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Best simultaneous diophantine approximations of some cubic algebraic numbers


<http://www.numdam.org/item?id=JTNB_2002__14_2_403_0>
Best simultaneous diophantine approximations of some cubic algebraic numbers

par Nicolas Chevallier

Résumé. Soit $\alpha$ un nombre algébrique réel de degré 3 dont les conjugués ne sont pas réels. Il existe une unité $\zeta$ de l’anneau des entiers de $K = \mathbb{Q}(\alpha)$ pour laquelle il est possible de décrire l’ensemble de tous les vecteurs meilleurs approximations de $\theta = (\zeta, \zeta^2)$.

Abstract. Let $\alpha$ be a real algebraic number of degree 3 over $\mathbb{Q}$ whose conjugates are not real. There exists an unit $\zeta$ of the ring of integer of $K = \mathbb{Q}(\alpha)$ for which it is possible to describe the set of all best approximation vectors of $\theta = (\zeta, \zeta^2)$.

1. Introduction

In his first paper ([10]) on best simultaneous diophantine approximations J. C. Lagarias gives an interesting result which, he said, is in essence a corollary of W. W. Adams’ results ([1] and [2]):

Let $[1, \alpha_1, \alpha_2]$ be a $\mathbb{Q}$ basis to a non-totally real cubic field. Then the best simultaneous approximations of $\alpha = (\alpha_1, \alpha_2)$ (see definition below) with respect to a given norm $N$ are a subset of

$$\{ q_m^{(j)} : m \geq 0, 1 \leq j \leq p \}$$

where the $q_m^{(j)}$ satisfy a third-order linear recurrence (with constant coefficients).

$$q_{m+3} + a_2 q_{m+2} + a_1 q_{m+1} \pm q_m = 0$$

for a finite set of initial conditions $q_0^{(j)}$, $q_1^{(j)}$, $q_2^{(j)}$, for $1 \leq j \leq p$. The fundamental unit $\xi$ of $K = \mathbb{Q}(\alpha_1, \alpha_2)$ satisfies

$$\xi^3 - a_2 \xi^2 - a_1 \xi \pm 1 = 0.$$ 

Now consider the particular case $X = (\xi, \xi^2) \in \mathbb{R}^2$ where $\xi$ is the unique real root of $\xi^3 + \xi^2 + \xi - 1 = 0$. The vector $X$ can be seen as a two-dimensional golden number. N. Chekhova, P. Hubert and A. Messaoudi were able to precise Lagarias’ result (cf. [7]):

Manuscrit reçu le 10 juillet 2000.
There exists a euclidean norm on $\mathbb{R}^2$ such that all best diophantine approximations of $X$ are given by the ‘Tribonacci’ sequence $(q_n)_{n \in \mathbb{N}}$ defined by

$$q_0 = 1, \ q_2 = 2, \ q_3 = 4, \ q_{n+3} = q_{n+2} + q_{n+1} + q_n.$$  

The aim of this work is to make precise Lagarias’ result in the same way as N. Chekhova, P. Hubert and A. Messaoudi.

**Definition** ([10],[8]). Let $N$ be a norm on $\mathbb{R}^2$ and $\theta \in \mathbb{R}^2$.

1) A strictly positive integer $q$ is a best approximation (denominator) of $\theta$ with respect to $N$ if

$$\forall k \in \{1, \ldots, q-1\}, \ \min_{P \in \mathbb{Z}^2} N(q\theta - P) < \min_{Q \in \mathbb{Z}^2} N(k\theta - Q)$$  

2) An element $q\theta - P$ of $\mathbb{Z}\theta + \mathbb{Z}^2$ is a best approximation vector of $\theta$ with respect to $N$ if $q$ is a best approximation of $\theta$ and if

$$N(q\theta - P) = \min_{Q \in \mathbb{Z}^2} N(q\theta - Q)$$

We will call $\mathcal{M}(\theta)$ the set of all best approximation vectors of $\theta$.

Using Dirichlet’s theorem it is easy to show that there exists a positive constant $C$ depending only on the norm $N$, such that for all $\theta$ in $\mathbb{R}^2$ and all best approximation vectors $q\theta - P$ of $\theta$

$$N(q\theta - P) \leq \frac{C}{q^{1/2}}.$$  

If $[1, \alpha_1, \alpha_2]$ is a $\mathbb{Q}$-basis of a real cubic field then $\theta = (\alpha_1, \alpha_2)$ is badly approximable (cf. [6] p. 79):

there exists $c > 0$ such that for all best approximation vectors $q\theta - P$ of $\theta$

$$N(q\theta - P) \geq \frac{c}{q^{1/2}}.$$  

Let $\theta \in \mathbb{R}^2 \setminus \mathbb{Q}^2$ and $\Lambda = \theta \mathbb{Z} + \mathbb{Z}^2$. Endow $\Lambda$ with its natural $\mathbb{Z}$-basis $\theta, e_1 = (1, 0), \ e_2 = (0, 1)$. For a matrix $B \in M_3(\mathbb{Z})$ and $X = x_0\theta + x_1e_1 + x_2e_2 \in \Lambda$, the action $BX = Y$ of $B$ on $X$ is naturally defined: the coordinates vector of $Y$ is the matrix product of $B$ by the coordinates vector of $X$.

We shall prove the following results.

**Proposition 1.** Let $a_1, \ a_2 \in \mathbb{N}^*$. Suppose $P(x) = x^3 + a_2x^2 + a_1x - 1$ has a unique real root $\zeta$. Call $\theta = (\zeta, \zeta^2)$ and $B$ the matrix

$$B = \begin{pmatrix} a_1 & -a_2 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$
There exist a norm $N$ on $\mathbb{R}^2$ and a finite number of best approximation vectors $X_i = q_i \theta - P_i$, $i = 1, \ldots, m$ such that

$$\mathcal{M}(\theta) \setminus \{B^n X_i : n \in \mathbb{N} \text{ and } i = 1, \ldots, m\}$$

is a finite set.

**Proposition 2.** Suppose $\alpha$ is a real algebraic number of degree 3 over $\mathbb{Q}$ whose conjugates are not real. There exist a unit $\zeta$ of the ring of integer of $K = \mathbb{Q}(\alpha)$, two positive integers $a_1$ and $a_2$ and euclidean norm on $\mathbb{R}^2$ such that the set of best approximation vectors of $\theta = (\zeta, \zeta^2)$, is

$$\mathcal{M}(\theta) = \{B^n \theta : n \in \mathbb{N}\}$$

where

$$B = \begin{pmatrix} a_1 & -a_2 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The proof of Proposition 1 is quite different from Chechkova, Hubert and Messaoudi's one. It is based on two simple facts:

1) Following G. Rauzy ([14]) we construct a euclidean norm $N$ on $\mathbb{R}^2$ and a linear contracting similarity $F$ on $\mathbb{R}^2$ (i.e. $N(F(x)) = r N(x)$ for all $x$ in $\mathbb{R}^2$ where the ratio $r \in [0, 1[$ which is one to one on $\Lambda = \mathbb{Z} \theta + \mathbb{Z}^2$.

2) Since $a_1, a_2 > 0$ the map $F$ preserves the positive cone $\Lambda^+ = \mathbb{N} \theta - \mathbb{N}^2$. We deduce from these observations that $F$ send best approximation vectors of $\theta$ to best approximation vectors of $\theta$ (see lemma 2) and proposition 1 follow easily. Our method cannot be extended to higher dimension, because for $F$ to be a similarity, it is necessary that $P$ has one dominant root, all other roots being of the same modulus, and H. Minkowski proved that this can only occur for polynomials of degree 2 or 3 ([12]).

The sequence of best approximation vectors of $\theta \in \mathbb{R}^2$ may be seen as a two-dimensional continued fraction 'algorithm'. In this case Proposition 1 means that the 'development' of $(\zeta, \zeta^2)$ becomes periodic when $\zeta$ is the unique real root of the polynomial $x^3 + a_2 x^2 + a_1 x - 1$ with $a_1, a_2 \in \mathbb{N}$. This may be compared to the following results about Jacobi-Perron's algorithm:

(0. Perron [13]) Let $\zeta$ be the root of $P \in \mathbb{Z}[X]$, deg $P = 3$. If the development of $(\zeta, \zeta^2)$ by Jacobi-Perron's algorithm becomes periodic and if this development gives good approximations, i.e.

$$\max(|q_n \zeta - p_{1,n}|, |q_n \zeta^2 - p_{2,n}|) \leq \frac{C}{q_n^{1/2}}$$

where $(p_{1,n}, p_{2,n}, q_n)_{n \in \mathbb{N}}$ are given by Jacobi-Perron's algorithm, then the conjugates of $\zeta$ are complex (see [4] p. 7).
(P. Bachman [1]) Let $\zeta = d^{1/3}$ where $d$ is a cube-free integer greater than 1. If the development by Jacobi-Perron's algorithm of $(\zeta, \zeta^2)$ turns out to be periodic it gives good approximations as above.

(E. Dubois - R. Paysant [9]) If $K$ is a cubic extension of $\mathbb{Q}$ then there exist $\beta_1, \beta_2$ in $K$, linearly independent with 1, such that the development of $(\beta_1, \beta_2)$ by Jacobi-Perron's algorithm is periodic.

O. Perron (see [13] Theorem VII or Brentjes [5] Theorem 3.4.) also gives some numbers with a purely periodic development of length 1.

We should also note that A. J. Brentjes gives a two-dimensional continued fraction algorithm which finds all best approximations of a certain kind and he uses it to find the coordinates of the fundamental unit in a basis of the ring of integers of a non-totally real cubic field (see Brentjes' book on multi-dimensional continued fraction algorithms [5] section 5F).

Finally, we shall give a proof of Chechkova, Hubert and Messaoudi's result using proposition 1 together with the set of best approximations corresponding to the equation $\zeta^3 + 2\zeta^2 + \zeta = 1$.

2. The Rauzy norm

Fix $a_1, a_2 \in \mathbb{N}^*$ and suppose that the polynomial $P(x) = -x^3 + a_1 x^2 + a_2 x + 1$ has a unique real root. Endow $\mathbb{R}^3$ with its standard basis $e_1, e_2, e_3$. Let $M$ be the matrix

$$M = \begin{pmatrix} a_1 & a_2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$  

The characteristic polynomial of $M$ is $-x^3 + a_1 x^2 + a_2 x + 1$, the unique positive eigenvalue of $M$ is $\lambda = \frac{1}{\zeta}$ and $\Theta = (\zeta, \zeta^2, \zeta^3)$ is the eigenvector associated with $\lambda$. Let $l$ be the linear form on $\mathbb{R}^3$ with coefficients $a_1, a_2, 1$; we have $l(\Theta) = l(e_3) = 1$. Put $\Delta(X) = X - l(X) \Theta$. $\Delta \circ M$ map $\ker l$ into itself and $\mathbb{R} \Theta \subseteq \ker \Delta \circ M$. The eigenvalues of the restriction of $\Delta \circ M$ to $\ker l$, are $\lambda_1$ and $\lambda_2 = \overline{\lambda_1}$, the two other eigenvalues of $M$. In fact, if $Z$ is an eigenvector of $M$ associated to $\lambda_1$ then $\Delta(Z) \in \ker l$ and

$$\Delta \circ M \circ \Delta(Z) = \Delta(\lambda_1 Z - l(Z) \lambda \Theta) = \lambda_1 \Delta(Z).$$

Call $p$ the projection $\mathbb{R}^3$ onto $\mathbb{R}^2$. $p$ is one to one from $\ker l$ onto $\mathbb{R}^2$, call $i$ its inverse map and consider the linear map

$$F : X \in \mathbb{R}^2 \rightarrow p \circ \Delta \circ M \circ i(X) \in \mathbb{R}^2.$$  

The linear maps $F$ and $\Delta \circ M$ are conjugate, therefore the eigenvalues of $F$ are $\lambda_1$ and $\lambda_2$.

**Lemma 3.** $F$ is one to one of $A = \mathbb{Z}\theta + \mathbb{Z}^2$ on itself, where $\theta = (\zeta, \zeta^2)$. 

Proof. Since \( i(\theta) = \Theta - e_3 \) we have

\[
F(\theta) = p \circ \Delta(\lambda \Theta - e_1) = p(l(e_1)\Theta - e_1) = a_1\theta - e_1 \in \Lambda.
\]

Similarly \( i(e_k) = e_k - l(e_k)e_3, \ k = 1, 2 \), then \( X_k = M \circ i(e_k) \in \mathbb{Z}^3 \) and

\[
F(e_k) = p(X_k - l(X_k)\Theta) = p(X_k) - l(X_k)\theta \in \Lambda.
\]

Since \( F \) maps \( \Lambda \) into itself, it remains to show that \( F \) is one to one. Call \( B \) the matrix of \( F \) with respect to the basis \((\theta, e_1, e_2)\). We have

\[
\begin{align*}
X_1 &= M(e_1 - l(e_1)e_3) = a_1e_1 + e_2 - l(e_1)e_1 = e_2, \\
X_2 &= M(e_2 - l(e_2)e_3) = a_2e_1 + e_3 - l(e_2)e_1 = e_3
\end{align*}
\]

so that

\[
B = \begin{pmatrix}
a_1 & -a_2 & -1 \\
-a_2 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}
\]

and

\[
det B = -1.
\]

Call \( \Lambda^+ = \{ q\theta - P : q \in \mathbb{N} \text{ and } P \in \mathbb{N}^2 \} \). Since \( a_1 \) and \( a_2 \) are positive, we have:

**Corollary 4.** \( F(\Lambda^+) \subseteq \Lambda^+ \).

Since \( \lambda_2 = \sqrt{\lambda_1} \) there exists a euclidean norm \( N \) on \( \mathbb{R}^2 \) such that \( F \) is a linear similar map for this norm (i.e. \( N(F(x)) = rN(x) \) for all \( x \) in \( \mathbb{R}^2 \), where \( r \) in \( \mathbb{R}^+ \) is call the ratio of \( F \)). The ratio of \( F \) is \( r = |\lambda_1| = \frac{1}{\sqrt{\lambda}} = \sqrt{\zeta} < 1 \). Now let us determine the matrix \( M \) of the bilinear form \( \langle x, y \rangle \) associated with \( N \), this is necessary for Proposition 2 but not for Proposition 1. \( M \) is unique up to a multiplicative constant. Since the ratio of \( F \) is \( \sqrt{\zeta} \),

\[
\langle F(e_1), F(e_2) \rangle = \zeta \langle e_1, e_2 \rangle,
\]

and computing \( F(e_1) \) and \( F(e_2) \), we find that \( \langle e_1, e_1 \rangle, \langle e_1, e_2 \rangle \) and \( \langle e_2, e_2 \rangle \) satisfy

\[
\begin{align*}
a_2\zeta\langle e_1, e_1 \rangle + (-2 + 2a_2\zeta^2)\langle e_1, e_2 \rangle + (-\zeta + a_2\zeta^3)\langle e_2, e_2 \rangle &= 0 \\
\zeta\langle e_1, e_1 \rangle + 2\zeta^2\langle e_1, e_2 \rangle + (\zeta^3 - 1)\langle e_2, e_2 \rangle &= 0.
\end{align*}
\]

Since \( 1 = a_1\zeta + a_2\zeta^2 + \zeta^3 \), we find

\[
\langle e_1, e_1 \rangle = 2(a_1 + \zeta^2), \quad \langle e_1, e_2 \rangle = a_2 - \zeta, \quad \langle e_2, e_2 \rangle = 2.
\]
3. Best diophantine approximations

We suppose \( \mathbb{R}^2 \) is endowed with the norm \( N \) defined in the previous section.

**Notations.**
1) \( \rho_0 = d(0, \{(x_1, x_2) \in \mathbb{R}^2 : \sup(|x_1|, |x_2|) \geq 1\}) \).
2) For \( x \in \mathbb{R} \) we denote the nearest integer to \( x \) by \( I(x) \) (it is well-defined for all irrational number \( x \)).

We will often use the simple fact:
Let \( X = (x_1, x_2) \in \mathbb{R}^2 \) and \( P = (p_1, p_2) \in \mathbb{Z}^2 \). If \( N(X - P) < \frac{1}{2} \rho_0 \) then \( p_1 = I(x_1), p_2 = I(x_2) \) and \( P \) is the nearest point of \( \mathbb{Z}^2 \) to \( X \) (for the norm \( N \)).

We will say that two best approximation vectors \( q_1 \theta - P_1 \) and \( q_2 \theta - P_2 \) are consecutive if \( q_1 \) and \( q_2 \) are consecutive best approximations.

**Lemma 5.**
1) If \( q \theta - P \) is a best approximation vector such that \( N(q \theta - P) < \frac{1}{2} \rho_0 \) then \( q' \theta - P' = F(q \theta - P) \) is a best approximation vector of \( \theta \).
2) Let \( q_1 \) and \( q_2 \) be two consecutive best approximations of \( \theta \) and \( q_1 \theta - P_1 \) and \( q_2 \theta - P_2 \) be two corresponding best approximation vectors. If \( N(q_2 \theta - P_2) < \frac{1}{2} \rho_0 \) and \( F(q_1 \theta - P_1) \) is a best approximation vector then \( F(q_1 \theta - P_1) \) and \( F(q_2 \theta - P_2) \) are consecutive best approximation vectors.

**Proof.**
1) Let \( Y = k' \theta - R' \in \Lambda \setminus \{(0,0)\} \) be such that \( N(Y) \leq N(q' \theta - P') \).
We have to prove that \( |k'| > q' \) or that \( k' \theta - R' = \pm(q' \theta - P') \). By Lemma 1, we have \( Y = F(X) \) with \( X = k' \theta - R \in \Lambda \). Since \( F \) is a similar map, we have \( N(X) \leq N(q \theta - P) \) and by the definition of best approximations \( |k| \geq q \). If \( k < 0 \) we can replace \( Y \) by \( -Y \) so we can suppose that \( k \geq q \). Since \( N(X) \leq N(q \theta - P) < \frac{1}{2} \rho_0 \), \( R = (I(k \zeta), I(k \zeta^2)) \) and \( P = (I(q \zeta), I(q \zeta^2)) \). The nearest integer function \( x \to I(x) \) is nondecreasing so \( I(k \zeta) \geq I(q \zeta) \) and \( I(k \zeta^2) \geq I(q \zeta^2) \). This shows that \( (k \theta - R) - (q \theta - P) \in \Lambda^+ \) and by corollary 4, \( F(k \theta - R) - F(q \theta - P) \in \Lambda^+ \). Therefore \( k' \geq q' \). If \( k' = q' \), we have \( R' = (I(k' \zeta), I(k' \zeta^2)) = (I(q' \zeta), I(q' \zeta^2)) = P' \).
2) Put \( F(q_1 \theta - P_1) = k_1 \theta - R_1, i = 1, 2 \). Suppose \( k \theta - R \) is a best approximation vector with \( k_1 < k \leq k_2 \). We want to prove that \( k \theta - R = k_2 \theta - R_2 \).
Put \( F^{-1}(k \theta - R) = q \theta - P \). On the one hand, since \( F \) is similar, we have \( N(q \theta - P) < N(q_1 \theta - P_1) \), so \( q > q_1 \). Furthermore \( q_1 \) and \( q_2 \) are consecutive best approximations, so \( q \geq q_2 \).
On the other hand, \( k_1 \theta - R_1 = F(q_1 \theta - P_1) \) is a best approximation with \( N(k_1 \theta - R_1) = N(F(q_1 \theta - P_1) < N(q_1 \theta - P_1) \), then \( k_1 \geq q_2 \) and \( N(k_1 \theta - P_1) \leq N(q_2 \theta - P_2) < \frac{1}{2} \rho_0 \). Therefore \( N(k_2 \theta - R_2) \) and \( N(k \theta - R) < \frac{1}{2} \rho_0 \). It follows that
\[
R = (I(k \zeta), I(k \zeta^2)), R_2 = (I(k_2 \zeta), I(k_2 \zeta^2)).
\]
We have $I(k\zeta) \leq I(k_2\zeta)$ for $k \leq k_2$. Using the matrix $B$ we see that $R = (q,.)$ and $R_2 = (q_2,.).$ This shows $q \leq q_2$ and $q = q_2$, which implies $q_\theta - P = q_2 \theta - P_2$ and $k \theta - R = k_2 \theta - R_2$.

The increasing sequence of all best approximations of $\theta$ will be denoted by $(q_n)_{n \in \mathbb{N}}$ ($q_0 = 1$).

**Proposition 6.** If $q_{n_0} \theta - P_{n_0}, \ldots, q_{n_0+m} \theta - P_{n_0+m}$ are (consecutive) best approximation vectors such that $F(q_{n_0} \theta - P_{n_0}) = q_{n_0+m} \theta - P_{n_0+m}$ and $N(q_{n_0+1} \theta - P_{n_0+1}) < \frac{1}{2} \rho_0$, then for all $j \geq 0$ and all $k \in 0, \ldots, m - 1$,

$$q_{n_0+jm+k} \theta - P_{n_0+jm+k} = F^j(q_{n_0+k} \theta - P_{n_0+k}).$$

**Proof.** Put $V_n = q_n \theta - P_n$. The previous lemma shows that $F(V_{n_0+k})$, $k = 0, \ldots, m$, are consecutive best approximation vectors. By induction on $j \geq 0$, we see that $F^j(V_{n_0+k}) = V_{n_0+jm+k}$, $k = 0, \ldots, m$ are consecutive best approximation vectors and $F(V_{n_0+jm}) = V_{n_0+(j+1)m}$.

**Proof of Proposition 1.** Since $\lim_{n \to \infty} \min_{P \in \mathbb{Z}^2} N(q_n \theta - P) = 0$, there exists an integer $n_0$ such that for each $n \geq n_0$, $N(q_n \theta - P_n) < \frac{1}{2} \rho_0$. By Lemma 4, 1), $F(q_n \theta - P_n)$ is a best approximation vector and Proposition 1 follows of Proposition 6.

4. **Proof of Proposition 2**

**Lemma 7.** Let $P \in \mathbb{Q}$ be an irreducible polynomial of degree 3 with a unique real root $\alpha$ and $K = \mathbb{Q}(\alpha)$. There exist infinitely many $\lambda \in K$ such that

i) $\lambda > 1$

ii) $\lambda$ is a root of $Q(x) = x^3 - a_1 x^2 - a_2 x - 1$

iii) $a_1, a_2 \in \mathbb{N}$ and $3a_1 \geq a_2^2$.

**Proof.** Since $P$ has a unique real root, Dirichlet's theorem shows that the group of unit of the integral ring of $K$ contains an abelian free sub-group $G$ of rank 1. Let $\xi \neq 1$ be in $G$. We can suppose $\xi > 1$ and the norm $N_K(\xi) = 1$. The conjugates of $\xi$ are not real because those of $\alpha$ are not. Call $\gamma$ and $\overline{\gamma}$ these conjugates. We have $\xi \gamma \overline{\gamma} = 1$ and $|\gamma| < 1$ since the norm of $\xi$ is 1 and $\xi > 1$. We will show that $\lambda = \xi^m$ satisfy i), ii) and iii) for infinitely many $m \in \mathbb{N}$.

The minimal polynomial of $\lambda$ is $Q(x) = x^3 - a_1 x^2 - a_2 x - 1$ with

$$a_1 = a_1(m) = \xi^m + \gamma^m + \overline{\gamma}^m$$
$$a_2 = a_2(m) = -[\xi^m(\gamma^m + \overline{\gamma}^m) + |\gamma|^{2m}]$$

Since $\xi > 1 > |\gamma|$, $a_1$ is positive for $m$ large and $a_2$ will be positive if the argument of $\gamma$ is well chosen. Call $\alpha$ the argument of $\gamma$ and $\rho = \frac{1}{\sqrt{\xi}}$ its modulus.
First case. $\frac{\alpha}{2\pi} \notin \mathbb{Q}$.
There exist infinitely many $m \in \mathbb{N}$ such that $m\alpha \in \left[\frac{2\pi}{3}, \frac{4\pi}{5}\right] \mod 2\pi$. Call $I$ the set of such $m$. For $m \in I$

$$a_1(m) = \xi^m + \frac{2}{\xi^{\frac{m}{2}}} \cos m\alpha$$
$$a_2(m) = -2\xi^{\frac{m}{2}} \cos m\alpha - \frac{1}{\xi^m} \geq -2\xi^{\frac{m}{2}} \cos \frac{2\pi}{3} - \frac{1}{\xi^m}$$

then

$$\lim_{m \to \infty, m \in I} a_1(m) = \lim_{m \to \infty, m \in I} a_2(m) = +\infty.$$ 
Moreover,

$$a_2(m) \leq -2\xi^{\frac{m}{2}} \cos \frac{4\pi}{5} - \frac{1}{\xi^m}$$
then

$$\liminf_{m \to \infty, m \in I} \frac{a_1(m)}{a_2^2(m)} \geq \frac{1}{4\cos^2 \frac{4\pi}{5}} > \frac{1}{3}.$$

Therefore the conditions i), ii) and iii) are satisfied for large $m$ in $I$.

Second case. $\frac{\alpha}{2\pi} = \frac{\varphi}{q} \in \mathbb{Q}$.
Since $\gamma \notin \mathbb{R}$, $q > 2$. First note that $q \neq 4$ for, if $q = 4$, we have

$$0 = \Re(\gamma^3 - a_1\gamma^2 - a_2\gamma - 1) = a_1\rho^2 - 1$$
$$0 = \Im(\gamma^3 - a_1\gamma^2 - a_2\gamma - 1) = \pm \rho(\rho^2 + a_2)$$
so $a_1 = -a_2 = \rho = 1$ and $\gamma = \pm i$. This is impossible because the degree of the minimal polynomial of $\gamma$ is 3. So $q \in \{3\} \cup \{5,6,\ldots\}$. If $q = 3,5$ or 6, it is easy to see that there exist infinitely many $m \in \mathbb{N}$ such that $m\alpha \in \left[\frac{4\pi}{5} - \frac{2\pi}{7}, \frac{4\pi}{5}\right] \mod 2\pi$ while a similar conclusion is obvious if $q \geq 7$.
Now, we can conclude as in the previous case for $\frac{4\pi}{5} - \frac{2\pi}{7} > \frac{\pi}{2}$.

From now on, $a_1, a_2 \geq 1$ are two integers such that $P(x) = -1 + a_1x + a_2x^2 + x^3$ has a unique real root $\zeta$. We use the notations of Sections 2 and 3, the norm $N$ as defined in Section 2 and $\rho_0$ as defined at the beginning of Section 3.

Lemma 8.

$$\rho_0^2 \geq \frac{4a_1 - a_2^2 + 2a_2\zeta + 3\zeta^2}{2(a_1 + \zeta^2)}$$

Proof. By definition

$$\rho_0^2 \geq \min_{x \in \mathbb{R}} \min N^2(e_1 + xe_2), \min_{x \in \mathbb{R}} N^2(e_2 + xe_1)).$$

We have

$$N^2(e_1 + xe_2) = \langle e_1, e_1 \rangle + 2x\langle e_1, e_2 \rangle + x^2\langle e_2, e_2 \rangle$$
then
\[ \min_{x \in \mathbb{R}} N^2(e_1 + xe_2) = (e_1, e_1) - \frac{(e_1, e_2)^2}{(e_2, e_2)} = \frac{4(a_1 + \zeta^2) - (a_2 - \zeta)^2}{2} \]
similarly
\[ \min_{x \in \mathbb{R}} N^2(e_2 + xe_1) = (e_2, e_2) - \frac{(e_1, e_2)^2}{(e_1, e_1)} = \frac{4(a_1 + \zeta^2) - (a_2 - \zeta)^2}{2(a_1 + \zeta^2)}, \]
and since \( a_1 \geq 1 \),
\[ \rho_0^2 \geq \frac{4a_1 - a_2^2 + 2a_2 \zeta + 3\zeta^2}{2(a_1 + \zeta^2)}. \]
\[ \square \]

**Lemma 9.** Suppose \( a_1 \) and \( a_2 \) satisfy condition iii) of Lemma 7. For \( a_1 \) sufficiently large, \( N(\theta) \leq \frac{1}{2} \rho_0 \) and \( \theta \) is a best approximation vector of \( \theta \).

**Proof.** Put \( \phi(a_1, a_2) = \frac{4a_1 - a_2^2 + 2a_2 \zeta + 3\zeta^2}{2(a_1 + \zeta^2)}. \) We have
\[ \lim_{a_1 \to \infty} \zeta(a_1, a_2) = 0 \]
whereby
\[ \lim_{a_1 \to \infty, 3a_1 \geq a_2} \phi(a_1, a_2) \geq \frac{1}{2} \]
and so
\[ N^2(\theta) = N^2(F(e_2)) = 2\zeta < \frac{1}{4} \phi(a_1, a_2) \leq \frac{1}{4} \rho_0^2 \]
for \( a_1 \) sufficiently large. Now if \( P \in \mathbb{Z}^2 \setminus \{(0, 0)\} \), then \( N(\theta - P) \geq N(P) - N(\theta) \geq \frac{1}{2} \rho_0. \)
\[ \square \]

**Lemma 10.** If \( q \in \{0, \ldots, a_1 - 1\} \) then \( N(q\theta - e_1) > N(\theta) \).

**Proof.**
\[ N^2(q\theta - e_1) > N^2(\theta) \]
\[ \iff (q^2 - 1)(\theta, \theta) - 2q(\theta, e_1) + (e_1, e_1) > 0 \]
\[ \iff (q^2 - 1)(F(e_2), F(e_2)) - 2q[2(a_1 + \zeta^2)\zeta + (a_2 - \zeta)\zeta^2] + 2(a_1 + \zeta^2) > 0 \]
\[ \iff 2(q^2 - 1)\zeta - 2q(a_1\zeta + 1) + 2(a_1 + \zeta^2) > 0 \]
\[ \iff a_1 - q + (q^2 - 1 - a_1q)\zeta + \zeta^2 > 0 \]
\[ \iff (a_1 - q)(a_1\zeta + a_2\zeta^2 + \zeta^3) + (q^2 - 1 - a_1q)\zeta + \zeta^2 > 0 \]
\[ \iff q^2 + a_1^2 - 2a_1q - 1 + a_2(a_1 - q)\zeta + (a_1 - q)\zeta^2 > 0. \]
\[ \square \]
Lemma 11. Suppose $a_1$ and $a_2$ satisfy condition iii) of Lemma 7. For $a_1$ sufficiently large, $\theta$ and $a_1\theta - e_1$ are the first two best approximation vectors.

Proof. Since $a_1\theta - e_1 = F(\theta)$, the only thing to prove is

$$\inf_{q \in \{2, \ldots, a_1-1\}} \inf_{P \in \mathbb{Z}^2} N(q\theta - P) > N(\theta).$$

If $N(q\theta - P) \leq \frac{1}{2}\rho_0$, then by definition of $\rho_0$

$$|q\zeta - p_1| \leq \frac{1}{2}$$

$$|q\zeta^2 - p_2| \leq \frac{1}{2}$$

where $P = (p_1, p_2)$. Furthermore, if $q < a_1$ and if $a_1$ is large, then $q\zeta \leq 1$ and $q\zeta^2 \leq \frac{1}{2}$. Therefore,

$$\inf_{P \in \mathbb{Z}^2} N(q\theta - P) = \inf(N(q\theta), N(q\theta - e_1))$$

$$\geq \inf(qN(\theta), N(q\theta - e_1)) > N(\theta)$$

for $q \in \{2, \ldots, a_1-1\}$. \qed

End of proof of Proposition 2. By Lemma 7 there exists a unit $\lambda \in \mathbb{Q}(\alpha)$ which satisfies conditions i), ii) and iii) with $a_1$ large. $\zeta = \frac{1}{\lambda}$ is also unit. By Lemma 9, $\theta = (\zeta, \zeta^2)$ is a best approximation vector and by Lemma 11, $F(\theta) = a_1\theta - e_1$ is the next best approximation vector. Since $N(a_1\theta - e_1) < N(\theta) < \frac{1}{2}\rho_0$, by Proposition 6 we have $\mathcal{M}(\theta) = \{F^n(\theta) : n \in \mathbb{N}\}$.

5. The equations $1 = x^3 + a_2x^2 + x$

The polynomial $P(x) = x^3 + a_2x^2 + x - 1$ has only one real root if $a_2 = 1$ or 2.

5.1. $a_2 = 1$. Call $\zeta$ the positive root of $1 = x^3 + x^2 + x$ and $\theta = (\zeta, \zeta^2)$. N. Chekhova, P. Hubert, A. Messaoudi have proved that $\mathcal{M}(\theta) = \{F^n(\theta - e_1) : n \in \mathbb{N}\}$. If we want to recover this result with Proposition 6, we just have to show:

i) $\theta - e_1$ is a best approximation vector,

ii) $F(\theta - e_1)$ is the next best approximation vector,

iii) $N(F(\theta - e_1)) < \frac{1}{2}\rho_0$.

First note that $N(\theta - e_1) = 2\theta - e_1 - e_2$ and $N(F(\theta - e_1)) = \zeta N(\theta - e_1) < N(\theta - e_1)$, so if i) is true then 2 is the next best approximation and if iii) is also true, then $2\theta - e_1 - e_2$ is a best approximation vector. Let us now prove iii) and afterward i):

$$N^2(F(\theta - e_1)) = N^2(F^3(e_2)) = 2\zeta^3 < \frac{3 + 2\zeta + 3\zeta^2}{8(1 + \zeta^2)} \leq \frac{1}{4}\rho_0^2.$$
for
\[
2\zeta^3 < \frac{3 + 2\zeta + 3\zeta^2}{8(1 + \zeta^2)}
\]
\[\Leftrightarrow 3 + 2\zeta + 3\zeta^2 - 16\zeta^3(1 + \zeta^2) > 0\]
\[\Leftrightarrow 3(\zeta + \zeta^2 + \zeta^3) + 2\zeta + 3\zeta^2 - 16\zeta^3(1 + \zeta^2) > 0\]
\[\Leftrightarrow 5 + 6\zeta - 13\zeta^2 - 16\zeta^4 > 0\]
\[\Leftrightarrow 11 - 8\zeta + 5\zeta^2 - 16\zeta^3 > 0\]
\[\Leftrightarrow 3 + 16\zeta - 5\zeta^2 > 0\]

and the last inequality is obvious. Since \(\zeta > \frac{1}{2}\), \(2\zeta^3 < \frac{1}{4}\rho_0^2 \Rightarrow N^2(\theta - e_1) = 2\zeta^2 < \frac{1}{2}\rho_0^2 \leq \rho_0^2\). Then the point \(P = (p_1, p_2) \in \mathbb{Z}^2\) which is the nearest to \(\theta\), is one of \((0, 0), e_1, e_2\) or \(e_1 + e_2\). We have
\[
N^2(\theta - e_1) = \zeta N^2(\theta) < N^2(\theta)
\]

and
\[
N^2(\theta - e_2) = N^2(\theta) - 2(\theta, e_2) + 2 = 2\zeta - 2\zeta(1 - \zeta) - 4\zeta^2 + 2
\]
\[
= 2(1 - \zeta^2) > 2\zeta^2 = N(\theta - e_1),
\]
\[
N^2(\theta - e_1 - e_2) = N^2(\theta - e_1) - 2(\theta - e_1, e_2) + 2
\]
\[
= 2\zeta^2 - 2\zeta(1 - \zeta) - 4\zeta^2 + 2(e_1, e_2) + 2
\]
\[
= \zeta^2 - 2\zeta(1 - \zeta) - 4\zeta^2 + 2(1 - \zeta) + 2 = 4 - 4\zeta > 2\zeta^2,
\]

so \(P\) must be \(e_1\) and this completes the proof of i).

5.2. \(a_2 = 2\). Call \(\zeta\) the positive root of \(1 = x^3 + 2x^2 + x\) and \(\theta = (\zeta, \zeta^2)\). The set of all best approximations is given by two initial points
\[
\mathcal{M}(\theta) = \{B^nX_i : n \in \mathbb{N}, i = 1, 2\}
\]
where \(X_1 = \theta\) and \(X_2 = 2\theta - e_1\). To prove this result, by Proposition 6, we have to check the following properties:

i) \(\theta - e_1\) is the best approximation vector,

ii) \(2\theta - e_1\) is the next best approximation vector,

iii) \(F(\theta - e_1) = 3\theta - e_1, F(2\theta - e_1) = 4\theta - 2e_1 - e_2,\)

iv) \(N(3\theta - e_1) < \frac{1}{2}\rho_0\).

This requires some tedious calculations very similar to the case \(a_2 = 1\).

References


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