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On octahedral extensions of \mathbb{Q} and quadratic \mathbb{Q} -curves

par JULIO FERNÁNDEZ

RÉSUMÉ. On donne une condition nécessaire pour qu'une représentation surjective $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{PGL}_2(\mathbb{F}_3)$ provienne de la 3-torsion d'une \mathbb{Q} -courbe. Nous étudions plus particulièrement le cas des \mathbb{Q} -courbes quadratiques.

ABSTRACT. We give a necessary condition for a surjective representation $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{PGL}_2(\mathbb{F}_3)$ to arise from the 3-torsion of a \mathbb{Q} -curve. We pay a special attention to the case of quadratic \mathbb{Q} -curves.

1. Introduction

Let C be a \mathbb{Q} -curve defined over a number field k , that is an elliptic curve over k without complex multiplication and isogenous to all its Galois conjugates. Throughout, we will denote by G_k and $G_{\mathbb{Q}}$ the absolute Galois groups $\text{Gal}(\overline{\mathbb{Q}}/k)$ and $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, respectively. Let

$$\rho_C : G_k \longrightarrow \text{GL}_2(\mathbb{F}_3)$$

be the representation given by the Galois action on the 3-torsion points of C . Then, there exists an odd representation

$$\varrho_C : G_{\mathbb{Q}} \longrightarrow \text{PGL}_2(\mathbb{F}_3)$$

whose restriction to G_k is lifted to $\text{GL}_2(\mathbb{F}_3)$ by ρ_C . The representation ϱ_C comes from the Galois action on the 3-torsion of the abelian varieties of GL_2 -type having the curve C as a quotient (see the proof of Theorem 2.1).

The fixed field of ϱ_C , which we will denote by K_C , has Galois group over \mathbb{Q} inside the symmetric group \mathcal{S}_4 , since this last group is isomorphic to $\text{PGL}_2(\mathbb{F}_3)$. The behaviour of the restriction of ϱ_C to G_k implies the following property for the field K_C : its compositum with k is the extension

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generated by the x -coordinates of the 3-torsion points of C , with respect to any Weierstrass equation for C over k . Whenever k/\mathbb{Q} is Galois and ρ_C is surjective, K_C is the only \mathcal{S}_4 -extension of \mathbb{Q} satisfying that property; a proof of this claim is given in the appendix. We recall that ϱ_C being odd amounts to K_C not being real.

We will say that a representation

$$\varrho : G_{\mathbb{Q}} \longrightarrow \mathrm{PGL}_2(\mathbb{F}_3)$$

arises from a \mathbb{Q} -curve C if $\varrho = \varrho_C$, where this last equality is considered up to conjugation inside $\mathrm{PGL}_2(\mathbb{F}_3)$. Any such representation ϱ must be odd.

In section 2 we give a necessary condition for a surjective representation ϱ as above to arise from a given \mathbb{Q} -curve, in terms of the trace quadratic form attached to any quartic subextension of the fixed field of ϱ . In section 3 we focus on the case of elliptic curves defined over quadratic fields.

2. The sign component in $\mathrm{Br}_2(\mathbb{Q})$ attached to a \mathbb{Q} -curve

Let C/k be a \mathbb{Q} -curve. From any locally constant set of isogenies from C to its $G_{\mathbb{Q}}$ -conjugates, one can attach to C an invariant $\xi_C \in H^2(G_{\mathbb{Q}}, \mathbb{Q}^*)$ of its isogeny class (see [5], Proposition 2.1, and also [7], section 6). The sign component of ξ_C , denoted by ξ_C^{\pm} , is an element in $\mathrm{Br}_2(\mathbb{Q})$, the 2-torsion of the Brauer group of \mathbb{Q} . This element is related to the *complete definition* of the \mathbb{Q} -curve (cf. [5] and [6]): assuming k to be the minimal field of definition for C up to isogeny, which is a polyquadratic extension of \mathbb{Q} , the existence of a k -twist C' of C with all the isogenies

$$\sigma C' \longrightarrow C' \quad \sigma \in \mathrm{Gal}(k/\mathbb{Q})$$

defined over k amounts to the existence of a double cover of the group $\mathrm{Gal}(k/\mathbb{Q})$ whose corresponding embedding problem has obstruction given by ξ_C^{\pm} .

The sign component ξ_C^{\pm} is explicitly given in [5], Theorem 3.1, as a product of quaternion algebras in terms of the minimal field of definition for C up to isogeny and the degrees of the isogenies between C and its Galois conjugates. Our first result gives another expression for ξ_C^{\pm} depending only on the octahedral extension K_C/\mathbb{Q} in the introduction.

Theorem 2.1. *Assume that ϱ_C is surjective, and let K_C be defined as in section 1. Then, the sign component of ξ_C is given by the following product in $\mathrm{Br}_2(\mathbb{Q})$:*

$$\xi_C^{\pm} = w_C(-2, d_C)(-1, -3),$$

where d_C and w_C are, respectively, the discriminant and the Witt invariant of the trace quadratic form attached to any quartic subextension of K_C/\mathbb{Q} .

Proof. For any character $\eta: G_{\mathbb{Q}} \rightarrow F^*$, with F an algebraically closed field, let $[\eta]$ be the element in $\text{Br}_2(\mathbb{Q})$ giving the obstruction to the existence of a character $\psi: G_{\mathbb{Q}} \rightarrow F^*$ such that $\psi^2 = \eta$. Let us consider these two particular cases:

- For the mod 3 cyclotomic character

$$\chi: G_{\mathbb{Q}} \longrightarrow \text{Gal}(\mathbb{Q}(\sqrt{-3})/\mathbb{Q}) \simeq \{\pm 1\},$$

$[\chi]$ is given by the quaternion algebra $(-1, -3)$.

- For any lifting

$$\rho: G_{\mathbb{Q}} \longrightarrow \text{GL}_2(\overline{\mathbb{F}}_3)$$

of the projective representation ϱ_C , the element $[\det \rho]$ gives, by Theorem 6 in [8] and Proposition 1.2 in [4], the obstruction to the solvability of the embedding problem

$$2S_4^- \longrightarrow S_4 \simeq \text{Gal}(K_C/\mathbb{Q}),$$

where $2S_4^-$ is the only double cover of S_4 which can be embedded into $\text{SL}_2(\overline{\mathbb{F}}_3)$. That obstruction can be expressed (see [9], Théorème 1, and [10], section 2) in terms of the trace quadratic form attached to any quartic subextension of K_C/\mathbb{Q} , so that we have the equality

$$[\det \rho] = w_C(-2, d_C)$$

in $\text{Br}_2(\mathbb{Q})$.

Every lifting of ϱ_C into $\text{GL}_2(\overline{\mathbb{F}}_3)$ is obtained, up to isomorphism, as follows. Let A/\mathbb{Q} be an abelian variety having the \mathbb{Q} -curve C as a quotient and with \mathbb{Q} -endomorphism algebra $\mathbb{Q} \otimes \text{End}_{\mathbb{Q}}(A)$ a number field of degree $\dim(A)$. By [7], Theorem 6.1, such an abelian variety exists, and we can also assume, replacing A by a \mathbb{Q} -isogenous abelian variety if necessary, that the \mathbb{Q} -endomorphism ring of A is the maximal order in $\mathbb{Q} \otimes \text{End}_{\mathbb{Q}}(A)$. For every prime ideal \mathfrak{p} over 3 in that order, the intersection $A[\mathfrak{p}]$ of the kernels of all the endomorphisms in \mathfrak{p} becomes then a 2-dimensional vector space over $\overline{\mathbb{F}}_3$, and the representation

$$\rho_{A,\mathfrak{p}}: G_{\mathbb{Q}} \longrightarrow \text{GL}_2(\overline{\mathbb{F}}_3)$$

given by the Galois action on $A[\mathfrak{p}]$ is a lifting of ϱ_C ; for another description of the representations $\rho_{A,\mathfrak{p}}$, we refer to [2], section 2.

From [1], Proposition 2.15 (see also [5], Theorem 4.2), we have the identity

$$[\chi \det \rho_{A,\mathfrak{p}}] = \xi_C^{\pm}.$$

By combining it with the above equalities, we obtain

$$\xi_C^{\pm} = [\det \rho_{A,\mathfrak{p}}] [\chi] = w_C(-2, d_C)(-1, -3)$$

in $\text{Br}_2(\mathbb{Q})$, as desired. \square

Remark 2.1. Given an octahedral extension K/\mathbb{Q} , let $d \in \mathbb{Q}^*/\mathbb{Q}^{*2}$ and $w \in \text{Br}_2(\mathbb{Q})$ be, respectively, the discriminant and the Witt invariant of the quadratic form $\text{Tr}_{K_1/\mathbb{Q}}(x^2)$ attached to a quartic subextension K_1/\mathbb{Q} of K/\mathbb{Q} . Then, w can easily be computed from any *reduced* polynomial $f(X) = X^4 + aX^2 + bX + c$ defining the extension K_1/\mathbb{Q} (cf. [2], Remark 4.2, and [3], section 2):

- If $a = 0$ or $a = 2d$ up to squares, then $w = (-1, -d)$.
- Otherwise, $\delta = 2a^3 + 9b^2 - 8ac$ is non-zero, and then $w = (-1, -d)(2ad, \delta)$.

Notice also that d is the discriminant of f up to squares.

The following corollary, which is just a restatement of Theorem 2.1, gives the necessary condition announced in the introduction.

Corollary 2.1. *Let $\varrho : G_{\mathbb{Q}} \rightarrow \text{PGL}_2(\mathbb{F}_3)$ be a surjective representation, and let d and w be the invariants defined by the fixed field of ϱ (as in Remark 2.1). If ϱ arises from a \mathbb{Q} -curve C , with attached sign component ξ_C^{\pm} in $\text{Br}_2(\mathbb{Q})$, then the equality*

$$\xi_C^{\pm} = w(-2, d)(-1, -3)$$

must hold.

3. Ellipticity over quadratic fields of projective mod 3 Galois representations

Given a projective representation

$$\varrho : G_{\mathbb{Q}} \rightarrow \text{PGL}_2(\mathbb{F}_3),$$

we will say that ϱ is *elliptic* over a quadratic field k if its restriction to G_k is given by

$$\varrho|_{G_k} = \bar{\rho}_E$$

for some elliptic curve E defined over k . Here ρ_E denotes as above the representation of G_k attached to the 3-torsion points of E , and $\bar{\rho}_E$ stands for its associated projective representation. It is clear from the definitions that ellipticity is a necessary condition for the representation ϱ to arise from a \mathbb{Q} -curve defined over k .

Whenever $\det \varrho : G_{\mathbb{Q}} \rightarrow \mathbb{F}_3^*$ is the cyclotomic character χ , the representation ϱ arises from an elliptic curve defined over \mathbb{Q} (and hence it is elliptic over any quadratic field) if and only if it can be lifted to $\text{GL}_2(\mathbb{F}_3)$ [3]. If this last condition is not fulfilled, there are still a priori infinitely many quadratic fields k over which ϱ could be elliptic.

On the other hand, the quadratic field k is uniquely determined by ϱ provided that $\det \varrho \neq \chi$: it corresponds necessarily to the quadratic character $\chi \det \varrho$, so that the restriction $\det \varrho|_{G_k}$ becomes cyclotomic. In this

case, the following result gives a characterization of those surjective representations ϱ which are elliptic over k .

Theorem 3.1. *Let $\varrho : G_{\mathbb{Q}} \rightarrow \mathrm{PGL}_2(\mathbb{F}_3)$ be a surjective representation with non-cyclotomic determinant, and let d and w be, respectively, the discriminant and the Witt invariant of the trace quadratic form attached to any quartic subextension of the fixed field of ϱ . The following conditions are equivalent:*

- (1) *The representation ϱ is elliptic over a quadratic field.*
- (2) *For every prime p that splits in $\mathbb{Q}(\sqrt{-3d})$, the local component at p of the quaternion algebra $w(-1, -d)$ is trivial.*
- (3) *Every prime p for which $w_p \neq (-1, -d)_p$ in $\mathrm{Br}_2(\mathbb{Q}_p)$ satisfies the following (where we regard d and $-3d$ as squarefree integers):*
 - *If p is odd, then the Legendre symbol $\left(\frac{-3d}{p}\right)$ is not 1.*
 - *If $p = 2$, then $d \not\equiv 5 \pmod{8}$.*
- (4) *The fixed field of ϱ is the splitting field of a polynomial of the form $X^4 - 6X^2 + bX + c$, for some $b, c \in \mathbb{Q}$.*

Proof. The hypothesis on $\det \varrho$ amounts to saying that $d \neq -3$ in $\mathbb{Q}^*/\mathbb{Q}^{*2}$. As we have noticed above, the only quadratic field k over which ϱ can be elliptic is the fixed field of $\chi \det \varrho$, namely $\mathbb{Q}(\sqrt{-3d})$.

Let K_1/\mathbb{Q} be a quartic subextension of the fixed field of ϱ . Consider the quadratic form $\mathrm{Tr}_{K_1/\mathbb{Q}}(x^2)$ on K_1 , with invariants $d \in \mathbb{Q}^*/\mathbb{Q}^{*2}$ and $w \in \mathrm{Br}_2(\mathbb{Q})$, and denote by \mathcal{T} its restriction to the 3-dimensional subspace

$$\{x \in K_1 \mid \mathrm{Tr}_{K_1/\mathbb{Q}}(x) = 0\}.$$

The quadratic form \mathcal{T} has also discriminant d and Witt invariant w . Regarded as a real quadratic form, its signature is $(r_1 + r_2 - 1, r_2)$, where r_1 (resp. $2r_2$) is the number of real (resp. non-real) embeddings of K_1 into $\overline{\mathbb{Q}}$ (see [9], 3.4); in particular, it represents any positive real number, which in terms of Hilbert symbols means that $w_{\infty} = (-1, -d)_{\infty}$ in $\mathrm{Br}_2(\mathbb{R})$ whenever d is negative.

Condition (4) amounts to the existence of an element $\alpha \in K_1$ with $\mathrm{Tr}_{K_1/\mathbb{Q}}(\alpha) = 0$ and $\mathrm{Tr}_{K_1/\mathbb{Q}}(\alpha^2) = 3$, i.e. to the representability of 3 by \mathcal{T} . In terms of Hilbert symbols, the obstruction to that is given by the equality

$$w_p = (-1, -d)_p$$

in $\mathrm{Br}_2(\mathbb{Q}_p)$ for every prime p such that $-3d$ is a square in \mathbb{Q}_p^* . This is in turn equivalent to condition (2), whose translation into Legendre-Kronecker symbols is given by (3).

Consider now the natural morphisms

$$\mathbb{Q}^*/\mathbb{Q}^{*2} \rightarrow k^*/k^{*2} \quad \mathrm{Br}_2(\mathbb{Q}) \rightarrow \mathrm{Br}_2(k).$$

The discriminant and the Witt invariant of the quadratic form (over k) $\text{Tr}_{K_1 k/k}(x^2)$ are the image of d and w , respectively, by these maps. By Theorem 3 in [3] and Theorem 4.2, Lemma 4.1 in [2], ϱ is elliptic over k if and only if $w = (-1, -d)$ in $\text{Br}_2(k)$. Since k is imaginary whenever $w_\infty \neq (-1, -d)_\infty$ in $\text{Br}_2(\mathbb{R})$, this amounts again to condition (2). \square

Remark 3.1. The representation ϱ need not be odd to satisfy the equivalent conditions in the proposition. Also, the surjectivity assumption can be relaxed by only asking the fixed field of ϱ to be the normal closure of a quartic extension of \mathbb{Q} , and the result remains the same.

Let us now apply the above result, along with the one in the previous section, to \mathbb{Q} -curves of degree N , that is to \mathbb{Q} -curves defined over a quadratic field, with non-rational j -invariant, and having an isogeny of degree N to its conjugate curve.

Proposition 3.1. *Let ϱ , d and w be as in Theorem 3.1. If ϱ arises from a \mathbb{Q} -curve of degree N , then the following two equivalent conditions are satisfied:*

(i) *The Witt invariant $w \in \text{Br}_2(\mathbb{Q})$ is given by*

$$w = (-1, -d)(2N, -3d).$$

(ii) *For every polynomial $X^4 - 6X^2 + bX + c \in \mathbb{Q}[X]$ having the fixed field of ϱ as splitting field, $\delta = 3(16c + 3b^2 - 144)$ is non-zero and the quaternion algebra $(2\delta N, -3d)$ is trivial in $\text{Br}_2(\mathbb{Q})$.*

Proof. The existence of polynomials as in (ii) is ensured by Theorem 3.1, and the equivalence between the two conditions is a straightforward consequence of Remark 2.1. Let C be a \mathbb{Q} -curve of degree N attached to ϱ . Since the quadratic field of definition for C is $\mathbb{Q}(\sqrt{-3d})$, the sign component ξ_C^\pm is given by the quaternion algebra $(N, -3d)$ [5]. Condition (i) follows then from Corollary 2.1. \square

Appendix

We look closer here at the uniqueness of the octahedral extension K_C/\mathbb{Q} attached in the introduction to a \mathbb{Q} -curve C defined over a Galois number field k , in the case of surjective 3-torsion. The precise statement amounts to the following *octahedral exercise*. Its proof is obtained directly from Galois theory and the lemma below.

Proposition. *Let K/\mathbb{Q} and k/\mathbb{Q} be normal extensions such that the Galois groups $\text{Gal}(K/\mathbb{Q})$ and $\text{Gal}(Kk/k)$ are isomorphic to the symmetric group S_4 . Then, there are no other S_4 -extensions of \mathbb{Q} having the same compositum with k as K .*

Lemma. *Let G_1 and G_2 be groups, and identify them with their respective canonical images inside the product group $G_1 \times G_2$. Assume the center of G_2 to be trivial. Let H be a normal subgroup of $G_1 \times G_2$ with the same order as G_1 and having trivial intersection with G_2 . Then, H must be equal to G_1 .*

Proof. From the assumptions on H , this subgroup must be of the form

$$\{ (g, \sigma_g) \mid g \in G_1 \}.$$

All we must see then is that $\sigma_g = 1$ for all g in G_1 . Assume that $\sigma_g \neq 1$ for some g . Since G_2 has trivial center, there must be some τ in G_2 such that $\tau^{-1}\sigma_g\tau \neq \sigma_g$. Then, the element

$$(g, \tau^{-1}\sigma_g\tau) = (1, \tau)^{-1}(g, \sigma_g)(1, \tau)$$

lies in H and is different from (g, σ_g) , which yields a contradiction. \square

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