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*Journal de Théorie des Nombres de Bordeaux*, tome 12, n° 1 (2000),  
p. 219-226

[http://www.numdam.org/item?id=JTNB\\_2000\\_\\_12\\_1\\_219\\_0](http://www.numdam.org/item?id=JTNB_2000__12_1_219_0)

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## Capitulation and Transfer Kernels

par K. W. GRUENBERG et A. WEISS

**RÉSUMÉ.** On sait que pour une extension galoisienne finie  $K/k$  d'un corps de nombres, le noyau du morphisme d'extension  $\text{Cl}_k \rightarrow \text{Cl}_K$  s'identifie au noyau  $X(H)$  du transfert  $H/H' \rightarrow A$ , où  $H = \text{Gal}(\tilde{K}/k)$ ,  $A = \text{Gal}(\tilde{K}/K)$  et  $\tilde{K}$  est le corps de classes de Hilbert de  $K$ . Lorsque le groupe  $G = \text{Gal}(K/k)$  est abélien, H. Suzuki a montré que  $|G|$  divise  $|X(H)|$ .

Nous appelons noyau de transfert pour  $G$  tout groupe abélien fini  $X$  qui s'écrit  $X(H)$  pour un certain groupe  $H$  tel que  $A \hookrightarrow H \twoheadrightarrow G$ . Après avoir caractérisé les noyaux de transfert en termes de représentations entières de  $G$ , nous montrons que  $X$  est un noyau de transfert pour le groupe abélien  $G$  si et seulement si on a  $|G|X = 0$  et  $|G|$  divise  $|X|$ , ce qui fournit une nouvelle démonstration du résultat de Suzuki.

**ABSTRACT.** If  $K/k$  is a finite Galois extension of number fields with Galois group  $G$ , then the kernel of the capitulation map  $\text{Cl}_k \rightarrow \text{Cl}_K$  of ideal class groups is isomorphic to the kernel  $X(H)$  of the transfer map  $H/H' \rightarrow A$ , where  $H = \text{Gal}(\tilde{K}/k)$ ,  $A = \text{Gal}(\tilde{K}/K)$  and  $\tilde{K}$  is the Hilbert class field of  $K$ . H. Suzuki proved that when  $G$  is abelian,  $|G|$  divides  $|X(H)|$ . We call a finite abelian group  $X$  a transfer kernel for  $G$  if  $X \cong X(H)$  for some group extension  $A \hookrightarrow H \twoheadrightarrow G$ .

After characterizing transfer kernels in terms of integral representations of  $G$ , we show that  $X$  is a transfer kernel for the abelian group  $G$  if and only if  $|G|X = 0$  and  $|G|$  divides  $|X|$ . Our arguments give a new proof of Suzuki's result.

Let  $K/k$  be a finite unramified Galois extension of number fields with Galois group  $G$ . The capitulation kernel for  $K/k$  is the kernel of the natural homomorphism of ideal class groups  $\text{Cl}_k \rightarrow \text{Cl}_K$ . Suzuki [S] proved that when  $G$  is abelian, its order  $|G|$  divides the order of the capitulation kernel. This remarkable result encapsulates much of the information previously available about capitulation. We refer to the surveys [J] and [M]

for relevant background. Our aim here is to explain a new approach to Suzuki's theorem.

The transition to group theory (reviewed in § 1) allows one to interpret capitulation kernels as transfer kernels, by which we mean the following: given a finite group  $G$ , then a finite abelian group  $X$  is a *transfer kernel* for  $G$  if there exists a group extension  $A \hookrightarrow H \twoheadrightarrow G$  with  $A$  finite abelian so that  $X$  is isomorphic to the kernel of the transfer homomorphism  $H/[H, H] \rightarrow A$ . We shall prove the following result.

**Theorem 1.** *If  $G$  is a finite abelian group, then the finite additive group  $X$  is a transfer kernel for  $G$  if, and only if,  $|G|X = 0$  and  $|G|$  divides  $|X|$ .*

We outline what follows. In §1 we translate the problem into an equivalent one on  $G$ -module extensions over  $\Delta G$ , the augmentation ideal of the integral group ring  $\mathbb{Z}G$ . Then §2, the core of the paper, is an analysis of the common structural properties of transfer kernels for  $G$ . This makes possible the proof of Theorem 1 in §3. In our final §4 we collect some comments and questions.

## 1. TRANSLATIONS

We begin with the classical result of E. Artin.

**Proposition 1.** *The capitulation kernel for  $K/k$  is a transfer kernel for  $G$ .*

Here is a sketch of the proof. Let  $\tilde{K}$  be the Hilbert class field of  $K$  and  $A = \text{Gal}(\tilde{K}/K)$ . If  $H = \text{Gal}(\tilde{K}/k)$  then there is a commutative square

$$\begin{array}{ccc} \text{Cl}_k & \xrightarrow{\simeq} & H/[H, H] \\ \text{capitulation} \downarrow & & \downarrow \text{transfer} \\ \text{Cl}_K & \xrightarrow{\simeq} & A \end{array}$$

from which the proposition follows by taking kernels.

**Proposition 2.** *The finite additive group  $X$  is a transfer kernel for  $G$  if, and only if, there exists a  $G$ -module extension  $A \twoheadrightarrow B \twoheadrightarrow \Delta G$  with  $A$  finite and  $X \simeq H^{-1}(G, B)$ .*

*Proof.* This result is clear from the functorial relationship between group extensions over  $G$  and  $G$ -module extensions over  $\Delta G$  (cf. [G] §10.5). As this is not the usual approach in the literature we sketch it here.

A group extension

$$(1) \quad A \xrightarrow{i} H \xrightarrow{\pi} G$$

yields the  $G$ -module extension

$$(2) \quad A \xrightarrow{j} B \xrightarrow{\tau} \Delta G$$

where  $B = \Delta H / \Delta A \cdot \Delta H$ ,  $\tau$  is induced from  $\pi : \Delta H \rightarrow \Delta G$  and  $j(a)$  is the appropriate coset of  $i(a) - 1$ .

Conversely, given (2), let

$$\tilde{H} = \{b \in B \mid \tau(b) = g - 1, \text{ for some } g \in G\}.$$

Then  $\tilde{H}$  is a group with multiplication  $x \cdot y = \tau(x)y + x + y$ , its identity element is 0 and the inverse of  $x$  is  $x^{-1} = -g^{-1}x$ , where  $\tau(x) = g - 1$ . The module homomorphism  $\tau$  gives the group homomorphism  $\tilde{H} \rightarrow G$  via  $x \mapsto \tau(x) + 1$  with kernel  $A$ .

If  $B$  arises from the group extension (1), then the *hidden group*  $\tilde{H}$  in  $B$  gives an extension equivalent to  $H$ :

$$(3) \quad \begin{array}{ccccc} & & H & & \\ & \nearrow & \downarrow \sigma & \searrow & \\ A & & & & G \\ & \searrow & \downarrow & \nearrow & \\ & & \tilde{H} & & \end{array}$$

where  $\sigma(h) = (h - 1) + \Delta A \cdot \Delta H$ . The  $G$ -coinvariants on  $B$ , namely  $B_G = B / (\Delta G)B$ , are naturally isomorphic to  $\Delta H / (\Delta H)^2$ , whence to  $H / [H, H]$  and so to  $\tilde{H} / [\tilde{H}, \tilde{H}]$ . Notice that  $\tilde{H} / [\tilde{H}, \tilde{H}] \simeq B_G$  is just  $x[\tilde{H}, \tilde{H}] \mapsto x + (\Delta G)B$ .

We claim there is a commutative square

$$\begin{array}{ccc} H/[H, H] & \xrightarrow{\simeq} & B_G \\ \text{transfer} \downarrow & & \downarrow \hat{G} \\ A^G & \xrightarrow{\simeq} & B^G \end{array}$$

where  $\hat{G}$  is the norm endomorphism  $\Sigma_{g \in G} g$  and the lower isomorphism is induced by  $j$ .

In view of (3) we may replace  $H$  by  $\tilde{H}$  and view  $j$  as inclusion. Take a transversal  $t_g$ ,  $g \in G$ , for  $A$  in  $\tilde{H}$ . If  $x \in \tilde{H}$  with  $\tau(x) = k - 1$ , then the image of  $x$  under transfer is  $\Pi_g t_{kg}^{-1} \cdot x \cdot t_g$ , which is the same as  $\Sigma_g t_{kg}^{-1} \cdot x \cdot t_g$ , because each factor is in  $A$ . Now

$$\begin{aligned} t_{kg}^{-1} \cdot x \cdot t_g &= (-(kg)^{-1} t_{kg}) \cdot (kt_g + x) \\ &= (kg)^{-1} (kt_g + x) - (kg)^{-1} t_{kg} \\ &= g^{-1} t_g + (kg)^{-1} x - (kg)^{-1} t_{kg} \end{aligned}$$

and so the transfer image is  $\widehat{G}x$  as required.

Proposition 2 follows by taking kernels.  $\square$

## 2. TRANSFER KERNELS

Let  $G$  be a finite group, not necessarily abelian. Put  $\Lambda = \mathbb{Z}G/(\widehat{G})$  and identify  $\Lambda_G$  with  $\mathbb{Z}/|G|\mathbb{Z}$ . If  $M$  is a  $\mathbb{Z}G$ -module, then  $d_G(M)$  denotes the minimum number of module generators of  $M$ .

**Theorem 2.** *The following are equivalent:*

- (a)  $X$  is a transfer kernel for  $G$ ;
- (b)  $X$  is isomorphic to the cokernel of a homomorphism  $\varphi : U_G \rightarrow \Lambda_G^m$ , where  $m \geq d_G(\Delta G)$  and  $U$  is a finitely generated  $G$ -submodule of  $\mathbb{Q}\Lambda^{m-1}$ ;
- (c)  $X$  is isomorphic to  $M_G$  for some finitely generated  $G$ -module  $M$ , where  $\widehat{G}M = 0$  and  $\mathbb{Q}M$  contains a  $\mathbb{Q}G$ -copy of  $\mathbb{Q}\Lambda$ ;
- (d)  $|G|X = 0$  and there exists a surjective homomorphism  $X \twoheadrightarrow M_G$  with  $M$  as in (c).

*Proof.* (a)  $\Rightarrow$  (b). Using Proposition 2 we may, and shall, assume  $X \simeq H^{-1}(G, B)$ , where  $A \mapsto B \twoheadrightarrow \Delta G$ . Take a free resolution of  $B$ , so determining  $m$  and  $S$  in the following diagram:

$$(4) \quad \begin{array}{ccccccc} S & \xrightarrow{=} & S & & & & \\ \downarrow & & \downarrow & & & & \\ R & \mapsto & \mathbb{Z}G^m & \twoheadrightarrow & \Delta G & & \\ \downarrow & & \downarrow & & \downarrow \parallel & & \\ A & \mapsto & B & \twoheadrightarrow & \Delta G & & \end{array}$$

Now  $H^{-1}(G, B) \simeq H^0(G, S)$  (we use Tate cohomology throughout) and the exact sequence  $S^G \hookrightarrow S \twoheadrightarrow U$  gives

$$H^{-1}(G, U) \xrightarrow{\delta} H^0(G, S^G) \longrightarrow H^0(G, S),$$

where  $\delta$  is the connecting homomorphism. Since  $A$  is finite,  $\mathbb{Q}S = \mathbb{Q}R \simeq \mathbb{Q} \oplus \mathbb{Q}G^{m-1}$ , whence  $S^G \simeq \mathbb{Z}^m$  and  $\mathbb{Q}U \simeq \mathbb{Q}\Lambda^{m-1}$ . Note that  $U$  is  $\mathbb{Z}$ -torsion-free and so  $U$  is contained in  $\mathbb{Q}U$ . Also  $\widehat{G}U = 0$  gives  $H^{-1}(G, U) = U_G$  and  $H^0(G, U) = 0$ . Thus  $U_G \xrightarrow{\delta} \Lambda_G^m \twoheadrightarrow X$  is exact.

(b)  $\Rightarrow$  (c). The exact sequence  $\Delta G \mapsto \Lambda \twoheadrightarrow \Lambda_G$  of  $G$ -modules stays exact when we apply  $\text{Hom}(U, -)$  because  $U$  is  $\mathbb{Z}$ -free. So we obtain the exact sequence

$$\text{Hom}_G(U, \Lambda^m) \longrightarrow \text{Hom}(U_G, \Lambda_G^m) \longrightarrow H^1(G, \text{Hom}(U, \Delta G^m)),$$

where the right hand term is 0 because it is isomorphic to  $H^0(G, \text{Hom}(U, \mathbb{Z}^m))$ , which is 0 because  $\text{Hom}_G(U, \mathbb{Z}) = 0$  as  $U_G$  is finite.

It follows that the given homomorphism  $\varphi : U_G \rightarrow \Lambda_G^m$  lifts to a  $G$ -homomorphism  $\beta : U \rightarrow \Lambda^m$  with  $\beta_G = \varphi$  giving the commutative diagram

$$\begin{array}{ccccccc} U & \xrightarrow{\beta} & \Lambda^m & \twoheadrightarrow & M \\ \downarrow & & \downarrow & & \\ U_G & \xrightarrow{\varphi} & \Lambda_G^m & \twoheadrightarrow & X \end{array}$$

with  $M = \text{Coker} \beta$ . This induces an epimorphism  $M \twoheadrightarrow X$  and hence, by taking  $G$ -coinvariants, an isomorphism  $M_G \simeq X$ . Finally,  $\mathbb{Q}M \oplus \mathbb{Q}U$  contains (a  $G$ -copy of)  $\mathbb{Q}\Lambda^m$ , whence  $\mathbb{Q}M$  contains  $\mathbb{Q}\Lambda$ .

(c)  $\Rightarrow$  (d) is clear since  $|G|$  annihilates  $M_G = H^{-1}(G, M)$ .

(d)  $\Rightarrow$  (b). Choose  $m \geq \max\{d_G(M), d_G(\Delta G), d(X)\}$  and take a  $G$ -free presentation  $L \hookrightarrow \mathbb{Z}G^m \twoheadrightarrow M$  of  $M$ . Since  $\widehat{G}M = 0$ ,  $L^G = (\mathbb{Z}G^m)^G$ , thereby giving the exact sequence  $U \hookrightarrow \Lambda^m \twoheadrightarrow M$ , where  $U = L/L^G$ , and hence the exact sequence  $U_G \xrightarrow{i_G} \Lambda_G^m \twoheadrightarrow M_G$ . Choose  $\alpha : \Lambda_G^m \twoheadrightarrow X$  and let  $\beta : X \rightarrow M_G$  be the given homomorphism. Thus  $\text{Coker } i_G \simeq M_G \simeq \text{Im} \beta \alpha$ , which implies, by the Lemma below, that  $\text{Im } i_G \simeq \text{Ker} \beta \alpha$ . Consequently  $\text{Ker} \alpha$  is isomorphic to a subgroup  $D$  of  $\text{Im } i_G$ . There exists a map of  $U_G$  onto  $D$  (remember we are dealing with finite abelian groups), giving the composite  $\varphi : U_G \twoheadrightarrow D \hookrightarrow \Lambda_G^m$ , with  $\text{Im} \varphi \simeq \text{Ker} \alpha$ . Again using the Lemma below,  $\text{Coker} \varphi \simeq \text{Im} \alpha \simeq X$  and so  $U_G \xrightarrow{\varphi} \Lambda_G^m \twoheadrightarrow X$  is exact. Finally,  $\mathbb{Q}U \oplus \mathbb{Q}M \simeq \mathbb{Q}\Lambda^m$  shows that  $\mathbb{Q}M \supseteq \mathbb{Q}\Lambda$  implies  $\mathbb{Q}U \subseteq \mathbb{Q}\Lambda^{m-1}$ .

**Lemma.** (i) Given epimorphisms  $f_1, f_2 : \Lambda_G^m \twoheadrightarrow X$ , then  $\text{Ker} f_1 \simeq \text{Ker} f_2$ .

(ii) Given epimorphisms  $g_i : \Lambda_G^m \twoheadrightarrow X_i$ ,  $i = 1, 2$ , with  $\text{Ker} g_1 \simeq \text{Ker} g_2$ , then  $X_1 \simeq X_2$ .

*Proof.* In (i) the homomorphisms  $f_1, f_2$  are free presentations of  $X$  as  $\mathbb{Z}/|G|\mathbb{Z}$ -module. So Schanuel's Lemma and the Krull-Schmidt property give the result. For (ii), dualise with respect to  $\mathbb{Q}/\mathbb{Z}$  and obtain  $X_1^* \simeq X_2^*$  by (i).

(b)  $\Rightarrow$  (a). Our aim is to prove that the  $X$  of (b) is a transfer kernel in the module-theoretic sense of Proposition 2.

We use the isomorphism

$$H^1(G, \text{Hom}(U, \mathbb{Z}^m)) \simeq \text{Hom}(H^{-1}(G, U), H^0(G, \mathbb{Z}^m))$$

given by integral duality:  $\xi \mapsto (x \mapsto \xi.x)$ . Hence  $\varphi$  corresponds to a uniquely determined extension  $\mathbb{Z}^m \twoheadrightarrow S \twoheadrightarrow U$  whose associated connecting homomorphism  $H^{-1}(G, U) \rightarrow H^0(G, \mathbb{Z}^m)$  is  $\varphi$  (e.g. 11.1 in [GW]). Thus  $U_G \xrightarrow{\varphi} \Lambda_G^m \twoheadrightarrow H^0(G, S)$  is exact and so  $X \simeq H^0(G, S)$ .

Take a free presentation  $R \hookrightarrow \mathbb{Z}G^m \twoheadrightarrow \Delta G$  of  $\Delta G$  and embed  $S$  in  $R$  with cokernel  $A$ . This can be done because

$$\mathbb{Q}S \simeq \mathbb{Q}^m \oplus \mathbb{Q}U \subseteq \mathbb{Q}^m \oplus \mathbb{Q}\Lambda^{m-1} \simeq \mathbb{Q} \oplus \mathbb{Q}G^{m-1} \simeq \mathbb{Q}R.$$

Taking the pushout along  $R \twoheadrightarrow A$  gives a diagram exactly like (4) except that  $A$  might not be finite. In any case  $H^0(G, S) \simeq H^{-1}(G, B)$ .

It remains to find a submodule  $L$  of  $A$  so that  $A/L$  is finite and  $H^{-1}(G, B) \simeq H^{-1}(G, B/L)$ . First note that  $\mathbb{Q}B^G = 0$  by the middle column of (4) and  $\mathbb{Q}S^G \simeq \mathbb{Q}^m$ . Hence  $B_G$  is finite and so  $A/A \cap \Delta G.B$  is finite. Pick a torsion-free  $G$ -submodule  $L$  of finite index in  $A \cap \Delta G.B$ . Then  $A/L$  is finite; also  $\widehat{G}L = 0$ , whence  $L^G = 0$ . The exact sequence  $L \hookrightarrow B \twoheadrightarrow B/L$  then gives the exact sequence

$$H^{-1}(G, L) \xrightarrow{0} H^{-1}(G, B) \longrightarrow H^{-1}(G, B/L) \longrightarrow H^0(G, L) = 0,$$

which finishes the proof.  $\square$

### 3. PROOF OF THEOREM 1

To prove Theorem 1 it suffices, in view of Proposition 2 and Theorem 2, to establish the equivalence of

(i)  $X$  is a finite additive group such that  $|G|X = 0$  and  $|G|$  divides  $|X|$ ;

with

(ii)  $X$  is isomorphic to  $M_G$  for some finitely generated  $G$ -module  $M$ , where  $\widehat{G}M = 0$  and  $\mathbb{Q}M$  contains a  $\mathbb{Q}G$ -copy of  $\mathbb{Q}\Lambda$ .

(i)  $\Rightarrow$  (ii). By (d) of Theorem 2 it suffices to prove  $X$  has a transfer kernel for  $G$  as a homomorphic image. We shall show that any image of  $X$  of order  $|G|$  is a transfer kernel. Change notation and call this image  $X$ . So we have  $|X| = |G|$  and shall use induction on  $|X|$ : when  $X = 0$ , then  $G = 1$  and so we can take  $M = 0$ .

Now let  $X = X_1 \oplus \mathbb{Z}/p^s\mathbb{Z}$ . Since  $|G| = |X|$ , so  $G$  has an image  $\overline{G} = G/G_1$  of order  $p^s$  and then  $|G_1| = |X_1|$ . By induction,  $X_1 \simeq (M_1)_{G_1}$  for an appropriate  $M_1$ . Define  $M = \text{Ind}_{G_1}^G(M_1) \oplus \bar{\Lambda}$ , where  $\bar{\Lambda} = \mathbb{Z}\overline{G}/(\widehat{\overline{G}})$  is a  $G$ -module by inflation. Then  $\widehat{G}M = 0$  since  $\widehat{G}_1M_1 = 0$ , whence

$$\begin{aligned} M_G &= H^{-1}(G, M) = H^{-1}(G_1, M_1) \oplus H^{-1}(G, \bar{\Lambda}) \\ &= (M_1)_{G_1} \oplus \bar{\Lambda}_G = X_1 \oplus \bar{\Lambda}_G = X. \end{aligned}$$

Also, since  $\mathbb{Q}M_1 \supseteq \mathbb{Q}\Lambda_1$  so  $\mathbb{Q}M \supseteq \mathbb{Q}(\text{Ind}_{G_1}^G \Lambda_1) \oplus \mathbb{Q}\bar{\Lambda} \simeq \mathbb{Q}\Lambda$ , as required.

(ii)  $\Rightarrow$  (i). We repeat the classical argument. Take a free  $\mathbb{Z}G$ -presentation  $F \twoheadrightarrow M$  of  $M$ , with  $F = \mathbb{Z}G^m$ . Since  $M_G$  is finite, the kernel of  $F_G \rightarrow M_G$  is isomorphic to  $F_G$  and so  $M_G$  is the cokernel of an endomorphism  $f$  of  $F_G$ . It follows that  $\det f = \pm|M_G|$ .

Since  $F \rightarrow F_G$  maps  $\text{Ker}(F \rightarrow M)$  onto the image of  $f$ , there is a  $\mathbb{Z}G$ -endomorphism  $\tilde{f}$  of  $F$  such that  $\tilde{f}_G = f$  and  $\text{Coker } \tilde{f}$  maps onto  $M$ . Now  $\det \tilde{f}$  annihilates  $\text{Coker } \tilde{f}$  (recall that  $\mathbb{Z}G$  is a commutative ring). So

$(\det \tilde{f})M = 0$  implies  $\det \tilde{f} = n\hat{G}$  for a suitable integer  $n$  (since  $\mathbb{Q}\Lambda \subseteq \mathbb{Q}M$ ).

Finally, with  $\varepsilon$  denoting the augmentation on  $\mathbb{Z}G$ ,  $\varepsilon \det \tilde{f} = \det f$  and thus  $n|G| = \varepsilon(n\hat{G}) = \pm|M_G|$ .

#### 4. REMARKS

(1) Is the converse of Proposition 1 true? This is a fundamental problem. An even stronger form of this is the following: given a group extension  $A \rightarrow H \rightarrow G$  with  $A$  abelian, does there exist an unramified Galois extension  $L$  with Galois group  $H$  so that  $L$  is the Hilbert class field  $\tilde{K}$  of the fixed field  $K$  of  $A$ ?

It should be noticed that any group  $H$  can be realised as the Galois group of an unramified extension  $L/k$  ([L], p. 121). Then  $L \subseteq \tilde{K}$  and the difficulty lies in ensuring that  $L = \tilde{K}$ .

(2) Suppose  $X$  is a finite additive group such that  $|G|X = 0$ . Then

- (a) if  $X$  is a transfer kernel for  $G$ ,  $|G/[G, G]|$  divides  $|X|$ ;
- (b) if  $|G|$  divides  $|X|$ , then  $X$  is a transfer kernel for  $G$ .

Both these facts are variations of §3; for (b) one must first show that if, for each prime  $p$ , the  $p$ -primary part of  $X$  is a transfer kernel for a Sylow  $p$ -subgroup of  $G$ , then  $X$  is one for  $G$ .

However, neither (a) nor (b) has a converse if  $G$  is a non-abelian  $p$ -group. This is obvious for (b) (take  $A = 1$ ). For (a), if  $X$  is  $\mathbb{Z}$ -cyclic and of order  $|G/[G, G]|$ , then  $X$  cannot be a transfer kernel for  $G$ : for if  $X \simeq M_G$  with  $M$  as in (c) of Theorem 2, then lifting a generator of  $M_G$  to  $M$  gives a  $G$ -homomorphism  $\Lambda \rightarrow M$  which becomes an isomorphism on coinvariants (by Nakayama's Lemma and  $\mathbb{Q}M \supseteq \mathbb{Q}\Lambda$ ); then  $|X| = |G|$  forcing  $G$  to be abelian.

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