SHIGEKI AKIYAMA

JÖRG M. THUSWALDNER

Topological properties of two-dimensional number systems

Journal de Théorie des Nombres de Bordeaux, tome 12, n° 1 (2000),
p. 69-79

http://www.numdam.org/item?id=JTNB_2000__12_1_69_0

© Université Bordeaux 1, 2000, tous droits réservés.

L'accès aux archives de la revue « Journal de Théorie des Nombres de Bordeaux » (<http://jtnb.cedram.org/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
<http://www.numdam.org/>

Topological Properties of Two-Dimensional Number Systems

par SHIGEKI AKIYAMA et JÖRG M. THUSWALDNER

RÉSUMÉ. Pour une matrice réelle M d'ordre 2 donnée, on peut définir la notion de représentation M -adique d'un élément de $\mathbb{R}^2$. On note $\mathcal{F}$ le domaine fondamental constitué des nombres de $\mathbb{R}^2$ dont le développement " M -adique" ne commence pas par 0. C'est l'analogue dans $\mathbb{R}^2$ des nombres q -adiques, où la matrice M joue le rôle de la base q . Kátai et Környei ont démontré que $\mathcal{F}$ est compact, et que $\mathbb{R}^2$ s'écrit comme la réunion dénombrable de certains translatés de $\mathcal{F}$, l'intersection de 2 quelconques d'entre eux étant de mesure nulle. Dans cet article, nous construisons des points qui appartiennent simultanément à trois translatés de $\mathcal{F}$, et nous montrons que $\mathcal{F}$ est connexe. Nous donnons aussi une propriété sur la structure des points intérieurs de $\mathcal{F}$.

ABSTRACT. In the two dimensional real vector space $\mathbb{R}^2$ one can define analogs of the well-known q -adic number systems. In these number systems a matrix M plays the role of the base number q . In the present paper we study the so-called fundamental domain $\mathcal{F}$ of such number systems. This is the set of all elements of $\mathbb{R}^2$ having zero integer part in their " M -adic" representation. It was proved by Kátai and Környei, that $\mathcal{F}$ is a compact set and certain translates of it form a tiling of the $\mathbb{R}^2$. We construct points, where three different tiles of this tiling coincide. Furthermore, we prove the connectedness of $\mathcal{F}$ and give a result on the structure of its inner points.

1. INTRODUCTION

In this paper we use the following notations: $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$ and $\mathbb{N}$ denote the set of real numbers, rational numbers, integers and positive integers, respectively. If $x \in \mathbb{R}$ we will write $[x]$ for the largest integer less than or equal to x . λ will denote the 2-dimensional Lebesgue measure. Furthermore, we write ∂A for the boundary of the set A and $\text{int}(A)$ for its interior.

$\text{diag}(\lambda_1, \lambda_2)$ denotes a 2×2 diagonal matrix with diagonal elements λ_1 and λ_2 .

Let $q \geq 2$ be an integer. Then each positive integer n has a unique q -adic representation of the shape $n = \sum_{k=0}^H a_k q^k$ with $a_k \in \{0, 1, \dots, q-1\}$ ($0 \leq k \leq H$) and $a_H \neq 0$ for $H \neq 0$. These q -adic number systems have been generalized in various ways. In the present paper we deal with analogs of these number systems in the 2-dimensional real vector space, that emerge from number systems in quadratic number fields. The first major step in the investigation of number systems in number fields was done by Knuth [13], who studied number systems with negative bases as well as number systems in the ring of Gaussian integers. Meanwhile, Kátai, Kovács, Pethő and Szabó invented a general notion of number systems in rings of integers of number fields, the so-called *canonical number systems* (cf. for instance [10, 11, 12, 15]). We recall their definition.

Let K be a number field with ring of integers Z_K . For an algebraic integer $b \in Z_K$ define $\mathcal{N} = \{0, 1, \dots, |N(b)| - 1\}$, where $N(b)$ denotes the norm of b over $\mathbb{Q}$. The pair $(b, \mathcal{N})$ is called a *canonical number system* if any $\gamma \in Z_K$ admits a representation of the shape

$$\gamma = c_0 + c_1 b + \dots + c_H b^H,$$

where $c_k \in \mathcal{N}$ ($0 \leq k \leq H$) and $c_H \neq 0$ for $H \neq 0$.

These number systems resemble a natural generalization of q -adic number systems to number fields. Each of these number systems gives rise to a number system in the n -dimensional real vector space. Since we are only interested in the 2-dimensional case, we construct these number systems only for this case. Consider a canonical number system $(b, \mathcal{N})$ in a quadratic number field K with ring of integers Z_K . Let $p_b(x) = x^2 + Ax + B$ be the minimal polynomial of b . It is known, that for bases of canonical number systems $-1 \leq A \leq B \geq 2$ holds (cf. [10, 11, 12]). Now consider the embedding $\Phi : K \rightarrow \mathbb{R}^2$, $\alpha_1 + \alpha_2 b \mapsto (\alpha_1, \alpha_2)$, where $\alpha_1, \alpha_2 \in \mathbb{Q}$. Kovács [14] proved, that $\{1, b\}$ forms an integral basis of Z_K . Thus we have $\Phi(Z_K) = \mathbb{Z}^2$. Furthermore, note that $\Phi(bz) = M\Phi(z)$ with

$$M = \begin{pmatrix} 0 & -B \\ 1 & -A \end{pmatrix}.$$

Since the elements of $\mathcal{N}$ are rational integers, for each $c \in \mathcal{N}$, $\Phi(c) = (c, 0)^T$. Summing up we see, that $(M, \Phi(\mathcal{N}))$ forms a number system in the two dimensional real vector space in the following sense (cf. also [8], where some properties of these number systems are studied). Each $g \in \mathbb{Z}^2$ has a unique representation of the form

$$g = d_0 + M d_1 + \dots + M^H d_H,$$

with $d_k \in \Phi(\mathcal{N})$ ($0 \leq k \leq H$) and $d_H \neq (0, 0)^T$ for $H \neq 0$. These number systems form the object of this paper. In particular, we want to study the so-called *fundamental domains* of these number systems. The *fundamental domain* of a number system $(M, \Phi(\mathcal{N}))$ is defined by

$$\mathcal{F} = \left\{ z \mid z = \sum_{j \geq 1} M^{-j} d_j, d_j \in \Phi(\mathcal{N}) \right\}.$$

Sloppily spoken, $\mathcal{F}$ contains all elements of $\mathbb{R}^2$, with integer part zero in their “ M -adic” representation. In Figure 1 the fundamental domain corresponding to the M -adic representations arising from the Gaussian integer $-1 + i$ is depicted. This so-called “twin dragon” was studied extensively by Knuth in his book [13].

FIGURE 1. *The fundamental domain of a number system*

Fundamental domains of number systems have been studied in various papers. Kátaí and Környei [9] proved, that $\mathcal{F}$ is a compact set that tessellates the plane in the following way.

$$(1) \quad \bigcup_{g \in \mathbb{Z}^2} (\mathcal{F} + g) = \mathbb{R}^2 \quad \text{where} \quad \lambda((\mathcal{F} + g_1) \cap (\mathcal{F} + g_2)) = 0$$

$$(g_1, g_2 \in \mathbb{Z}^2; g_1 \neq g_2).$$

Furthermore, we want to mention, that the boundary of $\mathcal{F}$ has fractal dimension. Its Hausdorff and box counting dimension has been calculated by Gilbert [4], Ito [7], Müller-Thuswaldner-Tichy [16] and Thuswaldner [17]. In the present paper we are interested in topological properties of $\mathcal{F}$. Before we give a survey on our results we shall define some basic objects. Let S be the set of all translates of $\mathcal{F}$, that “touch” $\mathcal{F}$, i.e.

$$S := \{g \in \mathbb{Z}^2 \setminus (0, 0)^T \mid \mathcal{F} \cap (\mathcal{F} + g) \neq \emptyset\}.$$

Then by (1) the boundary of $\mathcal{F}$ has the representation

$$(2) \quad \partial\mathcal{F} = \bigcup_{g \in S} (\mathcal{F} \cap (\mathcal{F} + g)).$$

Hence, the boundary of $\mathcal{F}$ is the set of all elements of $\mathcal{F}$, that are contained in $\mathcal{F} + g$ for a certain $g \neq (0, 0)^T$. Of course, $\partial\mathcal{F}$ may contain points, that belong to $\mathcal{F}$ and two other different translates of $\mathcal{F}$. These points we call *vertices* of $\mathcal{F}$. Thus the set of vertices of $\mathcal{F}$ is defined by

$$V := \{z \in \mathcal{F} \mid z \in (\mathcal{F} + g_1) \cap (\mathcal{F} + g_2), g_1, g_2 \in \mathbb{Z}^2; g_1 \neq g_2, g_1 \neq 0, g_2 \neq 0\}.$$

In Section 2 we study the set of vertices of $\mathcal{F}$. It turns out, that, apart from one exceptional case, $\mathcal{F}$ has at least 6 vertices. In some cases we derive that V is an infinite or even uncountable set. In Section 3 we prove the connectedness of $\mathcal{F}$ and show that each element of $\mathcal{F}$, which has a finite M -adic expansion, is an inner point of $\mathcal{F}$.

2. VERTICES OF THE FUNDAMENTAL DOMAIN $\mathcal{F}$

In this section we give some results on the set of vertices V of $\mathcal{F}$. For number systems emerging from Gaussian integers, similar results have been established with help of different methods in Gilbert [3]. We start with the definition of useful abbreviations. Let

$$(3) \quad g = M^{-H_1}d_{-H_1} + \cdots + M^{H_2}d_{H_2}$$

be the M -adic representation of g . Note, that the digits d_j ($-H_1 \leq j \leq H_2$) are of the shape $d_j = (c_j, 0)^T \in \Phi(\mathcal{N})$. Thus for the expansion (3) we will write

$$g = c_{H_2}c_{H_2-1} \cdots c_1c_0.c_{-1} \cdots c_{-H_1}.$$

If the string $c_1 \cdots c_H$ occurs j times in an M -adic representation, then we write $[c_1 \cdots c_H]_j$. If a representation is ultimately periodic, i.e. a string $c_1 \cdots c_H$ occurs infinitely often, we write $[c_1 \cdots c_H]_\infty$. First we show, that for $A > 0$ any fundamental domain $\mathcal{F}$ contains at least 6 vertices.

Theorem 2.1. *Let $(M, \Phi(\mathcal{N}))$ be a number system in $\mathbb{R}^2$, which is induced by the base b of a canonical number system. Let $p_b(x) = x^2 + Ax + B$ with $A > 0$ be the minimal polynomial of b . Then the set of vertices V of the fundamental domain $\mathcal{F}$ of this number system contains the points*

$$\begin{aligned} P_1 &= 0.[0(A-1)(B-1)]_\infty, & P_2 &= 0.[(A-1)(B-1)0]_\infty, \\ P_3 &= 0.[0(B-1)(B-A)]_\infty, & P_4 &= 0.[(B-1)(B-A)0]_\infty, \\ P_5 &= 0.[(B-A)0(B-1)]_\infty, & P_6 &= 0.[(B-1)0(A-1)]_\infty. \end{aligned}$$

Depending on the cases $A = 1$, $1 < A < B$ and $A = B$, the points P_j ($1 \leq j \leq 6$) belong to the following translates $\mathcal{F} + w$ of $\mathcal{F}$.

	values of w for $1 < A < B$	values of w for $A = B$
P_1	0, 1, $1A$	0, 1, $1(A-1)10$
P_2	0, $1(A-1)$, $1A(B-1)$	0, $1(A-1)$, $1(A-1)10(A-1)$
P_3	0, $1A$, $1(A-1)$	0, $1(A-1)$, $1(A-1)10$
P_4	0, $1A(B-1)$, $1(A-1)(B-A)$	0, $1A(A-1)$, $1(A-1)0$
P_5	0, $1(A-1)(B-A+1)$, 1	0, $1(A-1)1$, 1
P_6	0, $1(A-1)(B-A)$, $1(A-1)(B-A+1)$	0, $1(A-1)1$, $1(A-1)0$

The case $A = 1$ is very similar to the case $1 < A < B$; just replace the representation $1(A-1)(B-A+1)$ by $11(B-1)0$ in the above table.

Remark 2.1. Note, that we have $0 < A \leq B \geq 2$. Hence the digits of the 6 points indicated in Theorem 2.1 are all admissible.

Proof of the theorem. We will prove that each of the 6 points $P_1, \dots, P_6$ is contained in three different translates of $\mathcal{F}$, as indicated in the statement of the theorem. First we consider the point P_1 . Write $\bar{x} = -x$. By using $b^2 + Ab + B = 0$, we see that

$$(4) \quad 0.1(A-1)(B-A)\bar{B} = 0.1[(A-1)(B-A)\overline{(B-1)}]_\infty = 0$$

are formal representations of zero. Adding the second representation for 0 given in (4) twice, we have

$$\begin{aligned} P_1 &= 0.[0(A-1)(B-1)]_\infty + 1.[(A-1)(B-A)\overline{(B-1)}]_\infty \\ &= 1.[(A-1)(B-1)0]_\infty \\ &= 1.[(A-1)(B-1)0]_\infty + 1(A-1).[(B-A)\overline{(B-1)}(A-1)]_\infty \\ &= 1A.[(B-1)0(A-1)]_\infty. \end{aligned}$$

For $A < B$ this yields

$$P_1 \in \mathcal{F} \cap (\mathcal{F} + 1) \cap (\mathcal{F} + 1A).$$

For $A = B$ the last expansion $1A.[(B-1)0(A-1)]_\infty$ is not admissible since $A > B-1$. In order to settle this case we use the first representation of zero given in (4) to get $1A = 1B = 1B + 1(B-1)0\bar{B} = 1(A-1)10$. As a result, we have

$$P_1 \in \mathcal{F} \cap (\mathcal{F} + 1) \cap (\mathcal{F} + 1(A-1)10)$$

for $A = B$. Since $P_2 = MP_1$, we get the desired results also for P_2 . Now we treat

$$P_3 = 0.[0(B-1)(B-A)]_\infty.$$

In the same way as before, we get, using both representations of zero in (4)

$$\begin{aligned} P_3 &= 0.[0(B-1)(B-A)]_\infty + 1A.B - 0.1[(A-1)(B-A)\overline{(B-1)}]_\infty \\ &= 1A.[(B-1)(B-A)0]_\infty \\ &= 1A.[(B-1)(B-A)0]_\infty - 1.[(A-1)(B-A)\overline{(B-1)}]_\infty \\ &= 1(A-1).[(B-A)0(B-1)]_\infty, \end{aligned}$$

which implies

$$P_3 \in \mathcal{F} \cap (\mathcal{F} + 1A) \cap (\mathcal{F} + 1(A - 1))$$

for $A < B$ and

$$P_3 \in \mathcal{F} \cap (\mathcal{F} + 1(A - 1)10) \cap (\mathcal{F} + 1(A - 1))$$

for $A = B$. Since $\mathcal{F}$ permits an involution $\varphi : x \rightarrow \sum_{j \geq 1} M^{-j}(B-1, 0)^T - x$, $\mathcal{F}$ is symmetric with respect to the center $\frac{1}{2} \sum_{j \geq 1} M^{-j}(B-1, 0)^T$. For $w \in \mathbb{Z}^2$ this map sends each $\mathcal{F} + w$ to $\mathcal{F} - w$. Thus we have

$$\begin{aligned} \varphi(\mathcal{F} + 1) &= \mathcal{F} + 1A(B-1), \\ \varphi(\mathcal{F} + 1(A-1)) &= \begin{cases} \mathcal{F} + 1(A-1)(B-A+1) & \text{for } A > 1, \\ \mathcal{F} + 11(B-1)0 & \text{for } A = 1, \end{cases} \\ \varphi(\mathcal{F} + 1A) &= \mathcal{F} + 1(A-1)(B-A), \end{aligned}$$

for $A < B$ and

$$\begin{aligned} \varphi(\mathcal{F} + 1) &= \mathcal{F} + 1(A-1)10(A-1), \\ \varphi(\mathcal{F} + 1(A-1)) &= \mathcal{F} + 1(A-1)1, \\ \varphi(\mathcal{F} + 1(A-1)10) &= \mathcal{F} + 1(A-1)0, \end{aligned}$$

for $A = B$. Furthermore, it is easy to see, that $\varphi(P_1) = P_4$, $\varphi(P_2) = P_5$ and $\varphi(P_3) = P_6$. Thus also P_4 , P_5 and P_6 are vertices of $\mathcal{F}$ that are contained in the translates of $\mathcal{F}$ indicated in the statement of the theorem. $\square$

In the case $A = 0$ it is easy to see that $\mathcal{F}$ is a square. It has exactly 4 vertices. These are the “usual” vertices of the square. Thus we only have to deal with the case $A = -1$. We will formulate the corresponding result as a corollary.

Corollary 2.1. *Let the same settings as in Theorem 2.1 be in force, but assume now, that $A = -1$. Then the following table gives 6 points P_j ($1 \leq j \leq 6$), that are contained in the set of vertices V of $\mathcal{F}$. Furthermore, we give the translates $\mathcal{F} + w$, to which P_j belongs.*

P_j	translates w , for which $P_j \in \mathcal{F} + w$
$0.[0(B-1)(B-1)(B-1)00]_\infty$	$0, 10(B-1), 10(B-1)(B-1)$
$0.[000(B-1)(B-1)(B-1)]_\infty$	$0, 1, 10$
$0.[00(B-1)(B-1)(B-1)0]_\infty$	$0, 10, 10(B-1)$
$0.[(B-1)000(B-1)(B-1)]_\infty$	$0, 1, 10(B-1)(B-1)1$
$0.[(B-1)(B-1)000(B-1)]_\infty$	$0, 10(B-1)(B-1), 10(B-1)(B-1)1$
$0.[(B-1)(B-1)(B-1)000]_\infty$	$0, 10(B-1)(B-1)0, 10(B-1)(B-1)1$

Proof. Let $M_1 = \begin{pmatrix} 0 & -B \\ 1 & -1 \end{pmatrix}$ and $M_2 = \begin{pmatrix} 0 & -B \\ 1 & 1 \end{pmatrix}$ be bases of number systems in $\mathbb{R}^2$ and let $\mathcal{F}_1$ and $\mathcal{F}_2$ be the fundamental domains corresponding

to M_1 and M_2 , respectively. We know the vertices of $\mathcal{F}_1$ from Theorem 2.1 and will construct the vertices of $\mathcal{F}_2$ from it. To this matter let $M_1 = G_1 \text{diag}(b_1, b_2) G_1^{-1}$. It is easy to see, that then $M_2 = G_2 \text{diag}(-b_1, -b_2) G_2^{-1}$ with $G_2 = \text{diag}(-1, 1) G_1$. Now suppose, that $\sum_{k \geq 1} M_1^{-k} a_k \in \mathcal{F}_1 \cap \mathcal{F}_1 + (v_1, v_2)^T \cap \mathcal{F}_1 + (w_1, w_2)^T$ with $v_1, v_2, w_1, w_2 \in \mathbb{Z}$ is a vertex of $\mathcal{F}_1$. Using the fact, that $G_1^{-1} a_k = -G_2^{-1} a_k$ for $a_k \in \mathcal{M}$ and setting $d = 0.[0(B-1)]_\infty$ we easily derive that

$$(5) \quad Q := \sum_{k \geq 1} (-1)^{k+1} M_2^{-k} a_k + d \\ \in \text{diag}(-1, 1)(\mathcal{F}_1 \cap \mathcal{F}_1 + (v_1, v_2)^T \cap \mathcal{F}_1 + (w_1, w_2)^T) + d.$$

Observe, that by the selection of d , Q has an admissible M_2 -adic representation with integer part zero. Thus $Q \in \mathcal{F}_2$. Since any element of $\mathcal{F}_2$ can be constructed from elements of $\mathcal{F}_1$ in the same way we conclude, that $\mathcal{F}_2 = \text{diag}(-1, 1)\mathcal{F}_1 + d$. But with that (5) reads $Q \in \mathcal{F}_2 \cap \mathcal{F}_2 + (-v_1, v_2)^T \cap \mathcal{F}_2 + (-w_1, w_2)^T$. Thus Q is a vertex of $\mathcal{F}_2$. The representations in the table above, can now easily be obtained from the results for $A = 1$ in Theorem 2.1. $\square$

The following corollary is an immediate consequence of Theorem 2.1 and Corollary 2.1.

Corollary 2.2. *For $1 < A < B$ we have*

$$S \supset \{1, 1A, 1(A-1), 1A(B-1), 1(A-1)(B-A), 1(A-1)(B-A+1)\},$$

for $A = B$

$$S \supset \{1, 1(A-1)10, 1(A-1), 1(A-1)10(A-1), 1(A-1)0, 1(A-1)1\},$$

while for $A = 1$

$$S \supset \{1, 10, 10(B-1), 10(B-1)(B-1), 10(B-1)(B-1)0, \\ 10(B-1)(B-1)1\}$$

holds.

Remark 2.2. Note, that “ $\supset$ ” may be replaced by “ $=$ ” in Corollary 2.2 if $2A < B + 3$. This is shown for the Gaussian case in [16]. For arbitrary quadratic number fields this fact can be proved in a similar way.

Theorem 2.2. *Let the same settings as in Theorem 2.1 be in force. If $2A = B + 3$ then $\mathcal{F}$ has infinitely many vertices.*

Proof. Set $K = B - A + 1 = \frac{B-1}{2}$. Then, using $b^2 + Ab + B = 0$, we get ($j \geq 0$)

$$(6) \quad 0 = \sum_{k=2}^{\infty} (-1)^k \left(M^{-k+2}(1, 0)^T + M^{-k+1}(A, 0)^T + M^{-k}(B, 0)^T \right) \\ = 1.(A-1)[K\overline{K}]_{\infty}.$$

Here we set $\bar{x} = -x$, as before. We will show, that the points

$$(7) \quad Q_j = 1A.[(B-1)0(A-1)]_{2j}(B-1)0[K]_{\infty} \quad (j \in \mathbb{N})$$

are vertices of $\mathcal{F}$. Therefore we need the representation (6). With help of this representation we define the following representations of zero.

$$\begin{aligned} N_1 &:= 1(A-1).[K\overline{K}]_{\infty} = 0, \\ N_2 &:= 1.(A-1)[K\overline{K}]_{\infty} = 0, \\ X_j &:= 0.[0]_j 1AB = 0 \quad (j \geq 0). \end{aligned}$$

In the sequel we write kX_j ($k \in \mathbb{Z}$) if we want to multiply each digit of the representation X_j by k . Furthermore, addition and subtraction of representations is always meant digit-wise. After these definitions we define the following, more complicated representations of zero.

$$\begin{aligned} Z_1(j) &:= N_1 + \sum_{k=1}^j (X_{6k-1} - 2X_{6k-2} + 2X_{6k-3} - X_{6k-4}) + (1.AB) - 2(1A.B) \\ &= \overline{1A}.[(\overline{B-1})(A-1)(B-A)]_{2j}(\overline{B-1})(A-1)[K\overline{K}]_{\infty}, \\ Z_2(j) &:= N_2 + \sum_{k=1}^j (-X_{6k-3} + 2X_{6k-4} - 2X_{6k-5} + X_{6k-6}) - (1A.B) \\ &= \overline{1(A-1)}.(\overline{B-A})[(B-1)(\overline{A-1})(\overline{B-A})]_{2j-1}(B-1)(\overline{A-1})\overline{K}[K\overline{K}]_{\infty}. \end{aligned}$$

Finally, we observe, that for $j \in \mathbb{N}$

$$\begin{aligned} Q_j &= Q_j + Z_1(j) \\ &= 0.[0(A-1)(B-1)]_{2j}0(A-1)[(B-1)0]_{\infty} \\ &= Q_j + Z_2(j) \\ &= 1.(A-1)[(B-1)0(A-1)]_{2j-1}(B-1)0KK[0(B-1)]_{\infty}, \end{aligned}$$

and this implies $Q_j \in V$. It remains to show, that the elements Q_j , $j \geq 1$, are pairwise different. This follows from the following observation. Select $k \in \mathbb{N}$ arbitrary and let $j_1, j_2 \leq k$. Suppose, that Q_{j_1} and Q_{j_2} are represented by the representation (7) for $j = j_1$ and $j = j_2$, respectively. Then $Q_{j_1} = Q_{j_2}$ if and only if $M^{6k+2}Q_{j_1} = M^{6k+2}Q_{j_2}$. For $k \geq \max(j_1, j_2)$, $M^{6k+2}Q_{j_1}$ and $M^{6k+2}Q_{j_2}$ have the same digit string $[0(B-1)]_{\infty}$ after the comma. Hence, they can only be equal, if their integer parts are equal. But since $(M, \Phi(\mathcal{N}))$ is a number system, this can only be the case, if the digit strings of their integer parts are the same. This implies $j_1 = j_2$. So we have proved, that the points Q_j are pairwise different for $j \leq k$. Since k

can be selected arbitrary, the result follows. Thus we found infinitely many different vertices of $\mathcal{F}$. $\square$

Theorem 2.3. *Let the same settings as in Theorem 2.1 be in force. If $2A > B + 3$ then $\mathcal{F}$ has uncountably many vertices.*

Proof. Set $K = B - A + 1$ and $\xi = \lfloor (B - 1)/2 \rfloor$. As $\xi + K, \xi - K \in \mathcal{N}$, by using (6), we see that

$$\begin{aligned} 0.[\xi]_{\infty} &= 1(A - 1).[(\xi + K)(\xi - K)]_{\infty} \\ &= 1(A - 1)K.[(\xi - K)(\xi + K)]_{\infty}. \end{aligned}$$

Thus $0.[\xi]_{\infty}$ is a vertex of $\mathcal{F}$. Fix an integer k , such that all eigenvalues of M^k are greater than 2 (such an integer exists, since the eigenvalues of M are all greater than 1). This implies, that the representations $0.c_1[0]_k c_2[0]_k c_3[0]_k c_4 \dots$, $c_j \in \{0, 1\}$ ($j \geq 1$) represent pairwise different elements of $\mathbb{R}^2$ for different $\{0, 1\}$ sequences $\{c_j\}_{j \geq 1}$. Because $\xi + K < B - 1$, each of the uncountably many representations

$$0.[\xi]_{\infty} + 0.c_1[0]_k c_2[0]_k c_3[0]_k c_4 \dots \quad (c_j \in \{0, 1\}, j \geq 1)$$

corresponds to a vertex of $\mathcal{F}$. Since they are pairwise different, the theorem is proved. $\square$

3. CONNECTEDNESS AND INNER POINTS OF THE FUNDAMENTAL DOMAIN $\mathcal{F}$

In this section we will show, that the fundamental domain $\mathcal{F}$ is arcwise connected. To establish this result, we will apply a general theorem due to Hata (cf. [5, 6]) which assures arcwise connectedness for a large class of sets. The second result of this section is devoted to the structure of the inner points of $\mathcal{F}$. In particular, we prove, that each point with finite M -adic representation is an inner point of $\mathcal{F}$. In this section we will use the notation

$$\mathcal{F}_k := \left\{ z \mid z = \sum_{j=1}^k M^{-j} a_j, a_j \in \Phi(\mathcal{N}) \right\} \quad (k \in \mathbb{N}).$$

We start with the connectedness result.

Theorem 3.1. *Let $(M, \Phi(\mathcal{N}))$ be a number system in $\mathbb{R}^2$, which is induced by the base b of a canonical number system in a quadratic number field. Then the fundamental domain $\mathcal{F}$ of $(M, \Phi(\mathcal{N}))$ is arcwise connected.*

Proof. It is an easy consequence of the definition of $\mathcal{F}$, that

$$(8) \quad \mathcal{F} = \bigcup_{g \in \Phi(\mathcal{N})} M^{-1}(\mathcal{F} + g).$$

Furthermore, Theorem 2.1 implies that $\mathcal{F} \cap (\mathcal{F} + (1, 0)^T) \neq \emptyset$. Thus the sets contained in the union of (8) form a *chain* in the sense that $(\mathcal{F} + g) \cap (\mathcal{F} + (g + (1, 0)^T)) \neq \emptyset$ for $g \in \Phi(\mathcal{N}) \setminus (B - 1, 0)^T$. Thus $\mathcal{F}$ fulfills the conditions being necessary for the application of a theorem of Hata, namely [5, Theorem 4.6]. This theorem yields the arcwise connectedness of $\mathcal{F}$. $\square$

Now we prove the result on the inner points of $\mathcal{F}$. Note, that the existence of inner points is an immediate consequence of [9, Theorem 1].

Theorem 3.2. *Let $(M, \Phi(\mathcal{N}))$ be a number system in $\mathbb{R}^2$, which is induced by the base b of a canonical number system in a quadratic number field. Then for each $k \in \mathbb{N}$ we have*

$$\mathcal{F}_k \subset \text{int}(\mathcal{F}).$$

Proof. First we will show, that 0 is an inner point of $\mathcal{F}$. Suppose, that 0 is contained in the boundary of $\mathcal{F}$. Then by (2) there exists a representation of zero of the shape

$$(9) \quad 0 = c_{H_1} c_{H_1-1} \dots c_1 c_0 . c_{-1} c_{-2} \dots$$

This representation implies $0 \in \mathcal{F} + c_{H_1} c_{H_1-1} \dots c_1 c_0$. If we multiply (9) by M^j for $j \in \mathbb{N}$ arbitrary, we conclude, that $0 \in \mathcal{F} + c_{H_1} c_{H_1-1} \dots c_1 c_0 c_{-1} \dots c_{-j}$ for each $j \in \mathbb{N}$. Hence, 0 is contained in infinitely many different translates of $\mathcal{F}$. But since $\mathcal{F}$ is a compact set this is a contradiction to (1). Thus $0 \in \text{int}(\mathcal{F})$.

Now fix $k \in \mathbb{N}$ and $g \in \mathcal{F}_k$. Then $0 \in \text{int}(\mathcal{F})$ implies, that $g \in \text{int}(M^{-k}\mathcal{F} + g)$. The result now follows from the representation

$$\mathcal{F} = \bigcup_{g \in \mathcal{F}_k} (M^{-k}\mathcal{F} + g).$$

$\square$

There is a direct alternative proof of this theorem by using the methods of [1] and [2]. In these papers a similar result for the tiling generated by Pisot number systems is shown.

REFERENCES

- [1] S. Akiyama, *Self affine tiling and pisot numeration system*. Number Theory and its Applications (K. Györy and S. Kanemitsu, eds.), Kluwer Academic Publishers, 1999, pp 7–17.
- [2] S. Akiyama and T. Sadahiro, *A self-similar tiling generated by the minimal pisot number*. Acta Math. Info. Univ. Ostraviensis **6** (1998), 9–26.
- [3] W. J. Gilbert, *Complex numbers with three radix representations*. Can. J. Math. **34** (1982), 1335–1348.
- [4] ———, *Complex bases and fractal similarity*. Ann. sc. math. Quebec **11** (1987), no. 1, 65–77.
- [5] M. Hata, *On the structure of self-similar sets*. Japan J. Appl. Math **2** (1985), 381–414.

- [6] ———, *Topological aspects of self-similar sets and singular functions*. Fractal Geometry and Analysis (Netherlands) (J. Bélair and S. Dubuc, eds.), Kluwer Academic Publishers, 1991, pp. 255–276.
- [7] S. Ito, *On the fractal curves induced from the complex radix expansion*. Tokyo J. Math. **12** (1989), no. 2, 299–320.
- [8] I. Kátai, *Number systems and fractal geometry*. preprint.
- [9] I. Kátai and I. Környei, *On number systems in algebraic number fields*. Publ. Math. Debrecen **41** (1992), no. 3–4, 289–294.
- [10] I. Kátai and B. Kovács, *Kanonische Zahlensysteme in der Theorie der Quadratischen Zahlen*. Acta Sci. Math. (Szeged) **42** (1980), 99–107.
- [11] ———, *Canonical number systems in imaginary quadratic fields*. Acta Math. Hungar. **37** (1981), 159–164.
- [12] I. Kátai and J. Szabó, *Canonical number systems for complex integers*. Acta Sci. Math. (Szeged) **37** (1975), 255–260.
- [13] D. E. Knuth, *The art of computer programming, vol 2: Seminumerical algorithms*, 3rd ed. Addison Wesley, London, 1998.
- [14] B. Kovács, *Canonical number systems in algebraic number fields*. Acta Math. Hungar. **37** (1981), 405–407.
- [15] B. Kovács and A. Pethő, *Number systems in integral domains, especially in orders of algebraic number fields*. Acta Sci. Math. (Szeged) **55** (1991), 286–299.
- [16] W. Müller, J. M. Thuswaldner, and R. F. Tichy, *Fractal properties of number systems*. Periodica Mathematica Hungarica, to appear.
- [17] J. M. Thuswaldner, *Fractal dimension of sets induced by bases of imaginary quadratic fields*, Math. Slovaca **48** (1998), no. 4, 365–371.

Shigeki AKIYAMA
 Department of Mathematics
 Faculty of Science
 Niigata University
 NIIGATA - JAPAN
E-mail : akiyama@mathalg.ge.niigata-u.ac.jp

Jörg M. THUSWALDNER
 Department of Mathematics and Statistics
 Montanuniversität Leoben
 Franz-Josef-Str. 18
 LEOBEN - AUSTRIA
E-mail : Joerg.Thuswaldner@unileoben.ac.at