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On Minkowski units constructed by special values of Siegel modular functions

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Résumé. Nous construisons les unités de Minkowski du corps de classes de rayon modulo 6, k_6, de $\mathbb{Q}(\exp(2\pi i/5))$ en utilisant les valeurs spéciales des fonctions modulaires de Siegel. Notre travail utilise la décomposition en idéaux premiers des valeurs spéciales et la description de l'action du groupe de Galois $G(k_6/\mathbb{Q})$ sur ces valeurs spéciales. De plus, sous GRH, nous décrivons entièrement le groupe des unités de k_6 à partir unités modulaires et des unités circulaires.

Abstract. Using the special values of Siegel modular functions, we construct Minkowski units for the ray class field k_6 of $\mathbb{Q}(\exp(2\pi i/5))$ modulo 6. Our work is based on investigating the prime decomposition of the special values and describing explicitly the action of the Galois group $G(k_6/\mathbb{Q})$ for the special values. Futhermore we construct the full unit group of k_6 using modular and circular units under the GRH.

1. Theorems

In our previous paper [1], we constructed a group of units with full rank for the ray class field k_6 of $\mathbb{Q}(\exp(2\pi i/5))$ modulo 6 using special values of Siegel modular functions and circular units. In this paper, we construct Minkowski units in k_6 using only the special values of Siegel modular functions. In these works, it is essential that $\mathbb{Q}(\exp(2\pi i/5))$ is the CM-field corresponding to the Jacobian variety of the curve $y^2 = 1 - x^5$.

We use the same notations as in [1]. So we explain notations briefly. We denote as usual by $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ the rational integer ring, the rational number field, the real number field and the complex number field, respectively. For a positive integer n, let I_n be the unit matrix of degree n and $\zeta_n = \exp(2\pi i/n)$. Let $\mathfrak{S}_2$ be the set of all complex symmetric matrices of degree 2 with positive definite imaginary parts. For $u \in \mathbb{C}^2$, $z \in \mathfrak{S}_2$ and

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$r, s \in \mathbb{R}^2$, put as usual

$$\Theta(u, z; r, s) = \sum_{x \in \mathbb{Z}^2} e\left(\frac{1}{2}{}^t(x+r)z(x+r) + {}^t(x+r)(u+s)\right),$$

where $e(\xi) = \exp(2\pi i \xi)$ for $\xi \in \mathbb{C}$. Let N be a positive integer. If we define

$$\Phi(z; r, s; r_1, s_1) = \frac{2\Theta(0, z; r, s)}{\Theta(0, z; r_1, s_1)}$$

for $r, s, r_1, s_1 \in \frac{1}{N}\mathbb{Z}^2$, then $\Phi(z; r, s; r_1, s_1)$ is a Siegel modular function of level $2N^2$. Let

$$J = \begin{pmatrix} 0 & -I_2 \\ I_2 & 0 \end{pmatrix} \quad \text{and} \quad \Gamma_1 = S_p(2, \mathbb{Z}) = \{\, \alpha \in GL_4(\mathbb{Z}) \mid {}^t\alpha J \alpha = J \,\}.$$

We let every element $\alpha = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ act on $\mathfrak{S}_2$ by $\alpha(z) = (Az + B)(Cz + D)^{-1}$ for $z \in \mathfrak{S}_2$.

Let α be a matrix in $M_4(\mathbb{Z})$ such that ${}^t\alpha J \alpha = vJ$ and $\det(\alpha) = v^2$ with positive integer v prime to $2N$. Then there exists a matrix β_α in Γ_1 with

$$\alpha \equiv \begin{pmatrix} I_2 & 0 \\ 0 & vI_2 \end{pmatrix} \beta_\alpha \pmod{2N^2}$$

by the strong approximation theorem for $S_p(2, \mathbb{Z})$. We let α act on $\Phi(z; r, s; r_1, s_1)$ by $\Phi^\alpha(z\,;\, r, s\,;\, r_1, s_1) = \Phi(\beta_\alpha(z)\,;\, r, vs\,;\, r_1, vs_1)$. Then Φ^α is also a Siegel modular function of level $2N^2$.

Remark . Our definition of Φ^α differs from that of [5]. Let $GS_p(A)$ be the adelization of the group of the symplectic similitudes $S_p(2, \mathbb{Q})$. View $\alpha \in GS_p(\mathbb{Q}) \subset GS_p(A)$ and write $\alpha' \in GS_p(A)$ to be the projection of α to $\prod_{p|2N} GS_p(\mathbb{Q}_p) \subset GS_p(A)$. Then our action of α is Shimura's action by α'.

In what follows, we fix $\zeta = \zeta_5$ and $k = \mathbb{Q}(\zeta)$. Let σ be the element of the Galois group $G(k/\mathbb{Q})$ with $\zeta^\sigma = \zeta^2$ and define the endmorphism φ of $k^\times$ by $\varphi(a) = a^{1+\sigma^3}$ for $a \in k^\times$.

Moreover put

$$\begin{aligned} z_0 &= \begin{pmatrix} \zeta^2 + \zeta^4 & \zeta^3 \\ \zeta^4 + \zeta^3 & \zeta \end{pmatrix}^{-1} \begin{pmatrix} -\zeta & \zeta^4 \\ -\zeta^2 & \zeta^3 \end{pmatrix} \\ &= \frac{1}{5} \begin{pmatrix} 2 + \zeta - \zeta^3 - 2\zeta^4 & 2 - \zeta + \zeta^2 - 2\zeta^3 \\ 2 - \zeta + \zeta^2 - 2\zeta^3 & \zeta + 2\zeta^2 - 2\zeta^3 - \zeta^4 \end{pmatrix}. \end{aligned}$$

We note that z_0 is a CM-point of $\mathfrak{S}_2$ associated to a Fermat curve $y^2 = 1 - x^5$. For an element ω in the integer ring $\mathfrak{O}_k$ of k, let $R(\omega)$ be the the regular representation of ω with respect to the basis $\{\, -\zeta,\ \zeta^4,\ \zeta^2 + \zeta^4,\ \zeta^3 \,\}$.

Then $R(\varphi(\omega))z_0 = z_0$, ${}^tR(\varphi(\omega))JR(\varphi(\omega)) = vJ$ and $\det R(\varphi(\omega)) = v^2$, where $v = N_{k/\mathbb{Q}}(\omega)$. Futhermore we put

$$\Psi(z\,;\,r_1,r_2,r_3,r_4\,;\,s_1,s_2,s_3,s_4) = \Phi(z\,;\begin{pmatrix}r_1/6\\r_2/6\end{pmatrix},\begin{pmatrix}r_3/6\\r_4/6\end{pmatrix};\begin{pmatrix}s_1/6\\s_2/6\end{pmatrix},\begin{pmatrix}s_3/6\\s_4/6\end{pmatrix})$$

for $r_i, s_i \in \mathbb{Z}$

The main purpose of this paper is to prove the following:

Theorem 1.1. *Let $k = \mathbb{Q}(\exp(2\pi i/5))$ and k_6 the ray class field of k modulo 6. We put*

$$\varepsilon = \left(\frac{\Psi(z_0\,;\,2,0,0,0\,;\,0,0,0,0)}{\Psi(z_0\,;\,2,4,2,0\,;\,0,0,0,0)}\right)^3.$$

Then ε is a unit in k_6 and the conjugates of ε with respect to k_6 over $\mathbb{Q}$ generate a unit group of full rank 19.

We note that k_6 is a Galois extension of $\mathbb{Q}$ and $k_6 = \mathbb{Q}(\zeta_3, \sqrt[5]{24}, \zeta)$. Now we put

$$\theta_{ijk} = \zeta_3^i(24 - \sqrt[5]{24})^j(1-\zeta)^k/d_{jk}$$

for $0 \le i \le 1$, $0 \le j \le 4$, $0 \le k \le 3$, where

$$d_{jk} = \begin{cases} 1 & \text{if } (j,k) = (0,0),(0,1),(0,2),(0,3),(1,0),(1,1),(1,2), \\ 2 & \text{if } (j,k) = (2,0),(2,1),(3,0), \\ 5 & \text{if } (j,k) = (1,3), \\ 10 & \text{if } (j,k) = (2,2),(2,3),(3,1),(3,2),(3,3), \\ 20 & \text{if } (j,k) = (4,0),(4,1),(4,2),(4,3). \end{cases}$$

Let M be the $\mathbb{Z}$-module generated by θ_{ijk} in k_6. Then M is a free submodule of the integer ring $\mathfrak{O}_{k_6}$ whose rank is 40 (cf. [4]). It is easy to see that $\mathfrak{O}_{k_6}/M \cong (\mathbb{Z}/3\mathbb{Z})^8$ using PARI. Hence, if α is an integer of k_6, then 3α has rational integral coefficients with respect to $\{\theta_{ijk}\}$. Our second result is the algebraic expression of ε which was defined analytically.

Theorem 1.2. *Let ε be as in Theorem 1.1. Then we have*

$$\begin{aligned} 3\varepsilon = & -2858219\theta_{000} - 1424317\theta_{001} + 4914337\theta_{002} - 1993566\theta_{003} + 497437\theta_{010} \\ & + 189637\theta_{011} - 786337\theta_{012} + 1625677\theta_{013} - 6465\theta_{020} - 1791\theta_{021} \\ & + 471277\theta_{022} - 198787\theta_{023} + 186\theta_{030} + 171\theta_{031} - 1254\theta_{032} + 540\theta_{033} \\ & - 20\theta_{040} - 2\theta_{041} + 25\theta_{042} - 11\theta_{043} - 873162\theta_{100} + 1354434\theta_{101} \\ & - 721941\theta_{102} + 119499\theta_{103} + 145188\theta_{110} - 226944\theta_{111} + 122211\theta_{112} \\ & - 102510\theta_{113} - 18096\theta_{120} + 28494\theta_{121} - 77484\theta_{122} + 13164\theta_{123} \\ & + 501\theta_{130} - 3972\theta_{131} + 2181\theta_{132} - 375\theta_{133} - 52\theta_{140} + 83\theta_{141} \\ & - 46\theta_{142} + 8\theta_{143}\,. \end{aligned}$$

2. Proof of Theorem 1.1

We extend σ to the element of $G(k_6/\mathbb{Q}(\zeta_3, \sqrt[5]{24}))$ and let τ be the Frobenius automorphism $\left(\frac{k_6/k}{(\zeta+2)}\right)$. Then $G(k_6/\mathbb{Q}) =< \sigma, \tau >$. The action of τ for k_6 is given by

$$\zeta_3^\tau = \zeta_3^2 \quad \text{and} \quad \sqrt[5]{24}^\tau = \zeta^2 \sqrt[5]{24} \tag{1}$$

because $N_{k/\mathbb{Q}}(\zeta+2) = 11$, $-\zeta \equiv 2 \equiv 24 \pmod{(\zeta+2)}$ and $\sqrt[5]{24}^{11} = 24^2\sqrt[5]{24} \equiv (-\zeta)^2\sqrt[5]{24} = \zeta^2\sqrt[5]{24} \pmod{\zeta+2}$. Put $\alpha_1 = \Psi(z_0\,;\, 2,0,0,0\,;\, 0,0,0,0)^3$ and $\alpha_2 = \Psi(z_0\,;\, 2,4,2,0\,;\, 0,0,0,0)^3$. Then α_i is an algebraic integer in k_6 (cf. Prop. 2 of [3], [2]). The action of τ for α_i is determined by Shimura's reciprocity law. Namely we see that

$$\Phi(z_0\,;\, r,s\,;\, r_1,s_1)^{3\tau} = \Phi(\beta(z_0)\,;\, r, 11s\,;\, r_1, 11s_1)^3$$

for $r, s, r_1, s_1 \in \frac{1}{6}\mathbb{Z}^2$, where

$$\beta = \begin{pmatrix} 3 & 0 & -1 & 1 \\ 2 & 2 & 0 & -1 \\ -111 & 26 & 46 & -59 \\ 26 & -13 & -13 & 20 \end{pmatrix} \in \Gamma_1 .$$

On the other hand, it is known that

$$\Phi(\beta(z_0)\,;\, r, 11s\,;\, r_1, 11s_1)^3 = \Phi(z_0\,;\, r', s'\,;\, r_1', s_1')^3 \zeta_{24}^m \tag{2}$$

for some integer m. Here, r', s', r_1', s_1' are elements in $\frac{1}{6}\mathbb{Z}^2$ determined from r, s, r_1, s_1 explicitly by translation formula for theta series. Since the convergence of the left side of (2) is slow, we use the right side of (2) for high precision calculation after we determined the actual value of m by calculating approximately both sides of (2) with low precision. In this manner, we get rapid convergent formulae of $\alpha_1^{\tau^i}$. For example,

$$\begin{aligned} \alpha_1^\tau &= \Psi(z_0\,;\, 3,0,4,5\,;\, 3,0,0,3)^3, \\ \alpha_1^{\tau^2} &= \Psi(z_0\,;\, 2,3,4,4\,;\, 0,3,0,0)^3 \zeta_6^4, \\ \alpha_1^{\tau^3} &= \Psi(z_0\,;\, 1,2,4,2\,;\, 3,0,0,0)^3 \zeta_6^4. \end{aligned}$$

In [1], we showed that $N_{k_6/\mathbb{Q}}(\alpha_i) = 2^{48}$. This was done by examining all possibilities of α_i^σ, $\alpha_i^{\sigma^2}$ and $\alpha_i^{\sigma^3}$. In a similar manner, we can show that

$$N_{k_6/\mathbb{Q}(\sqrt{-15})}(\alpha_1) = N_{k_6/\mathbb{Q}(\sqrt{-15})}(\alpha_2) = 2^{24}$$

noting that $G(k_6/\mathbb{Q}(\sqrt{-15})) = \{\, \tau^{2i}, \sigma\tau^{2i+1}, \sigma^2\tau^{2i}, \sigma^3\tau^{2i+1} \mid 0 \le i \le 4 \,\}$.

Since the ramification index of 2 in k_6 over $\mathbb{Q}$ is 5 and the relative degree is 4, exactly two prime ideals $\mathfrak{P}_1$ and $\mathfrak{P}_2$ divide 2 in k_6. We put $\mathfrak{p}_i = \mathfrak{P}_i \cap \mathbb{Q}(\sqrt{-15})$ for $i = 1, 2$. Since 2 splits in $\mathbb{Q}(\sqrt{-15})$, $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are distinct.

This means $(\alpha_1) = (\alpha_2) = \mathfrak{P}_1^6\mathfrak{P}_2^6$ in k_6, which shows that $\varepsilon = \alpha_1/\alpha_2$ is a unit in k_6.

In the next section, we will prove Theorem 1.2 without assuming Theorem 1.1. We can calculate the approximate values of $\log|\varepsilon^\rho|$ $(\rho \in G(k_6/\mathbb{Q}))$ with arbitrary precision using Theorem 1.2. It is a routine work to verify that the rank of the 20×20 matrix $(\log|a_{ij}|)$ is 19, where

$$a_{ij} = \begin{cases} \varepsilon^{\tau^i\tau^j} = \varepsilon^{\tau^{i+j}} & \text{if } 0 \le i,j \le 9, \\ \varepsilon^{\tau^i\sigma\tau^j} = \varepsilon^{\sigma\tau^{7i+j}} & \text{if } 0 \le i \le 9,\ 10 \le j \le 19, \\ \varepsilon^{\sigma\tau^i\tau^j} = \varepsilon^{\sigma\tau^{i+j}} & \text{if } 10 \le i \le 19,\ 0 \le j \le 9, \\ \varepsilon^{\sigma\tau^i\sigma\tau^j} = \varepsilon^{\sigma^2\tau^{7i+j}} & \text{if } 10 \le i,j \le 19. \end{cases}$$

3. Proof of Theorem 1.2

Since α_1 is an algebraic integer in k_6, $3\alpha_1$ has rational integral coefficients with respect to $\{\theta_{ijk}\}$:

$$3\alpha_1 = \sum_{0\le i\le 1,\, 0\le j\le 4,\, 0\le k\le 3} a_{ijk}\theta_{ijk} \qquad (a_{ijk} \in \mathbb{Z}).$$

If we know the actions of σ, σ^2 and σ^3 for α_1, then we can easily determine a_{ijk} by solving numerically the simultaneous equations

$$(3) \qquad \sum_{0\le i\le 1,\, 0\le j\le 4,\, 0\le k\le 3} a_{ijk}\theta_{ijk}^\rho = 3\alpha_1^\rho \qquad (\rho \in G(k_6/\mathbb{Q})).$$

In the preceding section, we computed all possibilities of α_1^σ, $\alpha_1^{\sigma^2}$ and $\alpha_1^{\sigma^3}$. Our computation shows that

$$\begin{aligned} \alpha_1^\sigma &= \Psi(z_0\,;\,1,1,0,0\,;\,3,3,0,0)^{3\tau^{i_1}}\zeta_6^{m_1}, \\ \alpha_1^{\sigma^2} &= \Psi(z_0\,;\,2,0,0,0\,;\,0,0,0,0)^{3\tau^{i_2}}\zeta_6^{m_2}, \\ \alpha_1^{\sigma^3} &= \Psi(z_0\,;\,1,1,0,0\,;\,3,3,0,0)^{3\tau^{i_3}}\zeta_6^{m_3} \end{aligned}$$

for some $i_1, i_2, i_3, m_1, m_2, m_3 \in \mathbb{Z}$. We first determine m_1, m_2, m_3 so that all elementary symmetric polynomials of α_1^ρ are rational integers. We next determine i_1, i_2, i_3 so that (3) has a integral solution. Fortunately we have only one possibility $(i_1, i_2, i_3, m_1, m_2, m_3) = (4, 1, 1, 0, 0, 0)$ which means

$$\begin{aligned} \alpha_1^\sigma &= \Psi(z_0\,;\,3,1,1,3\,;\,3,3,3,3)^3\zeta_6^3, \\ \alpha_1^{\sigma^2} &= \Psi(z_0\,;\,3,0,4,5\,;\,3,0,0,3)^3, \\ \alpha_1^{\sigma^3} &= \Psi(z_0\,;\,2,2,5,3\,;\,0,0,3,3)^3\zeta_6^2 \end{aligned}$$

and hence

$$
\begin{aligned}
(4)\quad 3\alpha_1 = {} & 2161156\theta_{000} - 3650004\theta_{001} + 3404776\theta_{002} - 1246668\theta_{003} \\
& - 366684\theta_{010} + 622226\theta_{011} - 577762\theta_{012} + 1055534\theta_{013} + 4668\theta_{020} \\
& - 7950\theta_{021} + 367732\theta_{022} - 133398\theta_{023} - 132\theta_{030} + 1128\theta_{031} \\
& - 1038\theta_{032} + 378\theta_{033} + 14\theta_{040} - 24\theta_{041} + 22\theta_{042} - 8\theta_{043} \\
& + 6618000\theta_{100} - 9918336\theta_{101} + 8276888\theta_{102} - 2650068\theta_{103} \\
& - 1104448\theta_{110} + 1656000\theta_{111} - 1380096\theta_{112} + 2210400\theta_{113} + 138818\theta_{120} \\
& - 207724\theta_{121} + 86370\theta_{122} - 276642\theta_{123} - 384\theta_{130} + 2880\theta_{131} \\
& - 2400\theta_{132} + 768\theta_{133} + 40\theta_{140} - 60\theta_{141} + 50\theta_{142} - 16\theta_{143}\,.
\end{aligned}
$$

Similarly we have

$$
\begin{aligned}
\alpha_2^{\sigma} &= \Psi(z_0\,;\,4,0,3,5\,;\,0,0,3,3)^3\zeta_6^4, \\
\alpha_2^{\sigma^2} &= \Psi(z_0\,;\,0,4,1,2\,;\,0,0,3,0)^3\zeta_6^2, \\
\alpha_2^{\sigma^3} &= \Psi(z_0\,;\,3,3,1,1\,;\,3,3,3,3)^3\zeta_6^4
\end{aligned}
$$

and

$$
\begin{aligned}
(5)\quad 3\alpha_2 = {} & -1495568\theta_{000} + 1907704\theta_{001} - 1074124\theta_{002} + 162248\theta_{003} \\
& + 269824\theta_{010} - 349024\theta_{011} + 203584\theta_{012} - 171364\theta_{013} - 3630\theta_{020} \\
& + 4752\theta_{021} - 14250\theta_{022} + 2586\theta_{023} + 108\theta_{030} - 714\theta_{031} + 438\theta_{032} \\
& - 84\theta_{033} - 12\theta_{040} + 16\theta_{041} - 10\theta_{042} + 2\theta_{043} - 189312\theta_{100} \\
& + 3130444\theta_{101} - 2138524\theta_{102} + 573724\theta_{103} + 320224\theta_{110} - 530464\theta_{111} \\
& + 364864\theta_{112} - 498244\theta_{113} - 4056\theta_{120} + 6732\theta_{121} - 232298\theta_{122} \\
& + 64684\theta_{123} + 114\theta_{130} - 948\theta_{131} + 660\theta_{132} - 186\theta_{133} - 12\theta_{140} \\
& + 20\theta_{141} - 14\theta_{142} + 4\theta_{143}\,.
\end{aligned}
$$

It is straightforward to deduce Theorem 1.2 from (4) and (5).

4. Construting the Unit Group Using Modular Units

It is natual to ask how large subgroups are generated by our units. In [1], we constructed a subgroup of E_{k_6} with full rank. Namely, if we put

$$
\begin{aligned}
\varepsilon_1 &= \Psi(z_0\,;\,2,0,0,0\,;\,0,0,0,0)^{1-\tau^2}, \\
\varepsilon_2 &= \Psi(z_0\,;\,2,4,2,0\,;\,0,0,0,0)^{1-\tau^2}, \\
\varepsilon_3 &= 1-\zeta_{15}\,,
\end{aligned}
$$

then $E_1 =< \varepsilon_i^{\rho} \mid 0 \leq i \leq 2\,, \rho \in G(k_6/\mathbb{Q}) >$ is a subgroup of E_{k_6} of free rank 19. Furthermore $E_2 =< \varepsilon^{\rho} \mid \rho \in G(k_6/\mathbb{Q}) >$ is also a subgroup of free rank 19, where ε is the unit defined in Theorem 1.1. Let W be the torsion subgroup of E_{k_6}. Then W is a cyclic group generated by ζ_{30}. We note that E_2 contains W because $\varepsilon_3^{\sigma^2-\tau} = \zeta_{30}^{23}$. On the other hand, the torsion subgroup of E_1 seems to be $< \zeta_3 >$.

Now, note that $k_6 = \mathbb{Q}(\zeta_{15} + \sqrt[5]{24})$. We gave the minimal polynomial of $\zeta_{15} + \sqrt[5]{24}$ to PARI's function `bnfinit`. Then Alpha 21264 of 667MHz computed a free basis of E_{k_6} in 30 hours under the GRH (Generalized Riemann Hypothesis). It is then a routine work to find $E_{k_6}/WE_1 \cong (\mathbb{Z}/5\mathbb{Z})^3$, $E_{k_6}/E_2 \cong (\mathbb{Z}/3\mathbb{Z})^{16} \oplus (\mathbb{Z}/120\mathbb{Z})$ and $E_{k_6}/E_1E_2 \cong \mathbb{Z}/5\mathbb{Z}$ (under GRH). Unfortunately the function `bnfcertify` failed to remove the assumption of GRH.

Next we construct a new unit. In the same manner as preceeding sections, we see that $\beta_1 = \Psi(z_0\,; 3,0,1,0\,;\, 0,0,0,0)^{12}$ and $\beta_2 = \Psi(z_0\,; 5,1,1,0\,;\, 0,0,0,0)^{12}$ are integers of k_6 and satisfy

$$N_{k_6/\mathbb{Q}}(\beta_1) = N_{k_6/\mathbb{Q}}(\beta_2) = (2^{16}3^2)^{12}\,.$$

If β_1/β_2 is an integer of k_6, then β_1/β_2 is a unit of k_6. It is not easy to prove theoretically the integrality of β_1/β_2. However, by expressing β_1 and β_2 with θ_{ijk}, we can show computationally that β_1/β_2 is an integer of k_6. Hence we obtain a new unit $\varepsilon_4 = \beta_1/\beta_2$. The free rank of $< \varepsilon_4^{\rho} \mid \rho \in G(k_6/\mathbb{Q}) >$ is 15. The unit ε_4 fills the gap between E_{k_6} and E_1E_2.

Theorem 4.1. *If the GRH is valid, then the full unit group E_{k_6} of k_6 is generated by the conjugates of ε_2, ε_3 and ε_4 (ε and ε_1 are not needed).*

Namely we suceeded in construting E_{k_6} using modular units and cyclotomic units. Finally we remark that the class number of k_6 is 1 again under GRH. This seems to suggest some relations between modular units and the class number.

5. Conclusion

The essential part of our work consists in the explicit description of the action of the Galois group $G(k_6/\mathbb{Q}) =< \sigma, \tau >$ on the special values of Siegel modular functions. Shimura's theory permits us to describe the action of τ. But no theory is known for that of σ. In this paper, we determined the action of σ by experiment based on numerical calculations. It is the next problem to construct a theory which enables us to handle the action of σ.

In a similar manner, we can construct another Minkowski units

$$\left(\frac{\Psi(z_0\,;\,2,0,0,0\,;\,3,0,0,3)}{\Psi(z_0\,;\,2,4,2,0\,;\,0,3,0,0)}\right)^{12},\quad \left(\frac{\Psi(z_0\,;\,2,0,0,0\,;\,0,3,0,0)}{\Psi(z_0\,;\,2,4,2,0\,;\,3,0,0,3)}\right)^{12}$$

and so on.

High precision calculations for theta series were done by TC, an interpreter of multi-precision C language, which is available from `ftp://tnt.math.metro-u.ac.jp/pub/math-packs/tc/` .

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