JAROSLAV HANČL

Linear independence of continued fractions

Journal de Théorie des Nombres de Bordeaux, tome 14, n° 2 (2002),
p. 489-495

[<http://www.numdam.org/item?id=JTNB_2002__14_2_489_0>](http://www.numdam.org/item?id=JTNB_2002__14_2_489_0)

© Université Bordeaux 1, 2002, tous droits réservés.

L'accès aux archives de la revue « Journal de Théorie des Nombres de Bordeaux » (<http://jtnb.cedram.org/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
<http://www.numdam.org/>

Linear independence of continued fractions

par JAROSLAV HANČL

RÉSUMÉ. Nous donnons un critère d'indépendance linéaire sur le corps des rationnels qui s'applique à une famille donnée de nombres réels dont les développements en fractions continues satisfont certaines conditions.

ABSTRACT. The main result of this paper is a criterion for linear independence of continued fractions over the rational numbers. The proof is based on their special properties.

1. Introduction

Forty years ago Davenport and Roth in [2] proved that the continued fraction $[a_1, a_2, \dots]$, where $a_1, a_2, \dots$ are positive integers satisfying

$$\limsup_{n \rightarrow \infty} ((\log \log a_n) \frac{\sqrt{\log n}}{n}) = \infty,$$

is a transcendental number. The generalization of transcendence is algebraic independence and there are several results concerning the algebraic independence of continued fractions. See, for instance, Bundschuh [1] or Hančl [5]. On the other hand it is a well known fact that if a positive real number has a finite continued fractional expansion then it is a rational number, and if not it is an irrational number. Irrationality is a special case of linear independence and this paper deals with such a theory. By the way, as to linear independence of series, one can find the criterion in [4], for instance.

2. Linear independence

Theorem 2.1. *Let $\epsilon > 1$ be a real number, K be a natural number and $\{a_{j,n}\}_{n=1}^{\infty}$ ($j = 1, 2, \dots, K$) be K sequences of positive integers such that*

$$(1) \quad a_{j+1,n} > a_{j,n} \left(1 + \frac{\epsilon}{n \log n}\right)$$

and

$$(2) \quad a_{1,n+1} > a_{K,n}^{K-1} \left(1 + \frac{1}{n}\right)$$

hold for every sufficiently large positive integer n and $j = 1, 2, 3, \dots, K-1$. Then the continued fractions $\alpha_j = [a_{j,1}, a_{j,2}, \dots]$ ($j = 1, 2, \dots, K$) and the number 1 are linearly independent over the rational numbers.

Lemma 2.1. Let $a_{j,n}$, $j = 1, 2, \dots, K$, $n = 1, 2, \dots$ and $K > 2$ satisfy all conditions stated in Theorem 2.1. Then

$$\prod_{n=1}^{\infty} \left(1 + \frac{1}{a_{j,n}}\right) = C_j < \infty.$$

Proof of Lemma 2.1. From (1) and (2) we obtain

$$\begin{aligned} a_{j,n} &\geq a_{1,n} \left(1 + \frac{\epsilon}{n \log n}\right)^{j-1} > a_{K,n-1}^{K-1} \left(1 + \frac{1}{n-1}\right) \left(1 + \frac{\epsilon}{n \log n}\right)^{j-1} \\ &> a_{j,n-1}^{K-1} \left(1 + \frac{\epsilon}{(n-1) \log(n-1)}\right)^{(K-1)(K-j)} \left(1 + \frac{1}{n-1}\right) \left(1 + \frac{\epsilon}{n \log n}\right)^{j-1} \\ &\geq a_{j,n-1} \left(1 + \frac{1}{n-1}\right) \left(1 + \frac{\epsilon}{n \log n}\right) > \left(1 + \frac{1}{n}\right) \left(1 + \frac{\epsilon}{n \log n}\right) a_{j,n-1} \end{aligned}$$

for every sufficiently large positive integer n and $j = 1, 2, \dots, K$. By mathematical induction we get

$$a_{j,n} \geq Y \prod_{j=2}^n \left(1 + \frac{1}{j}\right) \left(1 + \frac{\epsilon}{j \log j}\right)$$

for every $n = 2, 3, \dots$ and $j = 1, 2, \dots, K$, where Y is a positive real constant which does not depend on n . It follows that

$$\prod_{n=2}^{\infty} \left(1 + \frac{1}{a_{j,n}}\right) \leq \prod_{n=2}^{\infty} \left(1 + \frac{1}{Y \prod_{i=2}^n \left(1 + \frac{1}{i}\right) \left(1 + \frac{\epsilon}{i \log i}\right)}\right) = C_j < \infty$$

because the series

$$\sum_{n=2}^{\infty} \frac{1}{\prod_{j=2}^n \left(1 + \frac{1}{j}\right) \left(1 + \frac{\epsilon}{j \log j}\right)}$$

is convergent. (To prove this last fact one can use Bertrand's criterion for convergent series, for instance. See [3] for example.) $\square$

Proof of Theorem 2.1. If $K = 1$, then α_1 has an infinite continued fraction expansion. In this case α_1 is irrational and Theorem 2.1 holds. Now we will consider the case in which $K \geq 2$ and n is a sufficiently large positive integer. Let us assume that there exist $K+1$ integers $A_1, A_2, \dots, A_K, A_{K+1}$ (not all of which equal zero) such that

$$(3) \quad A_{K+1} = \sum_{j=1}^K A_j \alpha_j.$$

We can write each continued fraction α_j ($j = 1, 2, \dots, K$) in the form

$$(4) \quad \alpha_j = \frac{p_{j,n}}{q_{j,n}} + R_{j,n}$$

where $\frac{p_{j,n}}{q_{j,n}} = [a_{j,1}, a_{j,2}, \dots, a_{j,n}]$ is the n -th partial fraction of α_j and $R_{j,n}$ is the remainder. For $R_{j,n}$ we have the estimation

$$(5) \quad |R_{j,n}| = |\alpha_j - \frac{p_{j,n}}{q_{j,n}}| < \frac{1}{a_{j,n+1}q_{j,n}^2}$$

and

$$(6) \quad |R_{j,n}| > \frac{c}{a_{j,n+1}q_{j,n}^2}$$

where $c > 0$ is a constant which depends only on $\alpha_1, \alpha_2, \dots, \alpha_K$. (For the proof see, for instance, [6].) Substituting (4) into (3) we obtain

$$A_{K+1} = \sum_{j=1}^K A_j \left(\frac{p_{j,n}}{q_{j,n}} + R_{j,n} \right).$$

Multiplying both sides of the last equation by $\prod_{j=1}^K q_{j,n}$ we obtain

$$A_{K+1} \prod_{j=1}^K q_{j,n} = \prod_{j=1}^K q_{j,n} \sum_{j=1}^K A_j \left(\frac{p_{j,n}}{q_{j,n}} + R_{j,n} \right).$$

This implies

$$(7) \quad M_n = (A_{K+1} - \sum_{j=1}^K A_j \frac{p_{j,n}}{q_{j,n}}) \prod_{j=1}^K q_{j,n} = \prod_{j=1}^K q_{j,n} \sum_{j=1}^K A_j R_{j,n}$$

where M_n is an integer.

First we will prove that $|M_n| > 0$. Let P be the least positive integer such that $A_P \neq 0$. (Such a P must exist because not every A_j is equal to zero.) Then we have

$$\begin{aligned} |M_n| &= \left| \prod_{j=1}^K q_{j,n} \sum_{j=1}^K A_j R_{j,n} \right| = \left| \prod_{j=1}^K q_{j,n} \sum_{j=P}^K A_j R_{j,n} \right| \\ &\geq \prod_{j=1}^K q_{j,n} (|A_P| |R_{P,n}| - \sum_{j=P+1}^K |A_j| |R_{j,n}|). \end{aligned}$$

This, (5) and (6) imply

$$|M_n| \geq \prod_{j=1}^K q_{j,n} (|A_P| \frac{c}{a_{P,n+1}q_{P,n}^2} - \sum_{j=P+1}^K |A_j| \frac{1}{a_{j,n+1}q_{j,n}^2}).$$

From this last inequality and (1) we obtain

$$\begin{aligned}
 (8) \quad |M_n| &\geq \prod_{j=1}^K q_{j,n} \left(|A_P| \frac{c}{a_{P,n+1} q_{P,n}^2} - \frac{\sum_{j=P+1}^K |A_j|}{a_{P+1,n+1} q_{P+1,n}^2} \right) \\
 &\geq \frac{\prod_{j=1}^K q_{j,n} |A_P| c}{a_{P+1,n+1} q_{P+1,n}^2} \left(\frac{a_{P+1,n+1} q_{P+1,n}^2}{a_{P,n+1} q_{P,n}^2} - \frac{\sum_{j=P+1}^K |A_j|}{|A_P| c} \right) \\
 &= B \left(\frac{a_{P+1,n+1} q_{P+1,n}^2}{a_{P,n+1} q_{P,n}^2} - C \right)
 \end{aligned}$$

where B is a positive real number and C is a constant which does not depend on n . We also have

$$(9) \quad \prod_{i=1}^n a_{j,i} < q_{j,n} < \prod_{i=1}^n (a_{j,i} + 1)$$

for every $j = 1, 2, \dots, K$, $n = 1, 2, \dots$ which can be proved by mathematical induction using

$$q_{j,n+1} = a_{j,n+1} q_{j,n} + q_{j,n-1}.$$

(This identity can be found, for instance, in [6].) (8) and (9) imply

$$\begin{aligned}
 |M_n| &\geq B \left(\frac{a_{P+1,n+1}}{a_{P,n+1}} \prod_{j=1}^n \left(\frac{a_{P+1,j}}{a_{P,j} + 1} \right)^2 - C \right) \\
 &= B \left(\frac{a_{P+1,n+1}}{a_{P,n+1}} \left(\prod_{j=1}^n \left(\frac{a_{P+1,j}}{a_{P,j}} \right) \frac{1}{\prod_{j=1}^n \left(1 + \frac{1}{a_{P,j}} \right)} \right)^2 - C \right).
 \end{aligned}$$

This, Lemma 2.1 and (1) imply

$$\begin{aligned}
 (10) \quad |M_n| &\geq B \left(E \frac{1 + \frac{\epsilon}{(n+1) \log(n+1)}}{\left(\prod_{j=1}^{\infty} \left(1 + \frac{1}{a_{P,j}} \right) \right)^2} \prod_{j=1}^n \left(1 + \frac{\epsilon}{j \log j} \right)^2 - C \right) \\
 &> B \left(D \prod_{j=1}^n \left(1 + \frac{\epsilon}{j \log j} \right) - C \right)
 \end{aligned}$$

where $D > 0$ is a constant which does not depend on n . From (10) and the fact that $\prod_{j=1}^{\infty} \left(1 + \frac{\epsilon}{n \log n} \right) = \infty$ we obtain

$$(11) \quad |M_n| > 0$$

for every sufficiently large positive integer n .

Now we will prove that $|M_n| < 1$ for n sufficiently large. From (7) we obtain

$$|M_n| = \prod_{j=1}^K q_{j,n} \left| \sum_{j=1}^K A_j R_{j,n} \right| \leq \prod_{j=1}^K q_{j,n} \sum_{j=1}^K |A_j| |R_{j,n}|.$$

This and (5) imply

$$|M_n| \leq \prod_{j=1}^K q_{j,n} \sum_{j=1}^K |A_j| \frac{1}{a_{j,n+1} q_{j,n}^2}.$$

From this and (1) we obtain

$$\begin{aligned} (12) \quad |M_n| &\leq \prod_{j=1}^K q_{j,n} \sum_{j=1}^K |A_j| \frac{1}{a_{1,n+1} q_{1,n}^2} \\ &= \frac{\prod_{j=2}^K q_{j,n}}{a_{1,n+1} q_{1,n}} \sum_{j=1}^K |A_j| = F \frac{\prod_{j=2}^K q_{j,n}}{a_{1,n+1} q_{1,n}} \end{aligned}$$

where $F = \sum_{j=1}^K |A_j|$ is a positive real constant which does not depend on n . (9) and (12) imply

$$|M_n| \leq F \frac{\prod_{j=2}^K q_{j,n}}{a_{1,n+1} q_{1,n}} \leq F \frac{\prod_{j=2}^K \prod_{i=1}^n (a_{j,i} + 1)}{\prod_{i=1}^{n+1} a_{1,i}}.$$

From this and Lemma 2.1 we obtain

$$\begin{aligned} (13) \quad |M_n| &\leq F \frac{\prod_{j=2}^K \prod_{i=1}^n (a_{j,i} + 1)}{\prod_{i=1}^{n+1} a_{1,i}} \\ &= F \frac{\prod_{j=2}^K \prod_{i=1}^n a_{j,i}}{\prod_{i=1}^{n+1} a_{1,i}} \prod_{j=2}^K \prod_{i=1}^n \left(1 + \frac{1}{a_{j,i}}\right) \\ &\leq F \frac{\prod_{j=2}^K \prod_{i=1}^n a_{j,i}}{\prod_{i=1}^{n+1} a_{1,i}} \prod_{j=2}^K \prod_{i=1}^{\infty} \left(1 + \frac{1}{a_{j,i}}\right) \\ &= F \frac{\prod_{j=2}^K C_j a_{j,1}}{a_{1,1} a_{1,2}} \frac{\prod_{j=2}^K \prod_{i=2}^n a_{j,i}}{\prod_{i=3}^{n+1} a_{1,i}} \\ &= H \frac{\prod_{j=2}^K \prod_{i=2}^n a_{j,i}}{\prod_{i=3}^{n+1} a_{1,i}} \end{aligned}$$

where $H > 0$ is a constant which does not depend on n . (1), (2) and (13) imply

$$\begin{aligned}
|M_n| &< H \frac{\prod_{j=2}^K \prod_{i=2}^n a_{j,i}}{\prod_{i=3}^{n+1} a_{1,i}} \leq G \frac{\prod_{j=2}^K \prod_{i=2}^n a_{K,i}}{\prod_{i=3}^{n+1} a_{1,i}} \\
&= G \frac{\prod_{i=2}^n a_{K,i}^{K-1}}{\prod_{i=3}^{n+1} a_{1,i}} = G \prod_{i=2}^n \frac{a_{K,i}^{K-1}}{a_{1,i+1}} \leq L \prod_{i=2}^n \frac{1}{1 + \frac{1}{i}} \\
&= \frac{L}{\prod_{i=2}^n (1 + \frac{1}{i})}
\end{aligned}$$

where L is a positive real constant which does not depend on n . It follows that $|M_n| < 1$ for every sufficiently large positive integer n . This and (11) imply that $0 < |M_n| < 1$ for every sufficiently large n , where M_n is an integer. This is impossible therefore the numbers $\alpha_1, \alpha_2, \dots, \alpha_K$ and 1 are linearly independent over the rational numbers. $\square$

3. Conclusion

Example 1. The continued fractions

$$[2^K, 2^{K^2}, 2^{K^3}, \dots], [2 \cdot 2^K, 2 \cdot 2^{K^2}, 2 \cdot 2^{K^3}, \dots], \dots, [K \cdot 2^K, K \cdot 2^{K^2}, K \cdot 2^{K^3}, \dots]$$

and the number 1 are linearly independent over the rational numbers.

Example 2. The continued fractions

$$[3^{K+1}, 3^{K^2+1}, 3^{K^3+1}, \dots], [3^{K+2}, 3^{K^2+2}, 3^{K^3+2}, \dots], \dots, [3^{2K}, 3^{K^2+K}, 3^{K^3+K}, \dots]$$

and the number 1 are linearly independent over the rational numbers.

Example 3. The continued fractions

$$[2^2, 2^{2^2}, 2^{2^3}, 2^{2^4}, \dots], [3^2, 3^{2^2}, 3^{2^3}, 3^{2^4}, \dots]$$

and the number 1 are linearly independent over the rational numbers.

Open Problem. It is not known if the continued fractions

$$[2^2, 2^{2^2}, 2^{2^3}, \dots], [3^2, 3^{2^2}, 3^{2^3}, \dots], [4^2, 4^{2^2}, 4^{2^3}, \dots]$$

and the number 1 are linearly independent or not over the rational numbers.

Example 4. Let $\{G_n\}_{n=1}^\infty$ be the linear recurrence sequence of the k -th order such that $G_1, G_2, \dots, G_k, b_0, \dots, b_k$ belong to positive integers, $G_1 < G_2 < \dots < G_k$ and for every positive integer n , $G_{n+k} = G_n b_0 + G_{n+1} b_1 + \dots + G_{n+k-1} b_{k-1}$. If the roots $\alpha_1, \dots, \alpha_s$ of the equation $x^k = b_0 + b_1 x + \dots + b_{k-1} x^{k-1}$ satisfy $|\alpha_1| \geq |\alpha_2| \geq \dots \geq |\alpha_s|$, $|\alpha_1| > 1$ and α_1/α_j is not a root of unity for every $j = 2, 3, \dots, s$, then the continued fractions

$$[G_j G_{k^1}, G_j G_{k^2}, G_j G_{k^3}, \dots]$$

($j = 1, 2, \dots, k$) and the number 1 are linearly independent over the rational numbers.

This is an immediate consequence of Theorem 2.1 and the inequality $|\alpha_1|^{n(1-\epsilon)} < G_n < |\alpha_1|^{n(1+\epsilon)}$ which can be found in [7], for instance.

Acknowledgments. We would like to thank you very much to Professor James Carter and Professor Atilla Pethő for their help with this article.

References

- [1] P. BUNDSCHUH, *Transcendental continued fractions*. J. Number Theory **18** (1984), 91–98.
- [2] H. DAVENPORT, K. F. ROTH, *Rational approximations to algebraic numbers*. Mathematika **2** (1955), 160–167.
- [3] G. M. FICHTENGOLC, *Lecture on Differential and Integrational Calculus II* (Russian). Fizmatgiz, 1963.
- [4] J. HANČL, *Linearly unrelated sequences*. Pacific J. Math. **190** (1999), 299–310.
- [5] J. HANČL, *Continued fractional algebraic independence of sequences*. Publ. Math. Debrecen **46** (1995), 27–31.
- [6] G. H. HARDY, E. M. WRIGHT, *An Introduction to the Theory of Numbers*. Oxford Univ. Press, 1985.
- [7] H. P. SCHLICKWEI, A. J. VAN DER POORTEN, *The growth conditions for recurrence sequences*. Macquarie University Math. Rep. 82-0041, North Ryde, Australia, 1982.

Jaroslav HANČL
Department of Mathematics
University of Ostrava
Dvořákova 7
701 03 Ostrava 1
Czech Republic
E-mail : hanc1@osu.cz