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## Representation of prime powers in arithmetical progressions by binary quadratic forms

par FRANZ HALTER-KOCH

RÉSUMÉ. Soit  $\Gamma$  une famille de formes quadratiques à deux variables de même discriminant,  $\Delta$  un ensemble de progressions arithmétiques et  $m$  un entier strictement positif. Nous nous intéressons au problème de la représentation des puissances de nombres premiers  $p^m$  appartenant à une progression de  $\Delta$  par une forme quadratique de  $\Gamma$ .

ABSTRACT. Let  $\Gamma$  be a set of binary quadratic forms of the same discriminant,  $\Delta$  a set of arithmetical progressions and  $m$  a positive integer. We investigate the representability of prime powers  $p^m$  lying in some progression from  $\Delta$  by some form from  $\Gamma$ .

### 1. Introduction and notations

By a *form*  $\varphi$  we always mean a primitive integral non-degenerated binary quadratic form, that is,  $\varphi = aX^2 + bXY + cY^2 \in \mathbb{Z}[X, Y]$ , where  $\gcd(a, b, c) = 1$ ,  $d = b^2 - 4ac$  is not a square, and  $a > 0$  if  $d < 0$ . We call  $d$  the *discriminant* of  $\varphi$ . More generally, any non-square integer  $d \in \mathbb{Z}$  with  $d \equiv 0 \pmod{4}$  or  $d \equiv 1 \pmod{4}$  will be called a *discriminant*. Two forms  $\varphi, \psi \in \mathbb{Z}[X, Y]$  are called (properly) *equivalent* if  $\varphi(X, Y) = \psi(\alpha X + \beta Y, \gamma X + \delta Y)$  for some  $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$  such that  $\alpha\delta - \beta\gamma = 1$ . For any discriminant  $d$ , we denote by  $\mathcal{H}(d)$  the (finite) set of equivalence classes of forms with discriminant  $d$ . If  $\varphi = aX^2 + bXY + cY^2$  is a form with discriminant  $d$ , we denote by  $[\varphi] = [a, b, c] \in \mathcal{H}(d)$  the equivalence class of  $\varphi$ . For any discriminant  $d$ , we call

$$I = I_d = \begin{cases} [1, 0, -d/4], & \text{if } d \equiv 0 \pmod{4}, \\ [1, 1, (1-d)/4], & \text{if } d \equiv 1 \pmod{4} \end{cases}$$

the *principal class* of  $\mathcal{H}(d)$ .

A form  $\varphi \in \mathbb{Z}[X, Y]$  is said to *represent* (properly) an integer  $q \in \mathbb{Z}$ , if  $q = \varphi(x, y)$  for some  $x, y \in \mathbb{Z}$  such that  $\gcd(x, y) = 1$ . Equivalent forms

represent the same integers, and we write  $C \rightarrow q$  if  $C = [\varphi]$  for some form  $\varphi$  representing  $q$ .

This paper is addressed to the representation of prime powers in arithmetical progressions. To be precise, we shall derive criteria for a form  $\varphi$  (or its class  $[\varphi]$ ) to represent all prime powers (of fixed exponent)  $p^m \in b + a\mathbb{Z}$  for given coprime positive integers  $a$  and  $b$ . If  $m = 1$  and if genera are considered instead of individual forms or classes, the problem is solved by Gauss' genus theory. A first result for individual classes was proved by A. Meyer [9] using Dirichlet's theorem. A complete solution for  $m = 1$  and fundamental discriminants was presented by T. Kusaba [8] (using class field theory). The case of arbitrary discriminants (for  $m = 1$  and  $p \neq 2$ ) was settled by P. Kaplan and K. S. Williams [7] (using elementary methods and Meyer's theorem).

In fact, in this paper we shall consider more generally a set  $\Gamma \subset \mathcal{H}(d)$  (for some discriminant  $d \in \mathbb{Z}$ ) and a set  $\Delta \subset (\mathbb{Z}/a\mathbb{Z})^\times$  of arithmetical progressions (for some distance  $a \geq 2$ ), and we shall deal with the problem whether every prime power  $p^m$  (with fixed exponent  $m$ ) satisfying  $p^m + a\mathbb{Z} \in \Delta$  is represented by some class  $C \in \Gamma$ . We shall throughout make use of class field theory, and in order to do so, we will also formulate Gauss' genus theory in a class field theoretic setting.

In section 2, we gather the necessary facts from genus theory and class field theory in a form which is suitable for our purposes. In section 3, we formulate and prove the main results of this paper.

## 2. Class field theory and genus theory

The main references for this section are [2] and [1], but see also [3] and [4]. For a discriminant  $d$ , we set

$$R_d = \begin{cases} \mathbb{Z}[(1 + \sqrt{d})/2], & \text{if } d \equiv 1 \pmod{4}, \\ \mathbb{Z}[\sqrt{d}/2], & \text{if } d \equiv 0 \pmod{4}. \end{cases}$$

Let  $\mathcal{C}^+(R_d)$  be the Picard group of  $R_d$  in the narrow sense (that is, the group of invertible fractional ideals modulo fractional principal ideals generated by totally positive elements). If  $\varphi = aX^2 + bXY + cY^2 \in \mathbb{Z}[X, Y]$  is a form with discriminant  $d$  and  $a > 0$ , then

$$\mathfrak{c}_\varphi = (a, \frac{b + \sqrt{d}}{2}) \triangleleft R_d$$

is a primitive invertible ideal with norm  $\mathcal{N}(\mathfrak{c}_\varphi) = (R_d : \mathfrak{c}_\varphi) = a$ . Every class  $C \in \mathcal{H}(d)$  contains a form  $\varphi = aX^2 + bXY + cY^2$  with  $a > 0$ , and the assignment  $\varphi \mapsto \mathfrak{c}_\varphi$  induces a bijective map

$$\theta_d : \mathcal{H}(d) \rightarrow \mathcal{C}^+(R_d).$$

For an invertible ideal  $\mathfrak{a}$  of  $R_d$  we denote by  $[\mathfrak{a}] \in C^+(R_d)$  its class (in the narrow sense). Gauss' composition is the group structure on  $\mathcal{H}(d)$  for which  $\theta_d$  is an isomorphism, and  $I_d = \theta_d^{-1}(R_d)$  is the unit element of  $\mathcal{H}(d)$ .

For a class  $C \in \mathcal{H}(d)$  and  $q \in \mathbb{N}$ , we have  $C \rightarrow q$  if and only if  $q = \mathcal{N}(\mathfrak{a})$  for some primitive ideal  $\mathfrak{a} \in \theta_d(C)$ . For a form  $\varphi = aX^2 + bXY + cY^2 \in \mathbb{Z}[X, Y]$ , we denote by  $\bar{\varphi} = aX^2 - bXY + cY^2$  its conjugate (or opposite) form, and for a quadratic surd  $\alpha = u + v\sqrt{d} \in \mathbb{Q}(\sqrt{d})$ , we denote by  $\bar{\alpha} = u - v\sqrt{d}$  its (algebraic) conjugate. Conjugation induces inversion, both on  $\mathcal{H}(d)$  and  $C^+(R_d)$  (that means,  $[\bar{\varphi}] = [\varphi]^{-1}$  for every form  $\varphi$ , and  $[\bar{\mathfrak{a}}] = [\mathfrak{a}]^{-1}$  for every invertible ideal  $\mathfrak{a}$ ).

For every class  $C \in \mathcal{H}(d)$ ,  $C$  and  $C^{-1}$  represent the same integers. A prime power  $p^m$  with  $p \nmid d$  is represented by some class  $C \in \mathcal{H}(d)$  if and only if  $\left(\frac{d}{p}\right) = 1$ . In this case,  $pR_d = \mathfrak{p}\bar{\mathfrak{p}}$  for some prime ideal  $\mathfrak{p}$  of  $R_d$  such that  $\mathfrak{p} \neq \bar{\mathfrak{p}}$ , and if  $C = \theta_d^{-1}([\mathfrak{p}^m])$ , then  $C$  and  $C^{-1}$  are precisely the classes from  $\mathcal{H}(d)$  representing  $p^m$ .

Associated with a discriminant  $d$ , there is an abelian field extension  $K_d/\mathbb{Q}(\sqrt{d})$ , together with an isomorphism

$$\alpha_d : \mathcal{H}(d) \xrightarrow{\sim} \text{Gal}(K_d/\mathbb{Q}(\sqrt{d})),$$

having the following properties:

1.  $K_d/\mathbb{Q}$  is a Galois extension which is unramified at all primes  $p \nmid d\infty$  and whose Galois group is given by the splitting group extension

$$1 \longrightarrow \mathcal{H}(d) \xrightarrow{\alpha_d} \text{Gal}(K_d/\mathbb{Q}) \xrightarrow{\rho_d} \langle \tau \rangle \longrightarrow 1,$$

where  $\langle \tau \rangle = \text{Gal}(\mathbb{Q}(\sqrt{d})/\mathbb{Q})$ ,  $\rho_d(\sigma) = \sigma|_{\mathbb{Q}(\sqrt{d})}$ , and  $\tau$  acts on  $\mathcal{H}(d)$  by  $C^\tau = C^{-1}$ .

2. For a class  $C \in \mathcal{H}(d)$  and a prime  $p$  with  $p \nmid d$ , we have  $C \rightarrow p$  if and only if  $\alpha_d(C)$  is the Frobenius automorphism of some prime divisor of  $p$  in  $K_d$ .

Let  $K_d^*$  be the maximal absolutely abelian subfield of  $K_d$ . Then  $\text{Gal}(K_d/K_d^*) = \alpha_d(\mathcal{H}(d)^2)$ , and there is an isomorphism

$$\alpha_d^* : \begin{cases} \mathcal{H}(d)/\mathcal{H}(d)^2 & \xrightarrow{\sim} \text{Gal}(K_d^*/\mathbb{Q}(\sqrt{d})) \\ C\mathcal{H}(d)^2 & \mapsto \alpha_d(C) | K_d^*. \end{cases}$$

The field  $K_d$  is called the *ring class field*, the field  $K_d^*$  is called the *genus field*, the cosets  $C\mathcal{H}(d)^2 \subset \mathcal{H}(d)$  are called the *genera* and  $\mathcal{H}(d)^2$  is called the *principal genus* of discriminant  $d$ .

An explicit generation of  $K_d^*$  was given in [5] as follows: Let  $p_1, \dots, p_t$  be the distinct odd prime divisors of  $d$ , set

$$p_i^* = \left(\frac{-1}{p_i}\right)p_i \quad \text{for } i \in \{1, \dots, t\} \quad \text{and} \quad K'_d = \mathbb{Q}(\sqrt{p_1^*}, \dots, \sqrt{p_t^*}).$$

Then we obtain

$$K_d^* = \begin{cases} K'_d, & \text{if } d \equiv 1 \pmod{4} \text{ or } d \equiv 4 \pmod{16}, \\ K'_d(\sqrt{-1}), & \text{if } d \equiv 12 \pmod{16} \text{ or } d \equiv 16 \pmod{32}, \\ K'_d(\sqrt{\pm 2}), & \text{if } d \equiv \pm 8 \pmod{32}, \\ K'_d(\sqrt{-1}, \sqrt{2}), & \text{if } d \equiv 0 \pmod{32}. \end{cases}$$

we define the *reduced discriminant*  $d^*$  associated with  $d$  by

$$d^* = \begin{cases} p_1 \cdots p_t, & \text{if } d \equiv 1 \pmod{4}, \\ 2p_1 \cdots p_t, & \text{if } d \equiv 4 \pmod{16}, \\ 4p_1 \cdots p_t, & \text{if } d \equiv 12 \pmod{16} \text{ or } d \equiv 16 \pmod{32}, \\ 8p_1 \cdots p_t, & \text{if } d \equiv 0, 8 \text{ or } 24 \pmod{32}. \end{cases}$$

For  $a \in \mathbb{N}$ , we denote by  $\mathbb{Q}^{(a)}$  the field of  $a$ -th roots of unity and by

$$\beta_a : (\mathbb{Z}/a\mathbb{Z})^\times \xrightarrow{\sim} \text{Gal}(\mathbb{Q}^{(a)}/\mathbb{Q})$$

the Artin isomorphism for  $\mathbb{Q}^{(a)}/\mathbb{Q}$ , that is, for a prime  $p \nmid a$ ,  $\beta_a(p + a\mathbb{Z})$  is the Frobenius automorphism of the prime divisors of  $p$  in  $\mathbb{Q}^{(a)}$ .

If  $d$  is a discriminant, then  $d^* \mid d$ , hence  $\mathbb{Q}^{(d^*)} \subset \mathbb{Q}^{(d)}$ , and  $d^*$  is the smallest positive integer divisible by all prime divisors of  $d$  and satisfying  $K_d^* \subset \mathbb{Q}^{(d^*)}$ . If  $m \in \mathbb{Z}$  and  $\gcd(m, d) = 1$ , we consider the Kronecker symbol, defined by

$$\left(\frac{d}{m}\right) = \text{sign}(d)^\varepsilon (-1)^{\frac{d-1}{4}\beta} \left(\frac{d}{m_1}\right),$$

if  $m = (-1)^\varepsilon 2^\beta m_1$ , where  $\varepsilon \in \{0, 1\}$ ,  $\beta \in \mathbb{N}_0$ ,  $m_1$  is odd and  $\left(\frac{d}{m_1}\right)$  is the Jacobi symbol (for details see [6], Ch. 5.5). The Kronecker symbol  $\left(\frac{d}{m}\right)$  depends only on the residue class  $m + d^*\mathbb{Z} \in (\mathbb{Z}/d^*\mathbb{Z})^\times$ , and

$$\chi_d = \left(\frac{d}{\cdot}\right) : (\mathbb{Z}/d^*\mathbb{Z})^\times \rightarrow \{\pm 1\}$$

is a quadratic character with the following property:

If  $C \in \mathcal{H}(d)$ ,  $k \in \mathbb{Z}$ ,  $\gcd(k, d) = 1$  and  $C \rightarrow k$ , then  $\left(\frac{d}{k}\right) = 1$ . Indeed, observe that  $C \rightarrow k$  implies  $C = [k, b, c]$  for some  $b, c \in \mathbb{Z}$ , and since  $d = b^2 - 4kc$ , it follows that  $\left(\frac{d}{k}\right) = \left(\frac{d}{|k|}\right) = 1$ .

We define

$$\varphi_1, \dots, \varphi_g : (\mathbb{Z}/d^*\mathbb{Z})^\times \rightarrow \{\pm 1\} \quad \text{by} \quad \varphi_i(m + d^*\mathbb{Z}) = \left(\frac{m}{p_i}\right).$$

If  $d^* \equiv 0 \pmod{4}$ , we define

$$\varepsilon : (\mathbb{Z}/d^*\mathbb{Z})^\times \rightarrow \{\pm 1\} \quad \text{by} \quad \varepsilon(m + d^*\mathbb{Z}) = \left(\frac{-1}{m}\right),$$

and if  $d^* \equiv 0 \pmod{8}$ , we define

$$\delta : (\mathbb{Z}/d^*\mathbb{Z})^\times \rightarrow \{\pm 1\} \quad \text{by} \quad \delta(m + d^*\mathbb{Z}) = \left(\frac{2}{m}\right).$$

Then the vector of genus characters

$$\varphi_d : (\mathbb{Z}/d^*\mathbb{Z}) \rightarrow \{\pm 1\}^{\mu(d)}$$

is defined by its components as follows.

$$\varphi_d = \begin{cases} (\varphi_1, \dots, \varphi_t), & \text{if } d \equiv 1 \pmod{4} \text{ or } d \equiv 4 \pmod{16}, \\ (\varphi_1, \dots, \varphi_t, \varepsilon), & \text{if } d \equiv 12 \pmod{16} \text{ or } d \equiv 16 \pmod{32}, \\ (\varphi_1, \dots, \varphi_t, \delta), & \text{if } d \equiv 8 \pmod{32}, \\ (\varphi_1, \dots, \varphi_t, \varepsilon\delta), & \text{if } d \equiv 24 \pmod{32}, \\ (\varphi_1, \dots, \varphi_t, \varepsilon, \delta), & \text{if } d \equiv 0 \pmod{32}. \end{cases}$$

By its very definition, for a prime  $p \nmid d$  the Frobenius automorphism  $\beta_{d^*}(p + d^*\mathbb{Z}) \mid K_d^*$  is uniquely determined by its genus character values  $\varphi_d(p + d^*\mathbb{Z}) \in \{\pm 1\}^{\mu(d)}$ . We have

$$(\mathbb{Z}/d^*\mathbb{Z})^{\times 2} \subset \text{Ker}(\varphi_d) \subset \text{Ker}(\chi_d) \subset (\mathbb{Z}/d^*\mathbb{Z})^\times,$$

and

$$(\text{Ker}(\varphi_d) : (\mathbb{Z}/d^*\mathbb{Z})^{\times 2}) = \begin{cases} 2, & \text{if } d \equiv \pm 8 \pmod{32}, \\ 1 & \text{otherwise.} \end{cases}$$

From the class field theoretic description of  $K_d$ ,  $K_d^*$  and  $\mathbb{Q}^{(d^*)}$  we derive immediately the following two assertions **3.** and **4.** which are usually quoted as the main theorems of Gauss' genus theory.

### 3. The group

$$X_d = \text{Ker}(\chi_d) = \beta_{d^*}^{-1} \left[ \text{Gal}(\mathbb{Q}^{(d^*)}/\mathbb{Q}(\sqrt{d})) \right] \subset (\mathbb{Z}/d^*\mathbb{Z})^\times$$

consists of all residue classes  $p + d^*\mathbb{Z} \in (\mathbb{Z}/d^*\mathbb{Z})^\times$  generated by primes  $p$  such that  $p \nmid d$  and  $C \rightarrow p$  for some  $C \in \mathcal{H}(d)$ .

### 4. The map $\omega_d : (\mathbb{Z}/d^*\mathbb{Z})^\times \rightarrow \mathcal{H}(d)/\mathcal{H}(d)^2$ , defined by

$$\omega_d(x) = \alpha_d^{*-1} \left[ \beta_{d^*}(x) \mid K_d^* \right]$$

is a group epimorphism, and  $\text{Ker}(\omega_d) = \text{Ker}(\varphi_d)$ . In particular,

$$(\mathcal{H}(d) : \mathcal{H}(d)^2) = [K_d^* : \mathbb{Q}(\sqrt{d})] = 2^{\mu(d)-1}.$$

If  $p$  is a prime,  $p \nmid d$  and  $C \in \mathcal{H}(d)$  with  $C \rightarrow p$ , then  $\omega_d(p + d^*\mathbb{Z}) = C\mathcal{H}(d)^2$ . Consequently, the genus representing  $p$  depends only on the residue class  $p + d^*\mathbb{Z}$ .

Meyer [9] proved that if a class  $C \in \mathcal{H}(d)$  represents some prime  $p \nmid d$  in a coprime arithmetical progression, then it represents infinitely many primes from this progression. We shall need the following refinement of this result.

**Proposition 1.** *Let  $d$  be a discriminant,  $a, b \in \mathbb{N}$  and  $\gcd(a, b) = 1$ . Let  $C_0 \in \mathcal{H}(d)$  be a class representing some prime  $p_0 \in b + a\mathbb{Z}$  with  $p_0 \nmid d$ .*

1. *The set of all primes  $p \in b + a\mathbb{Z}$  represented by  $C_0$  has positive Dirichlet density.*
2. *Let  $\Omega \subset \mathcal{H}(d)$  be the set of all classes representing primes  $p \in b + a\mathbb{Z}$  with  $p \nmid d$ . Then  $\Omega\mathcal{H}(d)^2 = \Omega$  (that means,  $\Omega$  consists of full genera).*

*Proof.* We may assume that  $d^* \mid a$ ,  $\gcd(d, b) = 1$  and  $(\frac{d}{b}) = 1$  (otherwise we replace  $a$  by  $ad^*$  and consider all residue classes  $b' + ad^*\mathbb{Z}$ , where  $(\frac{d}{b'}) = 1$  and  $b' \equiv b \pmod{a}$ ). Since  $C_0 \rightarrow p_0$ , we have

$$\beta^{(a)}(b + a\mathbb{Z}) \mid K_d^* = \alpha_d(C_0) \mid K_d^*.$$

For a prime  $p \in b + a\mathbb{Z}$ , we have  $C_0 \rightarrow p$  if and only if  $(\alpha_d(C_0) \mid K_d^*)^{\pm 1}$  is the Frobenius automorphism for a prime divisor of  $p$  in  $K_d^*$ . By Čebotarev's theorem, this set has positive Dirichlet density.

A class  $C \in \mathcal{H}(d)$  represents some prime  $p \in b + a\mathbb{Z}$  if and only if  $\alpha_d(C) \mid K_d^* = \alpha_d(C_0) \mid K_d^*$ , and since  $\text{Gal}(K_d/K_d^*) = \alpha_d(\mathcal{H}(d)^2)$ , this is equivalent to  $C \in C_0\mathcal{H}(d)^2$ . Hence we obtain  $\Omega = C_0\mathcal{H}(d)^2 = \Omega\mathcal{H}(d)^2$ .  $\square$

### 3. The main results

For a discriminant  $d$ , we denote by  $\mathcal{H}_2(d)$  the 2-Sylow subgroup and by  $\mathcal{H}'(d)$  the odd part of  $\mathcal{H}(d)$ , so that  $\mathcal{H}(d) = \mathcal{H}_2(d) \times \mathcal{H}'(d)$ .

**Theorem 1.** *Let  $d$  be a discriminant,  $m$  an odd positive integer and  $\Omega \subset \mathcal{H}(d)^m$  a set of classes satisfying  $\Omega\mathcal{H}(d)^{2m} = \Omega$ . Then there is a subset  $\Delta \subset X_d$  such that, for every prime  $p \nmid d$ , if  $p^m + d^*\mathbb{Z} \in \Delta$  then  $C \rightarrow p^m$  for some  $C \in \Omega$ .*

*Proof.* Suppose  $\Omega = \Omega_0^m$ , where  $\Omega_0 \subset \mathcal{H}(d)$ , and set  $\Delta = \omega_d^{-1}(\Omega_0\mathcal{H}(d)^2) \subset X_d$ . Let  $p$  be a prime such that  $p \nmid d$  and  $p^m + d^*\mathbb{Z} \in \Delta$ . Since  $m$  is odd, we obtain  $p + d^*\mathbb{Z} \in X_d$  and  $\omega_d(p + d^*\mathbb{Z}) = \omega_d(p^m + d^*\mathbb{Z}) = C_0\mathcal{H}(d)^2$  for some class  $C_0 \in \Omega_0$ . Hence there exists some  $A \in \mathcal{H}(d)$  such that  $C_0A^2 \rightarrow p$ , and if  $C = (C_0A^2)^m$ , then  $C \rightarrow p^m$  and  $C \in \Omega_0^m\mathcal{H}(d)^{2m} = \Omega$ .  $\square$

The assumption  $\Omega\mathcal{H}(d)^{2m} = \Omega$  made in Theorem 1 is very restrictive. But as we shall see in Theorem 2, it is necessary. We investigate its effect in the special case  $\Omega = \{C, C^{-1}\}$  for some  $C \in \mathcal{H}(d)$ . Note that the following

(simple) Proposition 2 remains true if we replace  $\mathcal{H}(d)$  by any finite abelian group.

**Proposition 2.** *Let  $d$  be a discriminant,  $m$  an odd positive integer and  $C \in \mathcal{H}(d)$  a class satisfying  $\{C, C^{-1}\}\mathcal{H}(d)^{2m} = \{C, C^{-1}\}$ . Then we have  $\mathcal{H}'(d)^m = \{1\}$ , and either  $\mathcal{H}_2(d)^2 = \{1\}$  or  $C^4 = I$  and  $\mathcal{H}_2(d) = \langle C \rangle \times \mathcal{H}_2^*(d)$ , where  $\mathcal{H}_2^*(d)^2 = \{1\}$ .*

*Proof.* By assumption,

$$|\mathcal{H}(d)^{2m}| \leq |\{C, C^{-1}\}\mathcal{H}(d)^{2m}| \leq 2,$$

and since  $\mathcal{H}(d)^{2m} = \mathcal{H}_2(d)^2 \times \mathcal{H}'(d)^m$ , we obtain  $\mathcal{H}'(d)^m = \{1\}$  and  $|\mathcal{H}_2(d)^2| \leq 2$ . Suppose that  $\mathcal{H}_2(d)^2 = \langle A^2 \rangle$  for some  $A \in \mathcal{H}_2(d)$  with  $A^4 = I$ ,  $A^2 \neq I$ . Then  $CA^2 = CA^{2m} \in \{C, C^{-1}\}$ , hence  $CA^2 = C^{-1}$  and therefore  $C^4 = I$ .  $\square$

Now we formulate our main results (Theorems 2, 3 and 4) which will be proved in a uniform way later on.

**Theorem 2.** *Let  $d$  be a discriminant, let  $a$  and  $m$  be positive integers, and let  $\Gamma \subset \mathcal{H}(d)$  and  $\Delta \subset (\mathbb{Z}/a\mathbb{Z})^\times$  be any subsets. Suppose that for every prime  $p$  (except possibly a set of Dirichlet density zero) the following holds: If  $\left(\frac{d}{p}\right) = 1$  and  $p^m + a\mathbb{Z} \in \Delta$ , then  $C \rightarrow p^m$  for some  $C \in \Gamma$ .*

*Let  $\Omega$  be the set of all classes  $C \in \mathcal{H}(d)$  representing some prime power  $p^m$  such that  $p \nmid d$  and  $p^m + a\mathbb{Z} \in \Delta$ , and assume that  $\Gamma \subset \Omega$ . Then*

$$\Omega = \Omega\mathcal{H}(d)^{2m} = \Gamma \cup \Gamma^{-1} \subset \mathcal{H}(d)^m,$$

*where  $\Gamma^{-1} = \{C \in \mathcal{H}(d) \mid C^{-1} \in \Gamma\}$ . In particular,  $\Omega$  consists of full cosets modulo  $\mathcal{H}(d)^{2m}$ .*

From a qualitative point of view, Theorem 2 asserts that either  $\Gamma$  is large or  $\mathcal{H}(d)^{2m}$  is small. This will become plain in Theorem 4, when we will consider the case  $|\Gamma| = 1$ . The subsequent Theorem 3 generalizes [7], Theorem 1.

**Theorem 3.** *Let assumptions be as in Theorem 2. Let  $K \in \mathcal{H}(d)$  and  $k \in \mathbb{Z}$  be such that  $K \rightarrow k$  and  $\gcd(k, ad) = 1$ .*

*Then for every prime  $p$  satisfying  $\left(\frac{d}{p}\right) = 1$  and  $p^m + a\mathbb{Z} \in k^m\Delta$ , there exists some  $C \in \Omega$  such that  $K^m C \rightarrow p^m$ .*

**Theorem 4.** *Let assumptions be as in Theorem 2, and suppose in addition that  $\Gamma = \{C\}$  consists of a single class. Then the following holds.*

1.  $|\Omega| = |\mathcal{H}(d)^{2m}| \leq 2$ .
2. Suppose that  $m = 2^t m'$ , where  $t \geq 0$  and  $m' \in \mathbb{N}$  is odd, and let  $\mathcal{H}_2(d)$  be of type  $(2^{t_1}, 2^{t_2}, \dots, 2^{t_s})$ , where  $s \geq 0$  and  $t_1 \geq t_2 \geq \dots \geq t_s \geq 1$ .



Then  $\mathcal{H}'(d)^{m'} = \{1\}$ ,  $t_2 \leq t + 1$  and

$$t_1 \leq \begin{cases} t + 2, & \text{if } C \neq C^{-1}, \\ t + 1, & \text{if } C = C^{-1}. \end{cases}$$

3. Suppose in addition that  $m = 1$ , and let  $\Delta'$  be the set of all residue classes  $p + d^*\mathbb{Z} \in X_d$  of primes  $p$  such that  $p + a\mathbb{Z} \in \Delta$ . Then we have  $|\omega_d(\Delta')| = 1$  and  $\mathbb{Q}^{(d^*)} \subset \mathbb{Q}^{(a)}(\sqrt{d})$ .

**Remark.** Kaplan and Williams [7] considered the case  $m = 1$ ,  $\Gamma = \{C\}$  and  $\Delta = \{b + a\mathbb{Z}\}$  for some  $b \in \mathbb{Z}$  such that  $\gcd(a, b) = 1$ . They assumed moreover that  $a$  is even and that every prime  $p \in b + a\mathbb{Z}$  with  $p \nmid d$  is represented by  $C$ . Then every prime  $p \in b + a\mathbb{Z}$  with  $p \nmid d$  satisfies  $\left(\frac{d}{p}\right) = 1$ . Therefore it follows that  $\mathbb{Q}(\sqrt{d}) \subset \mathbb{Q}^{(a)}$ , hence  $\mathbb{Q}^{(d^*)} \subset \mathbb{Q}^{(a)}$  and  $d^* \mid a$ , since  $a$  is even.

*Proof of the Theorems.* Let  $\Delta_0 \subset (\mathbb{Z}/a\mathbb{Z})^\times$  be the set of all residue classes  $p + a\mathbb{Z}$  of primes  $p$  such that  $\left(\frac{d}{p}\right) = 1$  and  $p^m + a\mathbb{Z} \in \Delta$ . We may assume that  $\Delta = \Delta_0^m \subset (\mathbb{Z}/a\mathbb{Z})^\times$  (the other residue classes of  $\Delta$  are of no interest). Let  $\Gamma_0$  be the set of all classes  $C \in \mathcal{H}(d)$  such that  $C^m \in \Gamma$ . Since  $\Gamma \subset \mathcal{H}(d)^m$  by assumption, we have  $\Gamma = \Gamma_0^m$ . For the same reason,  $\Omega = \Omega_0^m$ , where  $\Omega_0$  is the set of all classes  $C \in \mathcal{H}(d)$  representing some prime  $p$  satisfying  $p \nmid d$  and  $p + a\mathbb{Z} \in \Delta_0$ . Now  $\Omega_0 = \Omega_0 \mathcal{H}(d)^2$  consists of full genera by Proposition 1, and therefore  $\Omega = \Omega_0^m = \Omega_0^m \mathcal{H}(d)^{2m} = \Omega \mathcal{H}(d)^{2m}$ .

If  $C \in \Omega$ , then  $C = C_0^m$  for some  $C_0 \in \Omega_0$  and (by Proposition 1) the set of all primes  $p$  such that  $p + a\mathbb{Z} \in \Delta_0$  and  $C_0 \rightarrow p$  has positive Dirichlet density. Hence the set of all primes  $p$  such that  $p^m + a\mathbb{Z} \in \Delta$  and  $C \rightarrow p$  has positive Dirichlet density, too. Therefore there exists some  $C' \in \Gamma$  representing a prime power which is also represented by  $C$ , hence  $C \in \{C', C'^{-1}\} \subset \Gamma \cup \Gamma^{-1}$ , and  $\Omega \subset \Gamma \cup \Gamma^{-1}$  follows. The other inclusion is obvious, since  $\Gamma$  and  $\Gamma^{-1}$  represent the same prime powers. This argument completes the proof of Theorem 2.

For the proof of Theorem 3, let  $p$  be a prime satisfying  $\left(\frac{d}{p}\right) = 1$  and  $p^m + a\mathbb{Z} \in k^m \Delta$ . Let  $p_0$  be a prime satisfying  $p \equiv kp_0 \pmod{ad}$ . Then

$$1 = \left(\frac{d}{p}\right) = \left(\frac{d}{k}\right) \left(\frac{d}{p_0}\right) = \left(\frac{d}{p_0}\right),$$

and  $p^m \equiv k^m p_0^m \pmod{a}$  implies  $p_0 + a\mathbb{Z} \in \Delta_0$ , whence  $C_0 \rightarrow p_0$  for some  $C_0 \in \Gamma_0$ . Let  $C_1 \in \mathcal{H}(d)$  be such that  $C_1 \rightarrow p$ . Then  $C_1^m \rightarrow p^m$ ,

$$C_1 \mathcal{H}(d)^2 = \omega_d(p + d^*\mathbb{Z}) = \omega_d(k + d^*\mathbb{Z}) \omega_d(p_0 + d^*\mathbb{Z}) = KC_0 \mathcal{H}(d)^2,$$

and therefore  $C_1 = KC_0 A^2$  for some  $A \in \mathcal{H}(d)^2$ , which implies  $C_1^m = K^m C$ , where  $C = C_0^m A^{2m} \in \Omega$ .

It remains to prove Theorem 4. Suppose that  $\Gamma = \{C\}$  and  $C = C_1^m$ . Then

$$\Omega = \Omega\mathcal{H}(d)^{2m} = \{C, C^{-1}\} = C_1^m\mathcal{H}(d)^{2m},$$

and therefore

$$|\Omega| = |\mathcal{H}(d)^{2m}| = \begin{cases} 2, & \text{if } C \neq C^{-1}, \\ 1, & \text{if } C = C^{-1}. \end{cases}$$

Since  $\mathcal{H}(d)^{2m} = \mathcal{H}'(d)^{m'} \times \mathcal{H}_2(d)^{2^{t+1}}$  and  $\mathcal{H}_2(d)^{2^{t+1}}$  is of type  $(2^{((t_1-t-1))}, \dots, 2^{((t_s-t-1))})$ , where  $((r)) = \max\{r, 0\}$ , the assertions 1. and 2. of Theorem 4 follow.

If in addition  $m = 1$ , then clearly  $|\omega_d(\Delta')| = 1$ . Also, for every prime  $p$  with  $\left(\frac{d}{p}\right) = 1$ , the residue class  $p + d^*\mathbb{Z}$  is uniquely determined by  $p + a\mathbb{Z}$ . Therefore Čebotarev's theorem implies  $\mathbb{Q}^{(d^*)} \subset \mathbb{Q}^{(a)}(\sqrt{d})$ .  $\square$

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