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RÉSUMÉ. Titre français : Sur les corps abéliens dont le nombre de classes est égal au nombre de genres.
Nous savons qu'il n'existe qu'un nombre fini de corps abéliens imaginaires pour lesquels le nombre de classes est égal au nombre de genres. Ceux de ces corps qui sont cycliques et non quadratiques ont été classés dans [Lou2,4] et dans [CK]. Dans cet article, nous déterminons tous les corps abéliens non cycliques dont le nombre de classes est égal au nombre de genres. Cela achève la classification des corps abéliens possédant une classe par genre, sauf dans le cas des corps quadratiques imaginaires.

ABSTRACT. We know that there exist only finitely many imaginary abelian number fields with class numbers equal to their genus class numbers. Such non-quadratic cyclic number fields are completely determined in [Lou2,4] and [CK]. In this paper we determine all non-cyclic abelian number fields with class numbers equal to their genus class numbers, thus the one class in each genus problem is solved, except for the imaginary quadratic number fields.

1. Introduction

For an abelian number field N the narrow genus field G_N of N is the maximal abelian number field containing N such that the extension G_N/N is unramified at all finite places. The degree $[G_N : N]$ is called the genus class number of N and is denoted by g_N. The genus class number of N is easy to determine, namely, $g_N = \frac{1}{[N:\mathbb{Q}]} \prod_p e_p$, where e_p is the ramification index of p in N. In particular, when N is imaginary, the genus field of N is contained in the Hilbert class field of N, whence g_N divides the class number of N. In [Lou1], Louboutin has proved that there exist only finitely many imaginary abelian number fields with class numbers equal to their genus class numbers. Imaginary non-quadratic cyclic number fields

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with class numbers equal to their genus numbers have been recently classified([Lou2, 4], [CK]). It is known that under a suitable generalized Riemann Hypothesis, there are exactly 65 imaginary quadratic number fields with class numbers equal to their genus numbers([We] and [Lou5]). These fields are listed in Table **I**. The aim of this paper is to prove the following result:

Theorem 1. *There are exactly 424 imaginary non-quadratic abelian number fields with class numbers equal to their genus class numbers:77 out of them are cyclic and 347 are non-cyclic. Their degrees are less than or equal to 24, their class numbers are 1, 2 or 4. The conductors of these fields are less than or equal to 65689. These fields are listed in tables at the end of this paper.*

We note that in [M] under the assumption of the Generalized Riemann Hypothesis, Miyada has determined all imaginary abelian number fields N with class numbers equal to their genus class numbers such that the Galois groups are elementary 2-groups. (There seems to be one misprint in [M, Table 2]: the field $K = \mathbb{Q}\left(\sqrt{-2}, \sqrt{-37}\right)$ appears twice, i.e. for $g_K = h_K = 1$ and for $g_K = h_K = 2$. In fact, $g_K = h_K = 2$.) However, we do not need any assumption for the proof of Theorem 1. This paper is organized in the following way. Section 2 presents some well-known facts on imaginary abelian number fields. In Section 3 we illustrate our computations. Throughout this paper the following notation will be used. For an imaginary abelian number field K, let K^+, f_K, h_K, ω_K, G_K, g_K, h_K^-, Q_K be the maximal real subfield, the conductor, the class number, the number of roots of unity in K, the genus field, the genus class number, the relative class number and the Hasse unit index of K, respectively. For an odd prime p let χ_p be an odd Dirichlet character of conductor p, of degree $p-1$. When $\rho \geq 2$, let ψ_{p^ρ} be an even primitive Dirichlet character of conductor p^ρ, of order $p^{\rho-1}$ with $\psi_{p^\rho}^p = \psi_{p^{\rho-1}}$. For the prime $p = 2$, let χ_4 be the odd Dirichlet quadratic character of conductor 4. When $\rho \geq 3$, let ψ_{2^ρ} be an even primitive Dirichlet character of conductor 2^ρ, of order $2^{\rho-2}$ with $\psi_{2^\rho}^2 = \psi_{2^{\rho-1}}$.

2. Preliminaries

In this section we review some well-known facts concerning imaginary abelian number fields.

Proposition 1. (1) *Let F be an abelian number field. If $h_F = g_F$, then $h_M = g_M$ for any subfield M of F.*

(2) *Let F be an imaginary abelian number field, χ_F the group of primitive Dirichlet characters associated to F and χ_F^- the subset of $\chi \in \chi_F$ such that $\chi(-1) = -1$. For a $\chi \in \chi_F$ let us denote the conductor of χ by*

f_χ. *We have*

$$h_F^- = Q_F\omega_F \prod_{\chi\in X_F^-}\left(-\frac{1}{2}B_{1,\chi}\right),$$

where $B_{1,\chi} = \frac{1}{f_\chi}\sum_{a=1}^{f_\chi-1}\chi(a)a$.

(3) *If* F *is an imaginary cyclic number field, then* $Q_F = 1$. *Let* $F \subset K$ *be the two CM-fields. If* $[K : F]$ *is odd, then* $Q_K = Q_F$.

(4) *Let* F *be an imaginary cyclic number field of degree* $2m$. *If* $h_F = g_F$, *then* $1 \le m \le 10$.

Proof. (1) See Lemma 1 in [M]. (2) See Theorem 4.17 in [W]. (3) For the first statement see Satz 24 in [H]. For the second statement see Lemma 2 in [HY1]. (4) See [Lou2 and 4] and [CK]. □

3. Proof of Theorem 1

Let N be an imaginary abelian number field with Galois group G. When $G = \prod \mathbb{Z}/p_i^{n_i}\mathbb{Z}$, we say N is of type $(p_1^{n_1}, \cdots, p_i^{n_i}, \cdots)$. In addition we put $*$ when the corresponding subfield is imaginary. For example, if N is of type $(4^*, 2)$, then N is the compositum of an imaginary cyclic quartic field M_1 and a real quadratic field M_2 with $M_1 \cap M_2 = \mathbb{Q}$. Then the maximal real subfield N^+ is of type $(2, 2)$. If N is of type $(4^*, 2^*)$, then N is the compositum of an imaginary cyclic quartic field M_1 and an imaginary quadratic field M_2. In this case, N^+ is of type (4). A field N of type $(2^*, 2, 2)$ can be either of type $(2^*, 2^*, 2)$ or $(2^*, 2^*, 2^*)$. However we prefer to say that N is of type $(2^*, 2^*, 2^*)$ with as many as possible $*$ for a technical reason to be seen later.

Let N be an imaginary abelian number field of type $(2^{m_1^*}, \cdots, 2^{m_s^*}, n_1, \cdots, n_r)$ with $m_1 \ge \cdots \ge m_s$. We may assume that $n_i \lneq 2^{m_s}$ for n_i which is a 2-power. In order to describe N we give the associated group of Dirichlet characters $\langle \tau_1, \cdots, \tau_s, \varphi_1, \cdots, \varphi_r\rangle$. Here $\tau_1, \cdots, \tau_s$ are odd primitive Dirichlet characters of order 2^{m_i} for $1 \le i \le s$ and $\varphi_1, \cdots, \varphi_r$ are the even primitive Dirichlet characters of order n_j for $1 \le j \le r$. For each prime p let $G^{(p)}$ be the p-Sylow subgroup of G. Let $N^{(2)}$ be the maximal subfield of N of 2-power degree. The scheme of our proof of Theorem 1 is as follows. In 3.1 we determine all imaginary abelian number fields N with $h_N = g_N$ such that $G^{(2)}$ is an elementary 2-group. According to Theorem 2 below, if $G^{(2)}$ is an elementary 2-group, then the 2-rank of $G^{(2)}$ is less than or equal to 3. In 3.2 we determine all imaginary abelian number fields N with $h_N = g_N$ such that $N^{(2)}$ is of type $(2^{m_1^*}, 2^{m_2})$ with $m_1 > m_2 \ge 1$. Moreover we prove that if N is an imaginary abelian number field with $h_N = g_N$, then 2-rank of $G^{(2)}$ is less than or equal to 3. If it is equal to 3, then $G^{(2)}$ is either $(2^*, 2^*, 2^*)$ or $(4^*, 2^*, 2^*)$. In 3.3 we determine all imaginary abelian number fields N with $h_N = g_N$ such that $N^{(2)}$ is of type $(2^{m_1^*}, 2^{m_2^*})$ with $m_1 \geq 2$ and $m_1 \geq m_2$. Finally it remains to determine all fields N with $h_N = g_N$ such that $N^{(2)} = (4^*, 2^*, 2^*)$. These fields are treated in 3.4. Our determinations rely on the computations of the relative class numbers h_N^- of N. The class numbers h_{N^+} of N^+ which we need are known in [G], [L], [Ma] and [Mä]. In the remaining part of this paper we say briefly "character of order m" instead of "primitive Dirichlet character of order m".

3.1. $N^{(2)}$ is of type $(2^*, \cdots, 2^*)$.

Proposition 2. *Let N be an imaginary abelian number field, G the Galois group of $N/\mathbb{Q}$. Assume that G is not cyclic and that the 2-Sylow subgroup of G is cyclic. If $h_N = g_N$, then N is of type $(2^*, 3, 3)$ and N is associated with $\langle \chi_3, \psi_9, \chi_7^2 \rangle$.*

Proof. First, we claim that if $h_N = g_N$, then N is of type $(2^*, 3, \cdots, 3)$. According to [CK] there are 4 imaginary cyclic number fields of degree 10 with class numbers equal to their genus class numbers. By Proposition 1.(1) we cannot have a field N with $h_N = g_N$ such that N is of type $(2^*, 5, 5)$. By the same argument if N is of type $(2^*, 7, 7)$, $(2^*, 9, 9)$, $(4^*, 3, 3)$ or $(4^*, 5, 5)$, then $h_N \neq g_N$. According to Proposition 1.(1) it follows that if $h_N = g_N$, then N is of type $(2^*, 3, \cdots, 3)$. Let us now consider with the fields of type $(2^*, 3, 3)$. Let χ be an odd quadratic character, φ_1, φ_2 two cubic characters, N the field associated with $\langle \chi, \varphi_1, \varphi_2 \rangle$. Let M_1, M_2, M_3 and M_4 be the subfields associated with $\langle \chi, \varphi_1 \rangle$, $\langle \chi, \varphi_1\varphi_2 \rangle$, $\langle \chi, \varphi_1\varphi_2^2 \rangle$, and $\langle \chi, \varphi_2 \rangle$, respectively. The fact that $h_N = g_N$ implies that $h_{M_i} = g_{M_i}$ for $1 \leq i \leq 4$. According to [Lou4], there is only one N such that $N = M_1M_2M_3M_4$ with $h_{M_i} = g_{M_i}$ for $1 \leq i \leq 4$. That is, $N = M_1M_2M_3M_4 = M_1M_2$, where M_1, M_2, M_3 and M_4 are associated with $\langle \chi_3, \psi_9 \rangle, \langle \chi_3, \chi_7^2 \rangle, \langle \chi_3, \chi_7^4\psi_9 \rangle$, and $\langle \chi_3, \chi_7^2 \rangle$ respectively. We verify that this field has $h_N = g_N = 1$. Since there is only one field of type $(2^*, 3, 3)$ with class number equal to its genus class number, there is no field of type $(2^*, 3, 3 \cdots 3)$, of degree ≥ 54 with class number equal to its genus class number. □

Theorem 2. *Let N be an imaginary abelian number field such that the Galois group G is an elementary 2-group. Assume that $g_N = h_N$. Then we have*

(1) $[N : \mathbb{Q}] \leq 2^3$.
(2) *If $G = (2, 2)$, then $h_N = g_N = 1, 2$ or 4.*
(3) *If $G = (2, 2, 2)$, then $h_N = g_N = 1$ or 2.*

Proof. See Theorem 1 in [M]. □

Proposition 3. (1) *Let N be an imaginary bicyclic biquadratic number field and let k_1 and k_2 be two imaginary quadratic subfields. If $h_N = g_N$, then h_{k_1} and h_{k_2} are 1, 2 or 4.*

(2) *There are exactly 219 bicyclic biquadratic number fields with class numbers equal to their genus class numbers. These fields are listed in Table* **III**.

(3) *There exist exactly 42 imaginary abelian number fields N of degree > 4 with class numbers equal to their genus numbers such that $N^{(2)}$ is of type* $(2^*, 2^*)$. *These fields are listed in Table* **IV**.

Proof. (1) Suppose that $h_N = g_N$ and $h_{k_1} \geq 8$. Since $h_N^- = \frac{Q_N}{2} h_{k_1} h_{k_2}$, we must have $h_N^- = 4$, $Q_N = 1$, $h_{k_1} = 8$, $h_{k_2} = 1$ and $h_{N^+} = 1$. By [Corollary 1, M], there exist at most three rational primes ramified in N. The fact that $h_N = g_N = 4 = \frac{1}{4} \prod e_p$ implies that there exist exactly three ramified primes including 2 with $e_2 = 4$. This leads a contradiction. Since the fact that $e_2 = 4$, $h_{k_2} = 1$ and $h_{N_+} = 1$ implies $k_2 \in \{\mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-2})\}$ and $N_+ \in \{\mathbb{Q}(\sqrt{2}), \mathbb{Q}(\sqrt{p}), \mathbb{Q}(\sqrt{2p})\}$ with $p \equiv 1 \bmod 4$, there exist at most two ramified primes in $N = k_2 N_+$.

(2) There are 81 imaginary quadratic number fields with class numbers 1, 2 or 4([S1,2] and [A]). Among these 81 fields there are 51 fields with class numbers equal to their genus class numbers. In order to get all bicyclic biquadratic number fields with class numbers equal to their genus class numbers it is sufficient to examine the composita of two of those 51 quadratic fields. Note that we do not assume the Generalized Riemann Hypothesis.

(3) Let N be an imaginary abelian number fields of degree > 4. Assume that $h_N = g_N$ and $N^{(2)}$ is of type $(2^*, 2^*)$. Bearing in mind the imaginary cyclic fields with class numbers equal to their genus class numbers and Proposition 2 it remains to consider the fields N of type $(2^*, 2^*, 3)$ or $(2^*, 2^*, 5)$.

i) Let τ_1, τ_2 be two odd quadratic characters, φ a cubic characters, N the field associated with $\langle \tau_1, \tau_2, \varphi \rangle$. Let M_1 and M_2 be the imaginary sextic subfields associated with $\langle \tau_1, \varphi \rangle$ and $\langle \tau_2, \varphi \rangle$, respectively. Let E be the imaginary bicyclic biquadratic subfield associated with $\langle \tau_1, \tau_2 \rangle$, k_1 and k_2 the imaginary quadratic subfields associated with τ_1 and τ_2, respectively. Using [Lou4], we make a list of $(M_1, M_2)'s$ such that $h_{M_1} = g_{M_1}$, $h_{M_2} = g_{M_2}$ and $h_E = g_E$. There are 40 pairs of (M_1, M_2) satisfying $h_{M_1} = g_{M_1}, h_{M_2} = g_{M_2}$ and $h_E = g_E$. For these 40 fields we determine h_N^- using the fact that $h_N^- = Q_N \omega_N \frac{h_{M_1}^-}{\omega_{M_1}} \frac{h_{M_2}^-}{\omega_{M_2}}$ and $Q_N = Q_E$. The class numbers of real sextic number fields are known by [Mä]. There are 39 fields N of type $(2^*, 2^*, 3)$ with $h_N = g_N$. (The remaining field is associated with $\langle \chi_4 \psi_8, \chi_{11}^5, \chi_7^2 \rangle$. This field has class number 3 and genus class number one.)

ii) By the same method as i) we verify that there are three imaginary abelian number fields N of type $(2^*, 2^*, 5)$ with $h_N = g_N$.

□

Proposition 4. *There are exactly 19 imaginary abelian number fields N with class numbers equal to their genus class numbers such that the subfield $N^{(2)}$ is type of $(2^*, 2^*, 2^*)$. There are exactly 17 fields of type $(2^*, 2^*, 2^*)$, and two fields of type $(2^*, 2^*, 2^*, 3)$. These fields are listed in Table* **V**.

Proof. Let N be an imaginary abelian number field N with class number equal to its genus class number such that the subfield $N^{(2)}$ is type of $(2^*, 2^*, 2^*)$. According to [Lou4] and [CK], if $p \geq 11$, then $G^{(p)}$ is trivial. Let us begin with the fields of type $(2^*, 2^*, 2^*)$.

If N is of type $(2^*, 2^*, 2^*)$ and if $h_N = g_N$, then $h_N = g_N = 1$ or 2. Let k_1, k_2, k_3 and k_4 be the four imaginary quadratic subfields of N. Since $h_N^- = Q_N \omega_N \prod_{i=1}^{4} \frac{h_i}{\omega_i}$, where $h_i = h_{k_i}$ and ω_i is the number of roots of unity in k_i for $i = 1, 2, 3$ and 4, there are at least three imaginary quadratic subfields k_i with $h_{k_i} \leq 4$ among the four imaginary quadratic subfields of N. It remains us to consider the composita $k_1k_2k_3$ of three imaginary quadratic subfields k_i with class numbers 1, 2 or 4, $i = 1, 2$ and 3. There are 18 composita $N = k_1k_2k_3$ satisfying the following:

i) $h_{k_i} = 1, 2$ or 4 for $i = 1, 2, 3$.
ii) $h_{k_i} = g_{k_i}$ for $i = 1, 2, 3$.
iii) For the six imaginary bicyclic biquadratic subfields M of N we have $h_M = g_M$.

Computing the unit indices Q_N by [HY3] and h_{N^+} by [Satz 5, K] for these 18 fields N, we verify that there are exactly 17 octic fields N with $h_N = g_N$.

Consider now the fields of type $(2^*, 2^*, 2^*, 3)$. Let τ_1, τ_2, τ_3 be three odd quadratic characters, φ a cubic character, N the field associated with $\langle \tau_1, \tau_2, \tau_3, \varphi \rangle$. Let M_1, M_2, M_3 and M_4 be the four imaginary sextic subfields of N associated with $\langle \tau_1, \varphi \rangle, \langle \tau_2, \varphi \rangle, \langle \tau_3, \varphi \rangle$, and $\langle \tau_1\tau_2\tau_3, \varphi \rangle$, respectively. Let E be the subfield associated with $\langle \tau_1, \tau_2, \tau_3 \rangle$. According to [Lou4], there are only two fields N such that $h_{M_i} = g_{M_i}$ for $1 \leq i \leq 4$ and $h_E = g_E$. For these two fields we compute h_N^- and h_{N^+}. The results are summarized in Table **V**.

Finally, it is sufficient to notice that there is no field N of type $(2^*, 2^*, 2^*, 5)$ with $h_N = g_N$. □

3.2. $N^{(2)}$ is of type $(2^{m_1^*}, 2^{m_2})$ with $m_1 > m_2 \geq 1$.

Proposition 5. (1) *There are 15 imaginary abelian number fields of type $(4^*, 2)$ with class numbers equal to their genus class numbers. These fields are given in Table* **VIII**

(2) *Let m be an odd integer with $m \geq 3$. There is no imaginary abelian number field of degree $8m$ with class number equal to its genus class number such that the subfield $N^{(2)}$ is of type $(4^*, 2)$.*

(3) *There is no field of type $(8^*, 2)$ with class number equal to its genus class number. There is no field of type $(16^*, 2)$ with class number equal to its genus class number. Consequently, if N is an imaginary abelian number field with $h_N = g_N$ such that the subfield $N^{(2)}$ is of type $(2^{m_1^*}, 2^{m_2})$ with $m_1 > m_2 \geq 1$, then N is of type $(4^*, 2)$.*

(4) *There is no imaginary abelian number field of type $(4^*, 2, 2)$ with class number equal to its genus class number.*

(5) *If N is an imaginary abelian number field with $h_N = g_N$, then the 2-rank of $G^{(2)}$ is less than or equal to 3. Moreover, if the 2-rank of $G^{(2)}$ is equal to 3, then $G^{(2)} = (2^*, 2^*, 2^*)$ or $(4^*, 2^*, 2^*)$.*

Proof. (1) Let φ be an odd primitive character of order 4, χ an even primitive quadratic character, N the associated field with $\langle \varphi, \chi \rangle$. Let M_1 and M_2 be two imaginary cyclic quartic subfields of N associated with $\langle \varphi \rangle$ and $\langle \varphi\chi \rangle$, respectively. The real quadratic subfields of M_1 and M_2 coincide. In order to obtain N with $h_N = g_N$ it is sufficient to consider the composita M_1M_2 such that $h_{M_1} = g_{M_1}$ and $h_{M_2} = g_{M_2}$. There are 35 fields N such that $N = M_1M_2$, $h_{M_1} = g_{M_1}$ and $h_{M_2} = g_{M_2}$. The Hasse unit indices Q_N are easily obtained from Sätze 15 and 22 of [H]. The relative class number h_N^- can be expressed as

$$h_N^- = Q_N \frac{\omega_N}{\omega_{M_1}\omega_{M_2}} h_{M_1}^- h_{M_2}^-.$$

The class number h_{N_+} can be computed following [K]. It remains 15 fields N such that $h_N = g_N$. Similarly we verify (3).

(2) Let N be an imaginary abelian number field of degree $8m$, containing a subfield of type $(4^*, 2)$. Suppose $h_N = g_N$. According to [CK], we have $m = 3$. Let φ, χ and ω be an odd character of order 4, an even quadratic character and a cubic character, respectively. Assume that N is associated with $\langle \varphi, \chi, \omega \rangle$. Let M_1 and M_2 be the subfields associated with $\langle \varphi\omega \rangle$ and $\langle \varphi\chi\omega \rangle$, respectively. Using Table **II** in [CK], we verify that there is no pair of (M_1, M_2) such that $h_{M_1} = g_{M_1}$ and $h_{M_2} = g_{M_2}$.

(4) Let φ be an odd primitive character of order 4, χ_1 and χ_2 two even primitive quadratic character, N the associated field with $\langle \varphi, \chi_1, \chi_2 \rangle$. Let M_1, M_2, M_3 and M_4 be four imaginary cyclic quartic subfields of N associated with $\langle \varphi \rangle$, $\langle \varphi\chi_1 \rangle$, $\langle \varphi\chi_2 \rangle$ and $\langle \varphi\chi_1\chi_2 \rangle$, respectively. The fields $M_i, 1 \leq i \leq 4$, have the same real quadratic subfield. Suppose that $h_N = g_N$. Then we have $h_{M_i} = g_{M_i}$, for $1 \leq i \leq 4, h_{M_1M_2} = g_{M_1M_2}, h_{M_1M_3} = g_{M_1M_3}, h_{M_1M_4} = g_{M_1M_4}, h_{M_2M_3} = g_{M_2M_3}, h_{M_2M_4} =$

$g_{M_2M_4}$ and $h_{M_3M_4} = g_{M_3M_4}$. We verify that there is no such quadruple of (M_1, M_2, M_3, M_4).

(5) By (3) and (4) it is sufficient to verify that there is no field N of type $(4^*, 4^*, 2^*)$ with $h_N = g_N$. Suppose that there is a field N of type $(4^*, 4^*, 2^*)$ with $h_N = g_N$. Let φ_1 and φ_2 be two odd primitive characters of order 4, χ the odd quadratic character such that N is associated to the group $\langle \varphi_1, \varphi_2, \chi \rangle$. Then we would have three fields of type $(4^*, 4^*)$ with class numbers equal to their genus class numbers, i.e. the fields associated with $\langle \varphi_1, \varphi_2 \rangle, \langle \varphi_1, \varphi_1\varphi_2\chi \rangle$ and $\langle \varphi_2, \varphi_1\varphi_2\chi \rangle$. This contradicts Proposition 8.(1) below.

□

3.3. $N^{(2)}$ is of type $(2^{m_1^*}, 2^{m_2^*})$ with $m_1 \geq 2$ and $m_1 \geq m_2$.

Bearing in mind Proposition 5.(3) if $h_N = g_N$ and $N^{(2)}$ is of type $(2^{m_1^*}, 2^{m_2^*})$ with $m_2 \geq 2$, then $m_1 = m_2 = 2$. We need only consider N such that $N^{(2)}$ is of type $(4^*, 2^*)$, $(8^*, 2^*)$, $(16^*, 2^*)$ or $(4^*, 4^*)$.

Proposition 6. *There exist 39 imaginary abelian number fields N with $h_N = g_N$ such that the subfield $N^{(2)}$ is of type $(4^*, 2^*)$: 36 of them are of type $(4^*, 2^*)$ and 3 of them are of type $(4^*, 2^*, 3)$. These fields are listed in Table* **VI**.

Proof. According to [CK], if N is an imaginary abelian number field with $h_N = g_N$ such that the subfield $N^{(2)}$ is of type $(4^*, 2^*)$, then $[N : \mathbb{Q}] \leq 40$. Our first step is to determine all fields of type $(4^*, 2^*)$. Let φ be an odd character of order 4, χ an odd quadratic character, N the field associated with $\langle \varphi, \chi \rangle$. Let M be the field associated with $\langle \varphi \rangle$ and let k_1 and k_2 be the two imaginary quadratic subfields associated with $\langle \chi \rangle$ and $\langle \varphi^2 \chi \rangle$, respectively. Using [Lou2] we make a list of N such that $h_M = g_M$ and $h_E = g_E$, where E is the field associated $\langle \varphi^2, \chi \rangle$. We use the formula

$$h_N^- = Q_N \omega_N \prod_{\chi \in X_N^-} \left(-\frac{1}{2} B_{1,\chi} \right) = Q_N \omega_N \frac{h_M^-}{\omega_M} \frac{h_{k_1}}{\omega_{k_1}} \frac{h_{k_2}}{\omega_{k_2}}.$$

To determine Q_N we use the tables in [H], [YH1] and [YH2] for the fields of conductors ≤ 200, and Sätze 15 and 22 in [H] for the fields of conductor > 200. For example let us consider Q_N for the field N which is associated with $\langle \chi_5 \psi_8, \chi_7^3 \rangle$. We have $N_+ = \mathbb{Q}\left(\sqrt{7\left(5+\sqrt{5}\right)}\right)$ and $N = N_+\left(\sqrt{-7}\right)$. In order to determine Q_N it is sufficient to know whether the prime ideal of N lying above 7 is principal or not. Using the function IdealIsPrincipal of KASH([KT]) we know that this prime ideal is principal. Hence we conclude that $Q_N = 2$. On the other hand, the class number of real quartic fields associated with $\langle \varphi\chi \rangle$ are known by [G]. Using the above results we can easily determine all fields of type $(4^*, 2^*, 3)$ and verify that there is no field of type $(4^*, 2^*, 5)$ with class number equal to its genus class number. The computational results are compiled in Table **VI**. □

Proposition 7. (1) *Let N be an imaginary abelian number field with $h_N = g_N$. If the subfield $N^{(2)}$ is of type $(8^*, 2^*)$, then N is of type $(8^*, 2^*)$. There exist three imaginary abelian number fields of type $(8^*, 2^*)$ with class numbers equal to their genus class numbers. These fields are given in Table* **VII**.

(2) *There is no imaginary number field of type $(16^*, 2^*)$ with class number equal to its genus class number.*

Proof. (1) According to [CK] we know that for an odd integer $m \geq 3$ there is no imaginary cyclic number field of degree $8m$ with class number equal to its genus class number. Let φ be an odd character of order 8, χ an odd quadratic character, N the field associated with $\langle \varphi, \chi \rangle$. Let M, L, F, E, k_1, and k_2 be the subfields associated with $\langle \varphi \rangle, \langle \varphi^2 \chi \rangle, \langle \varphi^2, \chi \rangle, \langle \varphi^4, \chi \rangle, \langle \chi \rangle$ and $\langle \varphi^4 \chi \rangle$, respectively. By [Lou2], Propositions 3 and 6 we have three fields N such that $h_M = g_M, h_L = g_L, h_F = g_F$ and $h_E = g_E$. These fields are listed in Table **VII**. Using the formula

$$h_N^- = Q_N \frac{\omega_N}{\omega_M \omega_L \omega_{k_1} \omega_{k_2}} h_M^- h_L^- h_{k_1} h_{k_2}$$

we obtain immediately h_N^-. The class numbers of real cyclic fields of conductor ≤ 100 are known by [Ma]. The class number of the real abelian number fields of conductor 160 in Table **VII** is equal to 2 according to Theorem 3 of [L].

(2) It is clear from the fact that there is only one imaginary cyclic number field of degree 16 whose class number is equal to its genus class number([Lou2]).

□

Proposition 8. (1) *There are two imaginary abelian number fields of type $(4^*, 4^*)$ with class numbers equal to their genus class number. These fields are given in Table* **IX**.

(2) *Let m be an odd integer with $m \geq 3$. There is no imaginary abelian number field of degree $16m$ with class number equal to its genus class number such that the subfield $N^{(2)}$ is of type $(4^*, 4^*)$.*

Proof. (1) Let φ_1 and φ_2 be two odd quartic characters, N the field associated with $\langle \varphi_1, \varphi_2 \rangle$. Let M_1, M_2, M_3 and M_4 be the fields associated with $\langle \varphi_1 \rangle, \langle \varphi_2 \rangle, \langle \varphi_1 \varphi_2^2 \rangle$ and $\langle \varphi_1^2 \varphi_2 \rangle$, respectively. We have only two fields N such that $h_{M_i} = g_{M_i}$ for $1 \leq i \leq 4$, $h_{M_1 M_3} = g_{M_1 M_3}$ and $h_{M_2 M_4} = g_{M_2 M_4}$. These two fields have class number one([Y]).

(2) It follows immediately from Proposition 5.(2).

□

3.4. $N^{(2)}$ is of type $(4^*, 2^*, 2^*)$.

Proposition 9. (1) *There are 7 imaginary abelian number fields of type* $(4^*, 2^*, 2^*)$ *with class numbers equal to their genus class numbers. These fields are given in Table* **X**.

(2) *Let* m *be an odd integer* ≥ 3. *There is no imaginary number field of degree* $16m$ *with class number equal to its genus class number such that the subfield* $N^{(2)}$ *is of type* $(4^*, 2^*, 2^*)$.

Proof. (1) Let φ be an odd character of order 4, χ_1 and χ_2 two odd quadratic characters, N the field associated with $\langle\varphi, \chi_1, \chi_2\rangle$. Let M_1, M_2, F_1, F_2, L and E be the imaginary subfields associated with $\langle\varphi\rangle$, $\langle\varphi\chi_1\chi_2\rangle$, $\langle\varphi, \chi_1\rangle$, $\langle\varphi, \chi_2\rangle$, $\langle\varphi, \chi_1\chi_2\rangle$ and $\langle\varphi^2, \chi_1, \chi_2\rangle$, respectively. Using [Lou2], Propositions 4 and 6 we verify that there are 7 fields N with $h_{M_1} = g_{M_1}$, $h_{M_2} = g_{M_2}$, $h_{F_1} = g_{F_1}$, $h_{F_2} = g_{F_2}$, $h_L = g_L$ and $h_E = g_E$. For these 7 fields we compute h_N^- and h_{N^+}. Note that

$$h_N^- = \frac{Q_N}{Q_E}\omega_N \frac{h_{M_1}^-}{\omega_{M_1}} \frac{h_{M_2}^-}{\omega_{M_2}} \frac{h_E^-}{\omega_E}.$$

(2) The proof of (2) is similar to that of Proposition 5.(2).

□

To conclude, the proof of Theorem 1 is completed by Propositions 2-9. All computations were carried out using PARI-GP[P] and KANT V4[KT].
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4. Tables

TABLE I. Cyclic number fields

$$\chi_5(2) = e^{2\pi i/4}, \chi_7(3) = e^{2\pi i/6}, \psi_9(2) = e^{2\pi i/3}, \chi_{13}(2) = e^{2\pi i/12},$$
$$\chi_{17}(3) = e^{2\pi i/16}, \chi_{19}(2) = e^{2\pi i/18}, \chi_{31}(3) = e^{2\pi i/30}$$

Type	$h_N = g_N$	f_N
2^*	1	3, 4, 7, 8, 11, 19, 43, 67, 163
	2	15, 20, 24, 35, 40, 51, 52, 88, 91, 115, 123, 148, 187, 232
		235, 267, 403, 427
	4	84, 120, 132, 168, 195, 228, 280, 312, 340, 372, 408, 435, 483
		520, 532, 555, 595, 627, 708, 715, 760, 795, 1012, 1435
	8	420, 660, 840, 1092, 1155, 1320, 1380, 1428, 1540, 1848, 1995
		3003, 3315
	16	5460

TABLE I continued.

Type	N	f_N	h_N^-	h_{N^+}	g_N	Type	N	f_N	h_N^-	h_{N^+}	g_N
4^*	χ_5	5	1	1	1	$(2^*,3)$	χ_7^3, χ_7^2	7	1	1	1
	χ_{13}^3	13	1	1	1		χ_3, ψ_9	9	1	1	1
	$\chi_4\psi_{16}$	16	1	1	1		χ_{19}^9, χ_{19}^6	19	1	1	1
	χ_{29}^7	29	1	1	1		χ_3, χ_7^2	21	1	1	1
	χ_{37}^9	37	1	1	1		χ_4, χ_7^2	28	1	1	1
	$\chi_5\psi_8$	40	2	1	2		$\chi_5^2\chi_7^3, \chi_7^2$	35	2	1	2
	$\chi_3\psi_{16}$	48	2	1	2		χ_4, ψ_9	36	1	1	1
	χ_{53}^{13}	53	1	1	1		χ_3, χ_{13}^4	39	1	1	1
	$\chi_3\chi_4\chi_5$	60	4	1	4		$\chi_{43}^{21}, \chi_{43}^{14}$	43	1	1	1
	χ_{61}^{15}	61	1	1	1		$\chi_3\chi_5^2, \psi_9$	45	2	1	2
	$\chi_5\chi_{13}^6$	65	2	1	2		$\chi_4\chi_{13}^6, \chi_{13}^4$	52	2	1	2
	$\chi_5^2\chi_{13}^3$	65	2	1	2		$\chi_4\psi_8, \chi_7^2$	56	1	1	1
	$\chi_5^2\chi_4\psi_{16}$	80	2	1	2		$\chi_3, \chi_7^2\psi_9$	63	1	3	3
	$\chi_5\psi_{16}$	80	2	2	4		$\chi_3, \chi_7^4\psi_9$	63	1	3	3
	$\chi_5\chi_{17}^8$	85	2	1	2		χ_7^3, ψ_9	63	1	1	1
	$\chi_5\chi_{17}^{12}$	85	2	2	4		$\chi_7^3, \chi_7^4\psi_9$	63	1	3	3
	$\psi_8\chi_{13}^3$	104	2	1	2		$\chi_{67}^{33}, \chi_{67}^{22}$	67	1	1	1
	$\chi_3\chi_5\chi_7^3$	105	4	1	4		$\chi_3\psi_8, \psi_9$	72	2	1	2
	$\chi_7^3\chi_{17}^4$	119	2	1	2		χ_4, χ_{19}^6	76	1	1	1
	$\chi_3\chi_4\psi_8\chi_5$	120	4	1	4		χ_{11}^5, χ_7^2	77	1	1	1
	$\chi_4\chi_5\chi_7^3$	140	4	1	4		$\chi_3\chi_4\chi_7^3, \chi_7^2$	84	4	1	4
	$\chi_3\chi_5\chi_{13}^9$	195	4	2	8		χ_7^3, χ_{13}^4	91	1	1	1
	$\chi_3\chi_5^2\chi_{17}^4$	255	4	1	4		$\chi_7^3, \chi_7^4\chi_{13}^4$	91	1	3	3
$(4^*,3)$	χ_{13}^3, χ_{13}^4	13	1	1	1		$\chi_7^3\chi_{13}^6, \chi_7^2\chi_{13}^4$	91	2	3	6
	χ_5, χ_7^2	35	1	1	1		$\chi_7^3\chi_{13}^6, \chi_7^4\chi_{13}^4$	91	2	3	6
	$\chi_{37}^9, \chi_{37}^{12}$	37	1	1	1		χ_3, χ_{31}^{10}	93	1	1	1
	χ_5, ψ_9	45	1	1	1		$\chi_4\psi_8, \chi_{13}^4$	104	1	1	1
	$\chi_{61}^{15}, \chi_{61}^{20}$	61	1	1	1		$\chi_3\chi_5^2, \chi_7^2$	105	2	1	2
	$\chi_{13}^3, \chi_7^2\chi_{13}^4$	91	1	3	3		$\chi_3, \psi_9\chi_{13}^8$	117	1	3	3
$(2^*,7)$	$\chi_{43}^{21}, \chi_{43}^6$	43	1	1	1		χ_3, χ_{43}^{14}	129	1	1	1
	χ_7^3, ψ_{49}	49	1	1	1		$\chi_7^3, \chi_7^4\chi_{19}^6$	133	1	3	3
16^*	χ_{17}	17	1	1	1		$\chi_{19}^9, \psi_9\chi_{19}^{12}$	171	1	3	3
$(2^*,9)$	χ_3, ψ_{27}	27	1	1	1		$\chi_7^3, \chi_7^4\chi_{31}^{10}$	217	1	3	3
	χ_{19}^9, χ_{19}^2	19	1	1	1		$\chi_{19}^9, \chi_{13}^8\chi_{19}^6$	247	1	3	3
$(4^*,5)$	χ_5, ψ_{25}	25	1	1	1	8^*	$\chi_4\psi_{32}$	32	1	1	1
$(2^*,5)$	χ_{11}^5, χ_{11}^2	11	1	1	1		χ_{41}^5	41	1	1	1
	χ_3, χ_{11}^2	33	1	1	1		$\chi_3\chi_{17}^2$	51	2	1	2
	χ_4, χ_{11}^2	44	1	1	1		$\chi_5\chi_{17}^2$	85	2	2	4
	$\chi_3\chi_5^2, \psi_{25}$	75	1	2	2						

TABLE II. $(2^*,3,3)$

N	N^+	f_N	Q_N	h_N^-	h_{N^+}	g_N
χ_3, χ_7^2, ψ_9	χ_7^2, ψ_9	63	1	1	1	1

TABLE III. $(2^*, 2^*)$

$h_N = g_N = 1$

f_{k_1}	f_{k_2}
3	4, 7, 8, 11, 15, 19, 24, 43, 51, 67, 123, 163, 267
4	7, 8, 11, 19, 20, 43, 52, 67, 148, 163
7	8, 11, 19, 35, 43, 91, 163, 427
8	11, 19, 40, 43, 67, 232
11	19, 67, 88, 163, 187
19	67, 163
43	67, 163
67	163

$h_N = g_N = 2$

f_{k_1}	f_{k_2}
3	20, 35, 40, 84, 88, 115, 132, 168, 187, 228, 232, 235, 372, 483, 627, 708
4	15, 24, 35, 40, 84, 88, 91, 115, 132, 228, 232, 372, 403, 532, 708, 1012
7	15, 20, 40, 51, 52, 84, 115, 123, 168, 187, 235, 267, 403, 483, 532
8	15, 20, 24, 35, 52, 88, 91, 115, 148, 168, 235, 403, 427
11	24, 51, 52, 91, 123, 132, 232, 403, 427, 627, 1012
15	20, 35, 40, 43, 67, 115, 163, 235
19	24, 52, 88, 91, 123, 148, 228, 232, 403, 532, 627
20	35, 40, 43, 67, 115, 163, 235
24	43, 67, 88, 163
35	40, 43, 67, 115, 163, 235
40	43, 67, 115, 163, 235
43	88, 115, 148, 232, 235, 267, 427
51	163, 187
52	67, 91, 163, 403
67	88, 123, 235, 403
88	163
91	163, 403
115	163, 235
148	163
163	187, 232, 235, 267, 403

$h_N = g_N = 4$

f_{k_1}	f_{k_2}
4	120, 168, 280, 312, 760
8	84, 120, 280, 312, 372,532, 760
20	88, 120, 280, 760
24	52, 84, 132, 148, 168, 228, 372, 708
40	88, 120, 280, 760
52	312
88	132, 148, 1012

TABLE IV.

$(2^*, 2^*, 3)$

τ_1	τ_2	φ	N^+	f_N	$h^-_{M_1}$	$h^-_{M_2}$	Q_N	h^-_N	h_{N^+}	g_N	nb.
χ_7^3	χ_3	χ_7^2	$\chi_3\chi_7^3, \chi_7^2$	21	1	1	2	1	1	1	1
	χ_4	χ_7^2	$\chi_4\chi_7^3, \chi_7^2$	28	1	1	2	1	1	1	2
	$\chi_5^2\chi_7^3$	χ_7^2	χ_5^2, χ_7^2	35	1	2	1	1	1	1	3
	$\chi_4\psi_8$	χ_7^2	$\chi_4\chi_7^3\psi_8, \chi_7^2$	56	1	1	2	1	1	1	4
	χ_{11}^5	χ_7^2	$\chi_7^3\chi_{11}^5, \chi_7^2$	77	1	1	2	1	1	1	5
	$\chi_3\chi_4\chi_7^3$	χ_7^2	$\chi_3\chi_4, \chi_7^2$	84	1	4	1	2	1	2	6
	$\chi_3\chi_5^2$	χ_7^2	$\chi_3\chi_5^2\chi_7^3, \chi_7^2$	105	1	2	1	1	2	2	7
χ_3	χ_4	χ_7^2	$\chi_3\chi_4, \chi_7^2$	84	1	1	2	1	1	1	8
	$\chi_5^2\chi_7^3$	χ_7^2	$\chi_3\chi_5^2\chi_7^3, \chi_7^2$	105	1	2	1	1	2	2	9
	$\chi_4\psi_8$	χ_7^2	$\chi_3\chi_4\psi_8, \chi_7^2$	168	1	1	2	1	1	1	10
	χ_{11}^5	χ_7^2	$\chi_3\chi_{11}^5, \chi_7^2$	231	1	1	2	1	1	1	11
	$\chi_3\chi_4\chi_7^3$	χ_7^2	$\chi_4\chi_7^3, \chi_7^2$	84	1	4	1	2	1	2	12
	$\chi_3\chi_5^2$	χ_7^2	χ_5^2, χ_7^2	105	1	2	1	1	1	1	13
χ_4	$\chi_5^2\chi_7^3$	χ_7^2	$\chi_4\chi_5^2\chi_7^3, \chi_7^2$	140	1	2	1	1	2	2	14
	$\chi_4\psi_8$	χ_7^2	ψ_8, χ_7^2	56	1	1	1	1	1	1	15
	χ_{11}^5	χ_7^2	$\chi_4\chi_{11}^5, \chi_7^2$	308	1	1	2	1	1	1	16
	$\chi_3\chi_4\chi_7^3$	χ_7^2	$\chi_3\chi_7^3, \chi_7^2$	84	1	4	1	2	1	2	17
	$\chi_3\chi_5^2$	χ_7^2	$\chi_3\chi_4\chi_5^2, \chi_7^2$	420	1	2	1	1	2	2	18
$\chi_5^2\chi_7^3$	$\chi_4\psi_8$	χ_7^2	$\chi_4\psi_8\chi_5^2\chi_7^3, \chi_7^2$	280	2	1	1	1	2	2	19
	$\chi_3\chi_5^2$	χ_7^2	$\chi_3\chi_7^3, \chi_7^2$	105	2	2	1	2	1	2	20
$\chi_4\psi_8$	$\chi_3\chi_4\chi_7^3$	χ_7^2	$\chi_3\chi_7^3\psi_8, \chi_7^2$	168	1	4	1	2	2	4	21
	$\chi_3\chi_5^2$	χ_7^2	$\chi_3\chi_4\psi_8\chi_5^2, \chi_7^2$	840	1	2	1	1	2	2	22
χ_3	χ_4	ψ_9	$\chi_3\chi_4, \psi_9$	36	1	1	2	1	1	1	23
	$\chi_3\chi_5^2$	ψ_9	χ_5^2, ψ_9	45	1	2	1	1	1	1	24
	χ_7^3	ψ_9	$\chi_3\chi_7^3, \psi_9$	63	1	1	2	1	1	1	25
	$\chi_3\psi_8$	ψ_9	ψ_8, ψ_9	72	1	2	1	1	1	1	26
χ_4	$\chi_3\chi_5^2$	ψ_9	$\chi_3\chi_4\chi_5^2, \psi_9$	180	1	2	1	1	2	2	27
	χ_7^3	ψ_9	$\chi_4\chi_7^3, \psi_9$	252	1	1	2	1	1	1	28
	$\chi_3\psi_8$	ψ_9	$\chi_3\chi_4\psi_8, \psi_9$	72	1	2	2	2	1	2	29
$\chi_3\chi_5^2$	χ_7^3	ψ_9	$\chi_3\chi_5^2\chi_7^3, \psi_9$	315	2	1	1	1	2	2	30
χ_3	χ_7^3	χ_{13}^4	$\chi_3\chi_7^3, \chi_{13}^4$	273	1	1	2	1	1	1	31
	$\chi_4\psi_8$	χ_{13}^4	$\chi_3\chi_4\psi_8, \chi_{13}^4$	312	1	1	2	1	1	1	32
$\chi_4\chi_{13}^6$	χ_7^3	χ_{13}^4	$\chi_4\chi_7^3\chi_{13}^6, \chi_{13}^4$	364	2	1	1	1	2	2	33
	$\chi_4\psi_8$	χ_{13}^4	$\psi_8\chi_{13}^6, \chi_{13}^4$	208	2	1	1	1	2	2	34
χ_7^3	$\chi_4\psi_8$	χ_{13}^4	$\chi_4\psi_8\chi_7^3, \chi_{13}^4$	728	1	1	2	1	1	1	35
χ_4	χ_{19}^9	χ_{19}^6	$\chi_4\chi_{19}^9, \chi_{19}^6$	76	1	1	2	1	1	1	36
χ_3	χ_{43}^{21}	χ_{43}^{14}	$\chi_3\chi_{43}^{21}, \chi_{43}^{14}$	129	1	1	2	1	1	1	37
χ_3	χ_7^3	$\chi_7^4\psi_9$	$\chi_3\chi_7^3, \chi_7^4\psi_9$	63	1	1	2	1	3	3	38
χ_7^3	$\chi_7^3\chi_{13}^6$	$\chi_7^4\chi_{13}^4$	$\chi_{13}^6, \chi_7^4\chi_{13}^4$	91	1	2	1	1	3	3	39

$\chi_7(3) = e^{2\pi i/6}, \chi_{13}(2) = e^{2\pi i/12}$

$(2^*, 2^*, 5)$

N	N^+	f_N	Q_N	h^-_N	h_{N^+}	g_N
$\chi_3, \chi_{11}^5, \chi_{11}^2$	$\chi_3\chi_{11}^5, \chi_{11}^2$	33	2	1	1	1
$\chi_3, \chi_4, \chi_{11}^2$	$\chi_3\chi_4, \chi_{11}^2$	44	2	1	1	1
$\chi_4, \chi_{11}^5, \chi_{11}^2$	$\chi_4\chi_{11}^5, \chi_{11}^2$	132	2	1	1	1

TABLE V.

$(2^*, 2^*, 2^*)$

$h_N = g_N$	$(f_{k_1}, f_{k_2}, f_{k_3})$
1	$(3,4,7), (3,4,8), (3,4,11), (3,4,15), (3,4,19), (3,7,8), (3,7,15)$ $(3,8,15), (3,11,19), (3,11,24), (3,11,51), (4,7,19), (4,7,20),$ $(4,7,52), (4,8,11), (4,8,20), (7,8,35)$

$(2^*, 2^*, 2^*, 3)$

N	N^+	f_N	$h^-_{M_1}$	$h^-_{M_2}$	$h^-_{M_3}$	$h^-_{M_4}$	Q_N	h^-_N	h_{N^+}	g_N
$\chi_3, \chi_4, \chi_7^3, \chi_7^2$	$\chi_7^2, \chi_3\chi_4, \chi_3\chi_7^3$	84	1	1	1	4	2	1	1	1
$\chi_3, \chi_7^3, \chi_3\chi_5^2, \chi_7^2$	$\chi_7^2, \chi_5^2, \chi_3\chi_7^3$	105	1	1	2	2	2	1	1	1

TABLE VI.

$(4^*, 2^*)$

φ	χ	N^+	f_N	h^-_M	h_{k_1}	h_{k_2}	Q_N	h^-_N	h_{N^+}	g_N	nb.
χ_5	χ_3	$\chi_3\chi_5$	15	1	1	2	2	1	1	1	1
	χ_4	$\chi_4\chi_5$	20	1	1	2	2	1	1	1	2
	χ_7^3	$\chi_5\chi_7^3$	35	1	1	2	2	1	1	1	3
	$\chi_4\psi_8$	$\chi_4\psi_8\chi_5$	40	1	1	2	2	1	1	1	4
χ_{13}^3	χ_4	$\chi_4\chi_{13}^3$	52	1	1	2	2	1	1	1	5
	χ_7^3	$\chi_7^3\chi_{13}^3$	91	1	1	2	2	1	1	1	6
$\chi_4\psi_{16}$	χ_3	$\chi_3\chi_4\psi_{16}$	48	1	1	2	2	1	1	1	7
	χ_4	ψ_{16}	16	1	1	1	1	1	1	1	8
	χ_{11}^5	$\chi_4\psi_{16}\chi_{11}^5$	176	1	1	2	2	1	1	1	9
	$\chi_4\chi_5^2$	$\chi_5^2\psi_{16}$	80	1	2	2	1	1	2	2	10
χ_{37}^9	χ_4	$\chi_4\chi_{37}^9$	148	1	1	2	2	1	1	1	11
$\chi_5\psi_8$	χ_4	$\chi_4\psi_8\chi_5$	40	2	1	2	2	2	1	2	12
	$\chi_4\psi_8$	$\chi_4\chi_5$	40	2	1	2	2	2	1	2	13
χ_{29}^7	$\chi_4\psi_8$	$\chi_4\psi_8\chi_{29}^7$	232	1	1	2	2	1	1	1	14
$\chi_3\chi_4\chi_5$	χ_3	$\chi_4\chi_5$	60	4	1	2	1	2	1	2	15
	χ_4	$\chi_3\chi_5$	60	4	1	2	1	2	1	2	16
	$\chi_4\psi_8$	$\chi_3\chi_5\psi_8$	120	4	1	2	1	2	2	4	17
$\chi_4\psi_{16}\chi_5^2$	χ_4	$\chi_5^2\psi_{16}$	80	2	1	1	1	1	2	2	18
	$\chi_4\chi_5^2$	ψ_{16}	80	2	2	2	1	2	1	2	19
$\chi_5\psi_{16}$	χ_4	$\chi_4\psi_{16}\chi_5$	80	2	1	2	2	2	2	4	20
	$\chi_4\psi_8$	$\chi_4\psi_{16}\chi_5$	80	2	1	2	2	2	2	4	21
$\chi_3\psi_{16}$	χ_3	ψ_{16}	48	2	1	2	1	1	1	1	22
	χ_4	$\chi_3\chi_4\psi_{16}$	48	2	1	1	2	2	1	2	23
	χ_{11}^5	$\chi_3\chi_{11}^5\psi_{16}$	528	2	1	2	1	1	2	2	24
	$\chi_4\chi_5^2$	$\chi_3\chi_4\psi_{16}\chi_5^2$	240	2	2	2	1	2	2	4	25
$\chi_3\chi_5\chi_7^3$	χ_3	$\chi_5\chi_7^3$	105	4	1	2	1	2	1	2	26
	χ_7^3	$\chi_3\chi_5$	105	4	1	2	1	2	1	2	27
$\chi_7^3\chi_{17}^4$	χ_3	$\chi_3\chi_7^3\chi_{17}^4$	357	2	1	2	1	1	2	2	28
	χ_{11}^5	$\chi_7^3\chi_{11}^5\chi_{17}^4$	1309	2	1	2	1	1	2	2	29
$\psi_8\chi_{13}^3$	χ_4	$\chi_4\psi_8\chi_{13}^3$	104	2	1	2	2	2	1	2	30
$\chi_3\chi_4\psi_8\chi_5$	χ_3	$\chi_4\psi_8\chi_5$	120	4	1	2	1	2	1	2	31
	χ_4	$\chi_3\chi_5\psi_8$	120	4	1	2	1	2	2	4	32
	$\chi_4\psi_8$	$\chi_3\chi_5$	120	4	1	2	1	2	1	2	33
$\chi_4\chi_5\chi_7^3$	χ_4	$\chi_5\chi_7^3$	140	4	1	2	1	2	1	2	34
	χ_7^3	$\chi_4\chi_5$	140	4	1	2	1	2	1	2	35
	$\chi_4\psi_8$	$\chi_5\chi_7^3\psi_8$	280	4	1	2	1	2	2	4	36

TABLE VI continued.

$(4^*, 2^*, 3)$

N	N^+	f_N	Q_N	h_N^-	h_{N^+}	g_N
$\chi_5, \chi_7^3, \chi_7^2$	$\chi_5\chi_7^3, \chi_7^2$	35	2	1	1	1
χ_3, χ_5, ψ_9	$\chi_3\chi_5, \psi_9$	45	2	1	1	1
χ_3, χ_5, χ_7^2	$\chi_3\chi_5, \chi_7^2$	105	2	1	1	1

TABLE VII. $(8^*, 2^*)$

N	N^+	f_N	h_M^-	h_L^-	h_E^-	Q_N	h_N^-	h_{N^+}	g_N
$\chi_4\psi_{32}, \chi_4$	ψ_{32}	32	1	1	1	1	1	1	1
$\chi_4\psi_{32}, \chi_3$	$\chi_3\chi_4\psi_{32}$	96	1	2	1	2	1	1	1
$\chi_4\psi_{32}, \chi_4\chi_5^2$	$\chi_5^2\psi_{32}$	160	1	2	2	1	1	2	2

TABLE VIII. $(4^*, 2)$

φ	χ	N^+	f_N	$h_{M_1}^-$	$h_{M_2}^-$	Q_N	h_N^-	h_{N^+}	g_N	nb.
χ_5	ψ_8	χ_5^2, ψ_8	40	1	2	1	1	1	1	1
	$\chi_3\chi_4$	$\chi_5^2, \chi_3\chi_4$	60	1	4	1	2	1	2	2
	χ_{13}^6	χ_5^2, χ_{13}^6	65	1	2	1	1	1	1	3
	χ_{17}^8	χ_5^2, χ_{17}^8	85	1	2	1	1	1	1	4
	$\chi_3\chi_7^3$	$\chi_5^2, \chi_3\chi_7^3$	105	1	4	1	2	1	2	5
	$\chi_3\chi_4\psi_8$	$\chi_5^2, \chi_3\chi_4\psi_8$	120	1	4	1	2	1	2	6
	$\chi_4\chi_7^3$	$\chi_5^2, \chi_4\chi_7^3$	140	1	4	1	2	1	2	7
χ_{13}^3	χ_5^2	χ_{13}^6, χ_5^2	65	1	2	1	1	1	1	8
	ψ_8	χ_{13}^6, ψ_8	104	1	2	1	1	1	1	9
$\chi_4\psi_{16}$	$\chi_3\chi_4$	$\psi_8, \chi_3\chi_4$	48	1	2	2	2	1	2	10
	χ_5^2	ψ_8, χ_5^2	80	1	2	1	1	1	1	11
$\chi_5\psi_8$	$\chi_3\chi_4\psi_8$	$\chi_5^2, \chi_3\chi_4\psi_8$	120	2	4	1	4	1	4	12
	$\chi_3\chi_4$	$\chi_5^2, \chi_3\chi_4$	120	2	4	1	4	1	4	13
	$\chi_4\psi_8\chi_7^3$	$\chi_5^2, \chi_4\psi_8\chi_7^3$	280	2	4	1	4	1	4	14
$\chi_3\psi_{16}$	$\chi_3\chi_4\chi_5^2$	$\psi_8, \chi_3\chi_4\chi_5^2$	240	2	2	1	2	2	4	15

TABLE IX. $(4^*, 4^*)$

N	N^+	f_N	$h_{M_1}^-$	$h_{M_2}^-$	$h_{M_3}^-$	$h_{M_4}^-$	Q_N	h_N^-	h_{N^+}	g_N
χ_5, χ_{13}^3	$\chi_5^2, \chi_5\chi_{13}^3$	65	1	1	2	2	2	1	1	1
$\chi_5, \chi_4\psi_{16}$	$\chi_5^2, \chi_4\psi_{16}\chi_5$	80	1	1	2	2	2	1	1	1

TABLE X. $(4^*, 2^*, 2^*)$

N	f_N	$h_{M_1}^-$	$h_{M_2}^-$	h_E^-	Q_E	Q_N	h_N^-	h_{N^+}	g_N
$\chi_5, \chi_4, \chi_4\psi_8$	40	1	2	1	1	2	1	1	1
χ_5, χ_3, χ_4	60	1	4	1	2	2	1	1	1
χ_5, χ_3, χ_7^3	105	1	4	1	2	2	1	1	1
$\chi_5, \chi_3, \chi_4\psi_8$	120	1	4	1	2	2	1	1	1
χ_5, χ_4, χ_7^3	140	1	4	1	2	2	1	1	1
$\chi_4\psi_{16}, \chi_3, \chi_4$	48	1	2	1	2	2	1	1	1
$\chi_4\psi_{16}, \chi_4, \chi_4\chi_5^2$	80	1	2	1	1	1	1	1	1

References

[A] S. ARNO, *The imaginary quadratic fields of class number 4.* Acta Arith. **60** (1992), 321–334.

[CK] K.-Y. CHANG, S.-H. KWON, *On the imaginary cyclic number fields.* J. Number Theory **73** (1998), 318–338.

[G] M.-N. GRAS, *Classes et unités des extensions cycliques réelles de degré 4 de* $\mathbb{Q}$. Ann. Inst. Fourier (Grenoble) **29** (1979), 107–124.

[H] H. HASSE, *Über die Klassenzahl abelscher Zahlkörper.* Academie-Verlag, Berlin, 1952. Reprinted with an introduction by J. Martinet: Springer-Verlag, Berlin (1985).

[HY1] M. HIRABAYASHI, K. YOSHINO, *Remarks on unit indices of imaginary abelian number fields.* Manuscripta Math. **60** (1988), 423–436.

[HY2] M. HIRABAYASHI, K. YOSHINO, *Remarks on unit indices of imaginary abelian number fields II.* Mamuscripta Math. **64** (1989), 235–251.

[HY3] M. HIRABAYASHI, K. YOSHINO, *Unit indices of imaginary abelian number fields of type* $(2,2,2)$. J. Number Theory **34** (1990), 346–361.

[K] T. KUBOTA, *Über den bizyklischen biquadratischen Zahlkörper.* Nagoya Math. J. **10** (1956), 65–85.

[KT] M. DABERKOW, C. FIEKER, J. KLÜNERS, M. POHST, K. ROEGNER, K. WILDANGER, KANT V4 J. Symbolic Comp. **24** (1997), 267–283.

[L] F. . VAN DER LINDEN, *Class number computations of real abelian number fields.* Math. Comp. **39** 160 (1982), 693–707.

[Lou1] S. LOUBOUTIN, *A finiteness theorem for imaginary abelian number fields.* Manuscripta Math. **91** (1996), 343–352.

[Lou2] S. LOUBOUTIN, *The nonquadratic imaginary cyclic fields of 2-power degrees with class numbers equal to their genus class numbers.* Proc. Amer. Math. Soc. **27** (1999), 355–361.

[Lou3] S. LOUBOUTIN, *Minoration au point 1 des fonctions L et détermination des sextiques abéliens totalement imaginaires principaux.* Acta Arith. **62** (1992), 535–540.

[Lou4] S. LOUBOUTIN, *The imaginary cyclic sextic fields with class numbers equal to their genus class numbers.* Colloquium Math. **75** (1998), 205–212.

[Lou5] S. LOUBOUTIN, *Minorations (sous l'hypothèse de Riemann généralisée) des nombres de classes des corps quadratiques imaginaires.* Application, C. R. Acad. Sci. Paris Sér. I Math. **310** (1990), 795–800.

[M] I. MIYADA, *On imaginary abelian number fields of type* $(2,2,\cdots,2)$ *with one class in each genus.* Manuscripta Math. **88** (1995), 535–540.

[Ma] J.M. MASLEY, *Class numbers of real cyclic number fields with small conductor.* Compositio Math. **37** (1978), 297–319.

[Mä] S. MÄKI, *The determination of units in real cyclic sextic fields.* Lecture notes in Math. **797**, Springer-Verlag, Berlin and New York (1980).

[P] C. BATUT, D. BERNARDI, H. COHEN, M. OLIVIER, PARI-GP, version 1. 38.

[S1] H.M. STARK, *A complete determination of the complex quadratic fields of class number one.* Michigan Math. J. **14** (1967), 1–27.

[S2] H.M. STARK, *On complex quadratic fields with class number two.* Math. Comp. **29** (1975), 289–302.

[W] L.C. WASHINGTON, *Introduction to cyclotomic fields.* GTM **83**, 2nd Ed., Springer-Verlag (1996).

[We] P.J. WEINBERGER, *Exponents of the class groups of complex quadratic fields.* Acta Arith. **22** (1973), 117–124.

[Y] K. YAMAMURA, *The determination of imaginary abelian number fields with class number one.* Math. Comp. **62** (1994), 899–921.

[YH1] K. YOSHINO, M. HIRABAYASHI, *On the relative class number of the imaginary abelian number field I.* Memoirs of the College of Liberal Arts, Kanazawa Medical University, Vol. 9 (1981).

[YH2] K. Yoshino, M. Hirabayashi, *On the relative class number of the imaginary abelian number field II.* Memoirs of the College of Liberal Arts, Kanazawa Medical University, Vol. 10 (1982).

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