_h}{\partial x} + \nu \frac{\partial v_h}{\partial y} \right) v_h(G_K) = \sum_{i=1}^{4} \left( \mu \frac{y_{i+1} - y_i}{2} - \nu \frac{x_{i+1} - x_i}{2} \right) v_i^2 + \frac{1}{4} \left( \mu(y_1 - y_2 + y_3 - y_4) - \nu(x_1 - x_2 + x_3 - x_4) \right) \left( (v_1 - v_3)^2 - (v_4 - v_2)^2 \right).$$

As a consequence of Lemma 3.3, we get:

**Lemma 3.4:** We assume that hypothesis 3.2 holds and that $h^\leq h_0$, then scheme 3 (resp. scheme 4) has a unique solution $u_h \in V_h$ (resp. $W_h$) and satisfying:

$$(3.10) \quad |u_h|_h + <M^{1/2}u_h>_h \ (\text{resp. } |u_h|_h + [M^{1/2}u_h]_h) \leq c |f|_h,$$

where $c$ is a constant independent of the triangulation.

**Remark 3.2:** If we want hypothesis (3.2) to hold for any $K \in \mathcal{G}_h$, we have to perform the calculations from the bottom to the top and from the left to the right, starting from $\Gamma_0 \cup \Gamma_1$. The values of the flux $\varphi$ in any quadrilateral $K_{ij}$ will depend on the values of the flux in all quadrilaterals belonging to $\Omega_{ij}$, where $\Omega_{ij}$ is defined on figure 3.1.

**Remark 3.3:** Lemma 3.4 shows that hypothesis 3.2 is a sufficient condition for stability. Numerical results show that if hypothesis 3.2 does not hold, then we do not have stability [12], and the results are meaningless.

n° août 1974, R-2.
Before we give some estimations of the error between the exact solution and the approximate solution, we shall check that the conditions of neutron conservation, as expressed in [9], are satisfied.

**Neutron conservation**: First of all, it is easy to see on each element $K$ that if the solution $u_h$ is a constant, then the approximations of the derivatives $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ cancel identically. Now we must check that spatial integration of the approximations of the derivatives $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ in equations (2.14) and (2.18) results in a balance statement involving boundary terms only. So we have to calculate the following quantities:

\[
E(u_h) = \sum_{K \in \mathcal{T}_h} \int_K \left( \mu \frac{\partial u_h}{\partial x} + \nu \frac{\partial u_h}{\partial y} \right) \, dx \, dy,
\]

defined for $u_h \in V_h$ or $W_h$ (scheme 1 or 2), and

\[
E_h(u_h) = \sum_{K \in \mathcal{T}_h} \left( \mu \frac{\partial u_h}{\partial x} + \nu \frac{\partial u_h}{\partial y} \right) (G_K) \cdot \text{area } K
\]

defined for $u_h \in V_h$ or $W_h$ (scheme 3 or 4).

One may write:

\[
E(u_h) = \sum_{K \in \mathcal{T}_h} \int_{\partial K} \left( \mu n^K_x + \nu n^K_y \right) u_h \, dt,
\]

and

\[
E_h(u_h) = \sum_{K \in \mathcal{T}_h} \int_K \left( \mu \frac{\partial u_h}{\partial x} + \nu \frac{\partial u_h}{\partial y} \right) \, dx \, dy = \sum_{K \in \mathcal{T}_h} \int_{\partial K} \left( \mu n^K_x + \nu n^K_y \right) u_h \, dt
\]

where $n^K_x$ and $n^K_y$ are the components of the outer normal on $\partial K$, and where $t$ is a curvilinear abscissa on $\partial K$.

Now, for schemes 1 and 3, $u_h$ is a continuous function on $\tilde{\Omega}$, so we get:

\[
E(u_h) = \int_{\Gamma} (\mu n_x + \nu n_y) u_h \, dt,
\]

and

\[
E_h(u_h) = \int_{\Gamma} (\mu n_x + \nu n_y) u_h \, dt.
\]

For schemes 2 and 4, we also get the same result because $u_h$ is a polynomial of degree $\leq 1$ on each edge and $u_h$ is continuous at the mid-point of the edges.
IV. ERROR BOUNDS: NOTATIONS AND FUNDAMENTAL LEMMAS

Given an integer \( m \geq 0 \) and a real number \( p \geq 1 \), we let

\[ W^{m,p}(\Omega) = \{ v \mid vL^p(\Omega), \partial^\alpha v \in L^p(\Omega), |\alpha| \leq m \} \]

denote the usual Sobolev space, with the following norm

\[ \|v\|_{m,p,\Omega} = \left( \sum_{|\alpha| \leq m} \|\partial^\alpha v\|_{p,\Omega}^p \right)^{1/p} \]

where \( \|\cdot\|_{p,\Omega} \) represents the usual norm in \( L^p(\Omega) \), and where \( \alpha \) is a multi-index such that \( \alpha = (\alpha_1, \alpha_2), \alpha_i \geq 0, |\alpha| = \alpha_1 + \alpha_2 \).

We shall also use the following semi-norm on \( W^{m,p}(\Omega) \):

\[ |v|_{m,p,\Omega} = \left( \sum_{|\alpha| \leq m} \|\partial^\alpha v\|_{p,\Omega}^p \right)^{1/p} \]

The usual modifications in the preceding definitions will be done for \( p = \infty \), and we shall write \( W^{m,\infty}(\Omega) = H^m(\Omega) \), for \( m \geq 0 \).

The same definitions will be used for \( \Omega \) replaced by \( \Gamma \), when \( m = 0 \). In what follows, \( c \) will always denote a constant independent of the triangulation \( \mathcal{T}_h \). We have the following lemmas [1], [2], [18]:

**Lemma 4.1**: Let \( u \) be a function belonging to the space \( W^{2,p}(\Omega), p > 1 \) and let \( r_h u \in V_h \) (or \( W_h \)) be its interpolate, as defined in paragraph II. We have, for \( 0 \leq m \leq 2, 1 < p \leq + \infty \) and for all \( K \in \mathcal{T}_h \):

\[ \|u - r_h u\|_{m,p,K} \leq ch(K)^{2-m} |u|_{2,p,K} \]

**Lemma 4.2**: Let \( u \) be a function belonging to the space \( H^r(\Omega) \) for \( r = 2, 3 \), and let \( r_h u \in V_h \) be its interpolate. Then, we have:

\[ \|u - r_h u\|_{L^2(\partial K)} \leq c(h(K))^{-1} |u|_{r-1,2,\partial K} \] for all \( K \in \mathcal{T}_h \), and

\[ \|u - r_h u\|_{L^2(\Gamma)} \leq ch^{-1} (|u|_{r,2,\Omega} + |u|_{r-1,2,\Omega}) \] for \( r = 2, 3 \)

**Lemma 4.3**: Let \( u \) be a function belonging to the space \( H^2(\Omega) \), and let \( r_h u \in W_h \) be its interpolate. We have:

\[ \|u - r_h u\|_{L^2(\partial K)} \leq c(h(K))^{3/2} |u|_{2,2,K} \] for all \( K \in \mathcal{T}_h \), and

\[ \|u - r_h u\|_{L^2(\Gamma)} \leq ch^{3/2} |u|_{2,2,\Omega} \]

n° août 1974, R-2.
If we assume now that \( u \in W^{2,\infty}(\Omega) \), we have:

\[
\| u - r_h u \|_{L^2(\partial K)} \leq c(h(K))^{5/2} \| u \|_{2,\infty,K} \quad \text{for all } K \in \mathcal{G}_h,
\]

and

\[
\| u - r_h u \|_{L^2(\Omega)} \leq ch^2 \| u \|_{2,\infty,\Omega}.
\]

If one uses the techniques developed in [3], one can show the following results:

**Lemma 4.4**: Let \( u \) be a function belonging to the space \( H^3(\Omega) \), and let \( r_h u \in V_h \) (or \( W_h \)) be its interpolate, we have:

\[
\| u - r_h u \|_{L^2(\Omega)} \leq c h^2 \| u \|_{2,2,\Omega}, \quad \text{with } r_h u \in V_h \text{ or } W_h,
\]

\[
\| u - r_h u \|_{W^1(\Omega)} \leq c h^2 (\| u \|_{3,2,\Omega} + \| u \|_{2,2,\Omega}), \quad \text{with } r_h u \in V_h \text{ or } W_h,
\]

\[
\langle u - r_h u, \psi \rangle_h \leq c h^2 (\| u \|_{3,2,\Omega} + \| u \|_{2,2,\Omega}) \quad \text{with } r_h u \in V_h.
\]

According to lemma 4.1, we see that the \( L^2 \) norm of the first derivatives of \( u - r_h u \) is of order \( h \). We shall show that in certain circumstances, we can get an order \( h^2 \), which is a super convergence result.

**Lemma 4.5**: Let \( u \) be a function belonging to the space \( H^3(\Omega) \), and let \( r_h u \in V_h \) be its interpolate. We assume that the triangulation \( \mathcal{G}_h \) is made up of equal rectangles whose edges are respectively equal to \( \Delta x \) and \( \Delta y \). Let \( \psi_{ij} \) be the function of \( V_h \) equal to 1 at the point \((i\Delta x,j\Delta y)\) and equal to zero at all the other nodes, and let \( Q_{ij} \) be the support of \( \psi_{ij} \), for \( 1 \leq i \leq I - 1, 1 \leq j \leq J - 1 \).

We have:

\[
\left( \frac{\partial}{\partial x} (u - r_h u), \psi_{ij} \right)_{L^2(\Omega)} + \left( \frac{\partial}{\partial y} (u - r_h u), \psi_{ij} \right)_{L^2(\Omega)} \leq c h^3 |u|_{3,2,\Omega}
\]

for \( 1 \leq i \leq I - 1, 1 \leq j \leq J - 1 \).

**Proof**: We have \( Q_{i,j} = K_{i,j} \cup K_{i-1,j} \cup K_{i,j-1} \cup K_{i-1,j-1} \). We consider the isoparametric transformation \( F_{i,j} \) which maps the reference square \( \hat{K} \) as defined in paragraph II on to the rectangle \( Q_{i,j} \). To any function \( u \) defined on \( K \), we let correspond a function \( \hat{u} \) defined on \( \hat{K} \) by \( \hat{u}(\xi, \eta) = u(x, y) \) with \((x, y) = F_{i,j}(\xi, \eta)\).

We have:

\[
(4.1) \quad \left( \frac{\partial}{\partial x} (u - r_h u), \psi_{ij} \right)_{L^2(\Omega)} = \Delta y \left( \frac{\partial}{\partial \xi} (\hat{u} - r_h \hat{u}), \hat{\psi}_{ij} \right)_{L^2(\hat{K})}
\]

We can check that the application defined by \( \hat{v} \rightarrow \left( \frac{\partial}{\partial \xi} (\hat{v} - r_h \hat{v}), \hat{\psi}_{ij} \right)_{L^2(\hat{K})} \) is linear and continuous from \( H^3(\hat{K}) \) into \( R \) and is identically equal to zero for all \( \hat{v} \in \hat{P}(2) \) (the space of polynomials of degree \( \leq 2 \) in both variables \( \xi \) and \( \eta \)).
So we get:

\[(4.2) \quad \left| \left( \frac{\partial}{\partial x} (\hat{u} - r_h \hat{u}), \psi_{ij} \right) \right|_{L^2(K)} \leq c \left| \hat{u} \right|_{3,2,K} \text{ for all } \hat{u} \in H^3(K).
\]

We also have:

\[(4.3) \quad \left| \hat{u} \right|_{3,2,K} \leq c h^2 \left| u \right|_{3,2,K} \text{ for all } u \in H^3(K).
\]

Combining inequalities (4.1), (4.2) and (4.3), we get lemma 4.5.

It is easy to show the following two technical results:

**Lemma 4.6:** Let \( v_h^0 \) be any function of \( V_h \). We can write \( v_h^0 \) as follows:

\[
v_h^0 = \sum_{i,j=1}^{I-1,J-1} \alpha_{i,j} \psi_{ij},
\]

and we have:

\[
\left| v_h^0 \right|_{L^2(\Omega)} \geq c \left( \sum_{i,j=1}^{I-1,J-1} (\alpha_{i,j})^2 \right)^{1/2}.
\]

**Lemma 4.7:** Let \( v_h \) be any function of \( V_h \), which we write, as in paragraph II, as follows:

\[
v_h = v_h^0 + v_h^b, \quad v_h^b \in V_h^0.
\]

We have:

\[
\left| v_h^0 \right|_{L^2(\Omega)} \leq c \left| v_h \right|_{L^2(\Omega)},
\]

\[
\left| v_h \right|_{L^2(\Omega)} \leq c h \left| v_h \right|_{L^2(\gamma)}.
\]

Combining Lemmas 4.5, 4.6 and 4.7, we get:

**Lemma 4.8:** Let \( u \) be a function belonging to the space \( H^3(\Omega) \cap W^{2,\infty} \) and let \( r_h u \in V_h \) be its interpolate. We assume that the triangulation \( \mathcal{G}_h \) is made up of equal rectangles. Then we have, for any \( v_h \in V_h \):

\[
\left| \left( \frac{\partial}{\partial x} (u - r_h u), v_h \right) \right|_{L^2(\Omega)} + \left| \left( \frac{\partial}{\partial y} (u - r_h u), v_h \right) \right|_{L^2(\Omega)}
\]

\[
\leq c h^2 \left| u \right|_{3,2,\Omega} \left| v_h \right|_{L^2(\Omega)} + \left| u \right|_{2,\infty,\Omega} \left| v_h \right|_{L^2(\gamma)}.
\]

**Proof:** Let \( v_h \) be any function of \( V_h \), with \( v_h = v_h^0 + v_h^b \), and

\[
v_h^0 = \sum_{i,j=1}^{I-1,J-1} \alpha_{i,j} \psi_{i,j}.
\]

Let \( k_{i,j} \) be defined by:

\[
k_{i,j} = \left( \frac{\partial}{\partial x} (u - r_h u), \psi_{i,j} \right)_{L^2(\Omega)}, \quad \text{for } 1 \leq i \leq I - 1, 1 \leq j \leq J - 1.
\]

\( n^e \) août 1974, R-2.
Combining lemmas 4.5, 4.6 and 4.7, we get:

\[
\left| \left( \frac{\partial}{\partial x} (u - r_h u), v_h^0 \right)_{L^2(\Omega)} \right| \leq c h^{-1} \sum_{i,j} \alpha_{i,j} k_{i,j} \leq c h^2 |u|_{3,2,\Omega}.
\]

(4.4)

According to lemmas 4.1 and 4.7, we have:

\[
\left| \left( \frac{\partial}{\partial x} (u - r_h u), v_h^0 \right)_{L^2(\Omega)} \right| \leq c h |u|_{2,\infty,\Omega} |v_h^0|_{L^1(\Omega)}
\]

\[
\leq c h^2 |u|_{2,\infty,\Omega} |v_h|_{L^2(\Omega)}.
\]

(4.5)

Inequalities (4.4) and (4.5), along with inequalities of the same type for the term \( \frac{\partial}{\partial y} (u - r_h u) \) give us lemma 4.8.

We shall now consider the non-conforming case. We can show the following fundamental result, using exactly the same proof as for lemma 4.5:

**Lemma 4.9**: Let \( u \) be a function of \( H^3(\Omega) \) and let \( r_h u \in W_h \) be its interpolate. We assume that the triangulation \( T_h \) is made up of equal rectangles whose edges are respectively equal to \( \Delta x \) and \( \Delta y \). Let \( \psi_{i,j} \) be the function of \( W_h \) equal to one at the points \(((i + 1/2)\Delta x, j\Delta y), ((i - 1/2)\Delta x, j\Delta y), (i\Delta x, (j + 1/2)\Delta y) \) and \((i\Delta x, (j - 1/2)\Delta y) \) and equal to zero at all the other nodes, and let \( Q_{i,j} \) be the support of \( \psi_{i,j} \) for \( 1 \leq i \leq I - 1, 1 \leq j \leq J - 1 \). We have:

\[
\left| \left( \frac{\partial}{\partial x} (u - r_h u), \psi_{i,j} \right)_{L^2(\Omega)} \right| + \left| \left( \frac{\partial}{\partial y} (u - r_h u), \psi_{i,j} \right)_{L^2(\Omega)} \right| \leq c h^3 |u|_{3,2,Q_{i,j}},
\]

for \( 1 \leq i \leq I - 1, 1 \leq j \leq J - 1 \).

Using Lemma 4.1 with \( p = + W \) and \( m = 1 \), it is easy to show the following result:

**Lemma 4.10**: Let \( u \) be a function of \( W^{2,\infty}(\Omega) \) and let \( r_h u \in W_h \) be its interpolate. We assume that the triangulation \( T_h \) is made up of equal rectangles whose edges are respectively equal to \( \Delta x \) and \( \Delta y \). Let \( \psi_{0,j} \) be the function of \( W_h \) equal to one at the points \((0, (j + 1/2)\Delta y), (0, (j - 1/2)\Delta y) \) and \( \left( \frac{\Delta x}{2}, j\Delta y \right) \) and equal to zero at all the other nodes, and let \( Q_{0,j} \) be the support of \( \psi_{0,j} \), for \( 1 \leq j \leq J - 1 \). In the same way, we define \( \psi_{i,0} \), for \( 1 \leq i \leq J - 1, \psi_{1,0} \) and \( \psi_{i,1} \), for \( 1 \leq i \leq I - 1 \). Let \( \psi_{0,0} \) be the function of \( W_h \) equal to 1 at the points \( \left( \frac{\Delta y}{2}, 0 \right) \) and \( \left( \frac{\Delta x}{2}, 0 \right) \) and equal to zero at all the other nodes, and let \( Q_{0,0} \),
be the support of $\psi_{0,0}$. In the same manner, we also define $\psi_{1,0}$ and $\psi_{1,1}$. We have:

$$\left| \left( \frac{\partial}{\partial x} (u - r_h u), \psi_{i,j} \right) \right|_{L^2(\Omega)}^* + \left| \left( \frac{\partial}{\partial y} (u - r_h u), \psi_{i,j} \right) \right|_{L^2(\Omega)}^* \leq c h^3 |u|_{2,\infty,\Omega},$$

for all the indices $i$ and $j$ defined above.

Combining lemmas 4.9 and 4.10, we get:

**Lemma 4.11**: Let $u$ be a function of $H^3(\Omega) \cap W^{2,\infty}(\Omega)$ and let $r_h u \in W_h$ be its interpolate. We assume that the triangulation $\mathcal{T}_h$ is made up of equal rectangles. Then we have, for any $v_h \in W_h$:

$$\left| \left( \frac{\partial}{\partial x} (u - r_h u), v_h \right) \right|_{L^2(\Omega)}^* + \left| \left( \frac{\partial}{\partial y} (u - r_h u), v_h \right) \right|_{L^2(\Omega)}^* \leq c (h^2 |u|_{3,2,\Omega} + h^{3/2} |u|_{2,\infty,\Omega}) |v_h|_{L^2(\Omega)}.$$

**Proof**: The set $\{ \psi_{i,j}; 0 \leq j \leq J; 0 \leq i \leq I \} - \{ \psi_I, \} \} \} is a basis of $W_h$. Any function $v_h \in W_h$ can be written as $w_h = \sum_{i,j} \alpha_{i,j} \psi_{i,j}$, with $(i,j) \neq (I,J)$.

Then we have:

$$|v_h|_{L^2(\Omega)} \geq ch \left( \sum_{i,j} (\alpha_{i,j})^2 \right)^{1/2}.$$

Now we define $k_{i,j}$ by $k_{i,j} = \left( \frac{\partial}{\partial x} (u - r_h u), \psi_{i,j} \right)_{L^2(\Omega)}^*$, for $(i,j) \neq (I,J)$.

It is easy to show by applying lemmas 4.9 and 4.10 that:

$$\left| \left( \frac{\partial}{\partial x} (u - r_h u), v_h \right) \right|_{L^2(\Omega)}^* \leq ch^{-1} \left( \sum_{i,j} (k_{i,j})^2 \right)^{1/2} \leq c (h^2 |u|_{3,2,\Omega} + h^{3/2} |u|_{2,\infty,\Omega}),$$

which gives us lemma 4.11.

For the sake of completness, we shall give the proof of the following results, for convex quadrilaterals [20]:

**Lemma 4.12**: Let $u$ be a function of $H^3(\Omega)$ and let $r_h u \in V_h$ (or $W_h$) be its interpolate in the conforming (or non-conforming) case. Assume that hypothesis 3.1 holds. We have:

$$\left| \frac{\partial}{\partial x} (u - r_h u) \right|_h + \left| \frac{\partial}{\partial y} (u - r_h u) \right|_h \leq ch^2 (|u|_{3,2,\Omega} + |u|_{2,2,\Omega}).$$

n° août 1974, R-2.
Proof: We consider the conforming case. Let $K$ be any quadrilateral of the triangulation and let $F_K$ be the isoparametric transformation which maps the reference square $\hat{K}$ onto $K$. We have:

$$\text{area}(K) \left( \frac{\partial}{\partial x} (u - r_h u) \right)(G_K) = 4 \left( \frac{\partial}{\partial \xi} (\hat{u} - r_h \hat{u}) \frac{\partial y}{\partial \eta} - \frac{\partial}{\partial \eta} (\hat{u} - r_h \hat{u}) \frac{\partial y}{\partial \xi} \right)(0,0).$$

where $\hat{u}(\xi, \eta) = u(x, y)$ with $(x, y) = F_K(\xi, \eta)$.

We can easily check that the application defined by $\hat{u} \rightarrow \frac{\partial}{\partial \xi} (\hat{u} - r_h \hat{u})(0,0)$ is linear and continuous from $H^3(\hat{K})$ into $R$, and is identically equal to zero for all $\hat{u} \in \hat{P}(2)$. So we get:

$$\left| \left( \frac{\partial}{\partial \xi} (\hat{u} - r_h \hat{u}) \right)(0,0) \right| \leq c |\hat{u}|_{3,2,K}.$$

Going back to quadrilateral $K$, by using transformation $F_K^{-1}$, we get (see [3] lemma 1):

$$\text{area}(K) \left( \frac{\partial}{\partial x} (u - r_h u) \right)(G_K) \leq c((h(K))^3 |u|_{3,2,K} + h(K) \cdot Z(K) |u|_{2,2,K}).$$

for all $K \in \mathcal{T}_h$. Summing on all quadrilaterals $K$ of $\mathcal{T}_h$, we get lemma 4.12. The proof is exactly the same in the non conforming case.

V. ERROR BOUNDS, THEOREMS

Theorem 5.1: Let $u_h \in V_h$ be the solution of scheme 1. We assume that the exact solution $u$ belongs to $H^2(\Omega)$ and that the triangulation $\mathcal{T}_h$ is a regular family of arbitrary convex quadrilaterals. Then we have:

$$|u - u_h|_{L^2(\Omega)} + |(M)^{1/2}(u - u_h)|_{L^2(\Gamma)} \leq ch |u|_{2,2,\Omega}.$$

If we assume now that all the quadrilaterals are equal rectangles, and that the exact solution $u$ belongs to $H^3(\Omega) \cap W^{2,\infty}(\Omega)$, then we have:

$$|u - u_h|_{L^2(\Omega)} + |(M)^{1/2}(u - u_h)|_{L^2(\Gamma)} \leq ch^2(|u|_{2,\infty,\Omega} + |u|_{3,2,\Omega}).$$

Proof: If in lemma 1.1 we replace $v$ by $u_h - r_h u$, where $r_h u \in V_h$ is the interpolate of $u$, we get:

$$|u_h - r_h u|_{L^2(\Omega)} + |(M)^{1/2}(u_h - r_h u)|_{L^2(\Gamma)} \leq c \left( (u_h - r_h u, u_h - r_h u)_{L^2(\Omega)} - \left( B - M \right) \frac{(u_h - r_h u, u_h - r_h u)}{L^2(\Gamma)} \right).$$

Revue Française d'Automatique, Informatique et Recherche Opérationnelle
Using the definition of scheme 1, we easily get:

\[
|u_h - r_h u|_{L^2(\Omega)}^2 + |(M)^{1/2}(u_h - r_h u)|_{L^2(\Gamma)}^2 \\
\leq c \left( |(A(u - r_h u), u_h - r_h u)|_{L^2(\Omega)} + \left| \left( \frac{B - M}{2} (u - r_h u), u_h - r_h u \right)|_{L^2(\Gamma)} \right| \right).
\]

Combining inequality (5.3) with lemmas 4.1 and 4.2, we get inequality (5.1). When the quadrilaterals are equal rectangles, we use inequality (5.3) along with lemmas 4.1, 4.2 and 4.8 to get inequality (5.2).

**Theorem 5.2**: We assume that the triangulation \( T_h \) is a regular family of arbitrary quadrilaterals and that hypothesis 3.1 is satisfied. Let \( u_h \in W_h \) be the solution of scheme 2, and let the exact solution \( u \) belong to \( H^2(\Omega) \). We have:

\[
|u - u_h|_{L^2(\Omega)} + |(M)^{1/2}(u - u_h)|_{L^2(\Gamma_0 \cup \Gamma_1)} + [(M)^{1/2}(u - u_h)]_h^n \\
\leq c h |u|_{2,2,\Omega}.
\]

If we assume that all the quadrilaterals are equal rectangles and that the exact solution \( u \) belongs to \( H^3(\Omega) \cap W^{2,\infty}(\Omega) \), we then have:

\[
|u - u_h|_{L^2(\Omega)} + |(M)^{1/2}(u - u_h)|_{L^2(\Gamma_0 \cup \Gamma_1)}^2 + [(M)^{1/2}(u - u_h)]_h^n^2 \\
\leq c(h^2 |u|_{3,2,\Omega} + h^{3/2} |u|_{2,\infty,\Omega}).
\]

**Proof**: Starting from lemma 3.2 and from the definition of scheme 2, we get:

\[
|u_h - r_h u|_{L^2(\Omega)}^2 + |(M)^{1/2}(u_h - r_h u)|_{L^2(\Gamma_0 \cup \Gamma_1)}^2 + [(M)^{1/2}(u_h - r_h u)]_h^n^2 \\
\leq c \left( |(A(u - r_h u), u_h - r_h u)|_{L^2(\Omega)} + \left| \left( \frac{B - M}{2} (u - r_h u), u_h - r_h u \right)|_{L^2(\Gamma)} \right| \right).
\]

where \( r_h u \in W_h \) is the interpolate of \( u \).

Combining inequality 5.6 and lemmas 4.1 and 4.3, we get inequality (5.4). When the quadrilaterals are equal rectangles, hypothesis 3.1 is automatically satisfied; we use lemmas 4.1, 4.3 and 4.11, with inequality (5.6) to get inequality (5.5).

**Theorem 5.3**: We assume that the triangulation \( T_h \) is a regular family of arbitrary quadrilaterals and that hypothesis 3.1 and 3.2 hold. Then let \( u_h \in V_h \) (resp. \( W_h \)) be the solution of scheme 3 (resp. scheme 4). We assume that the exact
solution $u$ belongs to $H^3(\Omega)$. We then have:

\[(5.7) \quad |u - u_h|_h \leq c h^2 (|u|_{2,2,\Omega} + |u|_{3,2,\Omega}),\]

\[(5.8) \quad \max_{s \in U} |(M_s)^{1/2}(u - u_h)(G_s)| \leq c h^{3/2} (|u|_{2,2,\Omega} + |u|_{3,2,\Omega}),\]

where $U$ denotes the set of all the edges of the quadrilaterals $K$ of $\mathcal{G}_h$, where $G_s$ is the mid-point of the edge $s$ and where $M_s = \mu n_s^x + \nu n_s^y$, $n_s^x$ and $n_s^y$ being the components of a normal on $s$.

**Proof**: We consider the conforming case. Inequality (3.8) of lemma 3.3 holds with $v_h$ replaced by $u_h - r_h u$, where $r_h u$ belongs to $V_h$ and is the interpolate of $u$. We consider the following expression for any $K$ belonging to $\mathcal{G}_h$ and non adjacent to $\Gamma_0 \cup \Gamma_1$:

$$
\chi_h(K) = \text{area}(K) \left( \mu \frac{\partial (u_h - r_h u)}{\partial x} + \nu \frac{\partial (u_h - r_h u)}{\partial y} + \sigma (u_h - r_h u) \right) \cdot (u_h - r_h u)(G_K).
$$

We have, with the same notations as in lemma 3.3:

\[(5.9) \quad \chi_h(K) \geq \frac{1 - 2c_0 h(K)}{1 - c_0 h(K)} \left\{ \left( \mu \frac{y_1 - y_4 - x_1 - x_2}{2} \right)^2 (u_h - r_h u)(A_{14}) \right\}

+ \left( \mu \frac{y_2 - y_1 - x_2 - x_3}{2} \right)^2 (u_h - r_h u)(A_{12}) \right\}

+ \left( \frac{1}{1 - c_0 h(K)} \left\{ \left( \mu \frac{y_3 - y_2 - x_3 - x_4}{2} \right)^2 (u_h - r_h u)(A_{13}) \right\}

+ \left( \mu \frac{y_4 - y_3 + x_4 - x_3}{2} \right)^2 (u_h - r_h u)(A_{34}) \right\} + \sigma((u_h - r_h u)(G_K))^2 \right\},

where $A_{ij}$ denotes the mid-point of any edge $A_iA_j$.

According to the definition of scheme 3, we have:

\[
\chi_h(K) = \text{area}(K) \left( \mu \frac{\partial (u - r_h u)}{\partial x} + \nu \frac{\partial (u - r_h u)}{\partial y} + \sigma (u - r_h u) \right) (u - r_h u)(G_K)
\]

When we use lemmas 4.1 and 4.8, we get:

\[(5.10) \quad \chi_h(K) \leq c(h(K))^2 (|u|_{3,2,K} + |u|_{2,2,K}) \text{ (area}(K) \cdot ((u_h - r_h u)(G_K))^2)^{1/2}\]
Combining inequalities (5.9) and (5.10), we get for any $K \in \mathcal{G}_h$:

\[
\frac{1 - 2c_0(K)}{1 - c_0 h(K)} \left\{ \left( \mu \frac{y_1 - y_4}{2} - \nu \frac{x_1 - x_4}{2} \right) (u_h - r_h u)^2 (A_{14}) + \left( \mu \frac{y_2 - y_1}{2} - \nu \frac{x_2 - x_1}{2} \right) (u_h - r_h u)^2 (A_{12}) \right\} + c(u_h - r_h u)^2 (G_K) \\
\leq c(h(K))^4 \left( |u|_{3,2,K}^2 + |u|_{2,2,K}^2 \right) + \\
\frac{1}{1 - c_0 h(K)} \left\{ \left( \mu \frac{y_2 - y_3}{2} - \nu \frac{x_2 - x_3}{2} \right) (u_h - r_h u)^2 (A_{23}) + \left( \mu \frac{y_3 - y_4}{2} - \nu \frac{x_3 - x_4}{2} \right) (u_h - r_h u)^2 (A_{34}) \right\}.
\]

If $K$ is adjacent to $\Gamma_0 \cup \Gamma_1$, we must add up some boundary terms. If we combine inequalities (5.11) for all $K \in \mathcal{G}_h$, with appropriate weights, we get:

\[
(5.12) \quad c |u_h - r_h u|_h^2 + c h \left( \max_{s \in \Gamma} (M_s) \right) (u_h - r_h u)^2 (G_s) \\
\leq c h^4 (|u|_{3,2,\Omega}^2 + |u|_{2,2,\Omega}^2).
\]

Inequality (5.7) follows immediately. Now we can get an inequality like (5.12) for any $\Omega_{ij}$ as defined in remark 3.2. Particularly we get:

\[
(5.13) \quad \max_{s \in \Gamma_{ij}} (M_s)(u_h - r_h u)^2 (G_s) \leq c h^3 (|u|_{3,2,\Omega}^2 + |u|_{2,2,\Omega}^2).
\]

Inequality (5.8) follows immediately from inequality (5.13) and lemma 4.4. The proof is the same in the non-conforming case.

**Remark 5.1**: We define the following discrete norm $\| \cdot \|_h$ on $V_h$ or $W_h$ by

\[
(5.14) \quad \|v_h\|_h^2 = \sum_{K \in \mathcal{G}_h} (h(K))^2 \sum_{s \in \partial K} (v_h(G_s))^2
\]

where $G_s$ is the mid-point of the edge $s$. We assume that hypotheses 3.1 and 3.2 hold and that we have:

\[
(5.15) \quad |M_s| \geq \delta > 0
\]

for any $s \in U_s$, where $\delta$ is independent of $\mathcal{G}_h$. Then it is possible to show that for scheme 3 or for scheme 4, we have:

\[
(5.16) \quad \|u - u_h\|_h \leq c \frac{h^2}{\sqrt{\delta}} (|u|_{3,2,\Omega} + |u|_{2,2,\Omega}).
\]

Numerical results ([12]) show that we really get estimate (5.16) when hypotheses 3.1, 3.2 and 5.15 hold.
REMARK 5.2 : In the conforming case, it would seem natural to get the same estimate like (5.16) for the values at the vertices of the quadrilaterals. Numerical results give only an error of order \( h \). In fact, one can show that:

\[
|u_h - r_h u|_{L^2(\Omega)} \leq c h^{-1} \|u_h - r_h u\|
\]

This last inequality combined with inequality (5.16) gives the order \( h \) for the error at the vertices of the quadrilaterals.

We shall now give an estimate for the error due to both angular and spatial discretizations. We define the following discrete norm :

\[
\|\varphi\|_{h,\mu}^2 = \sum_{i=1}^{T} \text{area}(T_i) \sum_{K \in \mathcal{C}_h} \text{area}(K) \varphi^2(G_K, \mu, \nu_i)
\]

We then have :

**Theorem 5.4 :** Let \( \varphi \in H^3(\Omega \times Q) \) be the exact solution of problem (1.1), (1.2). We assume that the triangulation \( \mathcal{C}_h \) is made up of equal rectangles. Let \( \varphi_{h,\mu} \in V_h \times \mathcal{U}_\mu \) (resp. \( W_h \times \mathcal{U}_\mu \)) be the approximate solution when we use scheme 1 (resp. scheme 2). Then we have :

\[
\|\varphi - \varphi_{h,\mu}\|_\mu = O(h^2) + O(\Delta \mu^2).
\]

**Theorem 5.5 :** Let \( \varphi \in H^3(\Omega \times Q) \) be the exact solution of problem (1.1), (1.2). We assume that hypothesis 3.2 holds for any \((\mu_l, \nu_l)\) \( 1 \leq l \leq L \). Let \( \varphi_{h,\mu} \in V_h \times \mathcal{U}_\mu \) (resp. \( W_h \times \mathcal{U}_\mu \)) be the approximate solution when we use scheme 3 (resp. scheme 4). We then have :

\[
\|\varphi - \varphi_{h,\mu}\|_{h,\mu} = O(h^2) + O(\Delta \mu^2)
\]

**REMARK 5.2 :** Hypothesis 3.2 implies that we cannot choose any value for \((\mu_l, \nu_l)\). For example when the quadrilaterals \( K \) are very distorted, we cannot use a small value of \( \Delta \mu \).

We shall see in a forthcoming paper [13] that this problem of stability can be handled if we use discontinuous elements in space [17]: we can get an unconditionally stable quasi explicit (we have to invert a sequence of \( 4 \times 4 \) matrices when we use polynomials of degree \( \leq 1 \) in each spatial element) and rather accurate schemes.

**REFERENCES**


*Revue Française d'Automatique, Informatique et Recherche Opérationnelle*


n° août 1974, R-2.