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Continued fractions, multidimensional diophantine approximations and applications


<http://www.numdam.org/item?id=JTNB_1999__11_2_425_0>
Continued Fractions, Multidimensional Diophantine Approximations and Applications

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RÉSUMÉ. Cet article rassemble des résultats généraux d’approximation diophantienne, sur les meilleures approximations et leurs applications à la théorie de répartition uniforme.

ABSTRACT. This paper is a brief review of some general Diophantine results, best approximations and their applications to the theory of uniform distribution.

1. DIOPHANTINE APPROXIMATIONS.

1.1. One-dimensional approximations.

1.1.1. Lagrange spectrum. Let \( \alpha \) be an irrational number. Dirichlet’s theorem states that there are infinitely many positive integers \( q \) such that

\[
\| q \alpha \| < \frac{1}{q}
\]

holds, where \( \| \cdot \| \) denotes the distance to the nearest integer. Hurwitz obtained a more precise result: for any irrational number \( \alpha \), the inequality

\[
\| q \alpha \| < \frac{1}{\sqrt{5}q}
\]

has infinitely many solutions in \( q \). Moreover, there is a countable set of numbers \( \alpha \) for which this inequality is an exact one, that is, for any positive \( \varepsilon \) there are only finitely positive integers \( q \) such that the inequality

\[
\| q \alpha \| < \left( \frac{1}{\sqrt{5}} - \varepsilon \right) \frac{1}{q}
\]

holds.

We define the Lagrange spectrum to be the set of the real numbers \( \lambda \) for which there exists \( \alpha = \alpha(\lambda) \) such that the inequality

\[
\| q \alpha \| < \frac{\lambda}{q}
\]
has infinitely many solutions, and for any positive $\varepsilon$ the inequality

$$||q\alpha|| < (\lambda - \varepsilon) \frac{1}{q}$$

has only a finite number of solutions. It is well-known that Lagrange spectrum has a discrete part

$$\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{8}}, \ldots,$$

and the minimal $\lambda$ for which there are uncountably many $\alpha = \alpha(\lambda)$ is $\lambda = 1/3$. Also it is well-known that Lagrange spectrum contains an interval $[0, \lambda^*]$. Moreover, for any decreasing function $\psi$ satisfying $\psi(y) = o(y^{-1})$, as $y$ tends to infinity, there is an uncountable set of real numbers $\alpha$ such that the inequality

$$||q\alpha|| < \psi(q)$$

has infinitely many solutions, but for any $\varepsilon > 0$, the stronger inequality

$$||q\alpha|| < (1 - \varepsilon)\psi(q)$$

has only a finite number of solutions.

One can find the above results in [5]. All of them can be obtained from the continued fraction expansion [14].

1.1.2. Best approximations and continued fractions. Any real number $\alpha$ may be written as

$$\alpha = b_0 + \frac{1}{b_1 + \frac{1}{b_2 + \frac{1}{b_3 + \ldots}}}$$

where $b_0 \in \mathbb{Z}$ and, for $j > 0$, $b_j$ are a nonnegative integers. For convenience, we use the notation

$$\alpha = [b_0; b_1, b_2, b_3, \ldots].$$

This representation is infinite and unique when $\alpha$ is irrational. If $\alpha$ is rational, we have $\alpha = [b_0; b_1, b_2, b_3, \ldots, b_t]$, and this representation is unique if we impose the condition $b_t \neq 0, 1$.

Convergents to $\alpha$ of the order $\nu$ are defined as

$$\frac{p_\nu}{q_\nu} = [b_0; b_1, b_2, b_3, \ldots, b_\nu]$$

A simple theorem states that these fraction and only these form the best approximations, that is the relation

$$||q_\nu\alpha|| = \min_{q < q_\nu} ||q\alpha||$$

holds for the denominators $q_\nu$ and only for them (see [14]). We now give two other easy facts.
Theorem 1. We have
\[ ||q_{\nu}\alpha|| \asymp (q_{\nu+1})^{-1}, \quad \text{(in order of approximation)}. \]

Proposition 2. We have
\[ \Delta_\nu = \begin{vmatrix} p_\nu & q_\nu \\ p_{\nu+1} & q_{\nu+1} \end{vmatrix} = \pm 1. \]

1.1.3. Klein polygons. We now consider the integer lattice \( \mathbb{Z}^2 \subset \mathbb{R}^2 \). Let \((q, a) \in \mathbb{Z}^2\) be a primitive point \((\gcd(q, a) = 1)\) and \(q, a > 0\). We define the two angles \( \varphi_+ \) and \( \varphi_- \) by
\[ \varphi_+ = \left\{ Z = (x, y) \in \mathbb{R}^2 : x \geq 0, y \geq \frac{a}{q} \right\}, \]
\[ \varphi_- = \left\{ Z = (x, y) \in \mathbb{R}^2 : y \geq 0, y \leq \frac{a}{q} \right\}. \]

Klein polygons \( K_+(a, q) \) and \( K_-(a, q) \) are defined to be respectively the following borders
\[ \partial(\text{conv}(\varphi_+ \cap (\mathbb{Z}^2 \setminus \{0\}))) \]
and
\[ \partial(\text{conv}(\varphi_- \cap (\mathbb{Z}^2 \setminus \{0\}))) \]
which consist of finite (nontrivial) intervals.

We now define \( \Delta(a, q) \) to be the domain:
\[ \Delta(a, q) = \left\{ Z = (x, y) \in \mathbb{R}^2 : x > 0, y > 0, \right. \]
\[ \left. Z \not\in \text{conv}(\varphi_+ \cap (\mathbb{Z}^2 \setminus \{0\})), \quad Z \not\in \text{conv}(\varphi_- \cap (\mathbb{Z}^2 \setminus \{0\})) \right\}. \]

We have

Theorem 3 ([7, 9]). 1. The vertices of \( K_-(a, q) \) (different from \((q, a)\)) are integer points of the form \((q_{2\mu}, p_{2\mu})\), where \((p_{2\mu}/q_{2\mu})\) is the \(2\mu\)-th convergent to \(a/q\).
2. The vertices of \( K_+(a, q) \) (different from \((q, a)\)) are integer points of the form \((q_{2\nu+1}, p_{2\nu+1})\), where \(p_{2\nu+1}/q_{2\nu+1}\) is the \((2\nu + 1)\)-th convergent to \(a/q\).
3. If \((u, v) \in (K_+(a, q) \cup K_-(a, q)) \cap \mathbb{Z}^2\) is an integer point then \(v/u\) is a convergent to \(a/q\) or one of the intermediate fractions \((wp_{\nu + 1} + p_{\nu-1})/(wq_{\nu + q_{\nu-1}}), 1 \leq w \leq b_{\nu+1}\).
4. \( \Delta(a, q) \cap \mathbb{Z}^2 = \emptyset \).

One can easily verify the same results for infinite continued fractions (i.e. for irrational numbers).

Recently, several papers [1, 21, 45, 46, 27] devoted to multidimensional generalization of Klein polygons have appeared. Unfortunately one must notice that there is something incorrect in papers [45, 46].
1.1.4. Representation of rationals. The rationals \( a/q \) with bounded partial quotients \( b_j \) are of great interest (see [22, 23, 24, 11]).

Let \( N(k, q) \) be the number of integers \( A, 1 \leq A < q, \gcd(A, q) = 1 \) such that any component \( b_i \) of the continued fraction expansion

\[
\frac{A}{q} = [0; b_1, \ldots, b_{n(A)}]
\]

is bounded by \( k: b_i \leq k, \ i = 1, \ldots, n(A) \). It is known ([22, 4, 52]) that if \( k > \gamma \log q \) with \( \gamma \) sufficiently large, then \( N(k, q) \geq 1 \). Moreover we can show that for almost all positive integers \( q \) and \( A \) with \( 1 \leq A < q \), all partial quotients are bounded by \( O(\log q) \).

By the way we may recall a famous and still open conjecture which asserts that for any \( q \geq 1 \), we have \( N(6, q) \geq 1 \). However it is known that the conjecture holds when \( q = 2^a \) or \( q = 3^a \) ([39]).

Sergei Konyagin (see [17]), by means of Farey fractions, proved the following upper bound for \( N(k, q) \):

**Theorem 4.** For any \( \gamma < 1 \) and for any \( k \geq k(\gamma) \) we have

\[
N(k, q) \ll \varphi(q)q^{-\frac{\gamma}{k \log k}},
\]

where \( \varphi \) denotes the Euler function.

We define the sequence \( A_1 < A_2 < \cdots < A_d \) to be an almost arithmetic progression if

\[
\exists w > 1: \ w \leq A_{j+1} - A_j \leq 3w, \ j = 1, \ldots, d - 1.
\]

In [32], the author shows that numbers with bounded partial quotients cannot appear very regularly: they cannot form long almost arithmetic progressions. The following theorem improves the result from [32].

**Theorem 5.** For \( d \geq 3 \), let \( A_0, \ldots, A_d \) be positive integers. Suppose

(i) \( 0 < A_0 < \cdots < A_d \) form an almost arithmetic progression;

(ii) \( \gcd(A_i, q) = 1, \ i = 0, \ldots, d. \)

Let \( A_\nu/q = [b_{\nu, 1}, \ldots, b_{\nu, s(\nu)}] \). Then there exist \( \nu_0 \) and \( \mu_0 \) such that

\[
0 \leq \nu_0 \leq d, \ 1 \leq \mu_0 \leq s(\nu_0)
\]

and

\[
b_{\nu_0, \mu_0} \gg d^{1/2}.
\]

Theorem 5 is proved by means of Klein polygons. The same result is true for real-valued (not integer) almost arithmetic progressions and in the last case S. Konyagin showed that the result for real-valued progressions is exact in order.
1.2. Simultaneous approximations. Let \( \psi : \mathbb{R}_+ \to \mathbb{R}_+ \) be a positive and real-valued function. For given \( \alpha = (\alpha_1, \ldots, \alpha_s) \in \mathbb{R}^s \), a positive integer \( p \) is said to be a \( \psi \)-approximation of \( \alpha \), if

\[
\max_{j=1,\ldots,s} ||p\alpha_j|| = \max_{j=1,\ldots,s} \min_{a \in \mathbb{Z}} |p\alpha_j - a| \leq \psi(p).
\]

1.2.1. Dirichlet and Liouville's theorems. Dirichlet's theorem states that for any \( \alpha = (\alpha_1, \ldots, \alpha_s) \in \mathbb{R}^s \), where \( 1, \alpha_1, \ldots, \alpha_s \) linearly independent over \( \mathbb{Z} \), there are infinitely many \( \psi \)-approximations of \( \alpha \) with \( \psi(y) = y^{-1/s} \).

On the other hand, Liouville's theorem ([2], ch.5) shows that for any \( \alpha = (\alpha_1, \ldots, \alpha_s) \in \mathbb{R}^s \) such that \( 1, \alpha_1, \ldots, \alpha_s \) form a basis of a real algebraic field of degree \( s + 1 \), there exists \( C(\alpha) \) such that

\[
\max_{j=1,\ldots,s} ||p\alpha_j|| \geq C(\alpha)p^{-1/s}, \quad \forall p \in \mathbb{N}.
\]

One can see that there are only countably many algebraic \( \alpha = (\alpha_1, \ldots, \alpha_s) \).

1.2.2. Theorem by Cassels and Davenport and the result by Jarnik. In [3, 6] the following result is obtained.

**Theorem 6.** There exists a constant \( C_s \) for which there exists an uncountable set of elements \( \alpha \in \mathbb{R}^s \) which do not have any \( \psi \)-approximation where \( \psi(y) = C_s y^{-1/s} \).

V. Jarnik [12, 13] proved another result:

**Theorem 7.** Let \( \psi \) and \( \lambda \) be positive real-valued functions such that \( \psi(y)y^{1/s} \) decreases as \( y \to \infty \) and \( \lambda(y) \to 0 \) as \( y \to \infty \). Then there exists an uncountable set of elements \( \alpha \in \mathbb{R}^s \) for which there are infinitely many \( \psi \)-approximations, but only finitely many \( \psi \lambda \)-approximations.

A review of other results can be found in [43, 10, 2].

1.2.3. Exact results in terms of the order of approximation. Generalizing the work [3] by means of chains of parallelepipeds [28, 50, 7, 8] we improve Jarnik's result.

**Theorem 8.** For \( y \geq 1 \), let \( \psi \) and \( \omega \) such that \( \psi(y)y^{1/s} \) decreases as \( y \to \infty \) and \( \lambda(y) \to 0 \) as \( y \to \infty \). Then there exists a vector \( \alpha = (\alpha_1, \ldots, \alpha_s) \) which has infinitely many \( \psi \)-approximations but not any \( 2^{-((s+1)(s+2))} \)-approximation.
Theorem 9. Let $\omega$ and $\psi$ be as in Theorem 8 and suppose that
$$\omega(1) \leq 2^{-(s+1)(s+3)}(s!)^{-1/s}.$$ 
Then there exists an uncountable set of vectors $\alpha = (\alpha_1, \ldots, \alpha_s)$, each of them having infinitely many $\psi$-approximations but not any $2^{-(s+3)\psi}$-approximation.

It follows that in Cassels Theorem 6 we may put
$$C_s = 2^{-(s+2)(s+3)}(s!)^{-1/s}.$$ 
We say that $\alpha = (\alpha_1, \ldots, \alpha_s)$ satisfies the $\psi$-condition if $\alpha$ has infinitely many $\psi$-approximations but not any $c_\psi$-approximation for some $c = c(\alpha)$

Theorem 10. Let $\psi$ be defined by $\psi(y) = y^{-1/\epsilon} \omega(y)$ where $\omega$ is decreasing positive function. Then in any Jordan $s$-dimensional domain $\Omega$ with Vol $\Omega > 0$, there exists an uncountable set of $\alpha \in \mathbb{R}^s$ satisfying the $\psi$-condition.

Theorems 8, 9, 10 are discussed in [33].

1.2.4. Successive best approximations. Let $\alpha = (\alpha_1, \ldots, \alpha_s) \in \mathbb{R}^s$. We define a best simultaneous approximation (b.a) of $\alpha$ to be any integer point $\zeta = (p, a_1, \ldots, a_s) \in \mathbb{Z}^{s+1}$ such that $\forall q, (b_1, \ldots, b_s) \in \mathbb{Z}^s, 1 \leq q \leq p, (q, b_1, \ldots, b_s) \neq (p, a_1, \ldots, a_s)$, we have
$$D(\zeta) = \max_{j=1,\ldots,s} |p\alpha_j - a_j| < \max_{j=1,\ldots,s} |q\alpha_j - b_j|.$$ 
Let $\alpha_j \notin \mathbb{Q}, j = 1, \ldots, s$. Then all b.a. of $\alpha$ form infinite sequences
$$\zeta^\nu = (p^\nu, a_1^\nu, \ldots, a_s^\nu), \ \nu = 1, 2, \ldots$$
where $p^1 < p^2 < \ldots < p^\nu < p^{\nu+1} < \ldots$ and
$$D(\zeta^1) > D(\zeta^2) > \ldots > D(\zeta^\nu) > D(\zeta^{\nu+1}) > \ldots.$$ 
Let
$$M_\nu[\alpha] = \begin{pmatrix} p^\nu & a_1^\nu & \ldots & a_s^\nu \\ \vdots & \vdots & \ldots & \vdots \\ p^{\nu+s} & a_1^{\nu+s} & \ldots & a_s^{\nu+s} \end{pmatrix}.$$ 
For $\alpha = (\alpha_1, \ldots, \alpha_s)$ satisfying $\alpha_j \notin \mathbb{Q}, j = 1, \ldots, s$, we define $R(\alpha) \in [2, s+1]$ to be the integer
$$R(\alpha) = \min \{ n : \text{there exist a lattice } \Lambda \subseteq \mathbb{Z}^{s+1} \text{ with dim } \Lambda = n \text{ and a natural } \nu_0 \text{ such that } \zeta^\nu \in \Lambda, \ \forall \nu > \nu_0 \}.$$ 

Proposition 11. Let $s = 1$. Then for any $\nu \geq 1$ we have $\det M_\nu[\alpha] = \pm 1$ (rank $M_\nu[\alpha] = 2, \ \forall \nu$).

Proposition 12. For any $s \geq 1$ we have $R(\alpha) = \dim_{\mathbb{Z}} (\alpha_1, \ldots, \alpha_s, 1)$. 

Proposition 13. Let \( s = 2 \) and \( \alpha_1, \alpha_2 \) such that \( 1, \alpha_1, \alpha_2 \) are linearly independent over \( \mathbb{Z} \). Then for infinitely many \( \nu \) we have
\[
\text{rank } M_\nu[\alpha] = 3 = \dim_{\mathbb{Z}}(\alpha_1, \alpha_2, 1).
\]

Proposition 11 – 13 can be easily verified. The following result is proved in [36].

Theorem 14. Let \( s \geq 3 \). There exists an uncountable set of elements \( \alpha = (\alpha_1, \ldots, \alpha_s) \in \mathbb{R}^s \) such that

(i) \( 1, \alpha_1, \ldots, \alpha_s \) are linearly independent over \( \mathbb{Z} \),

(\text{thus } \dim_{\mathbb{Z}}(\alpha_1, \ldots, \alpha_s, 1) = s + 1),

and

(ii) \( \text{rank } M_\nu[\alpha] \leq 3, \ \forall \nu \geq 1, \)

(\text{Hence for all } \nu \geq 1 \text{ we have } \det M_\nu[\alpha] = 0).\)

Theorem 14 represents a counterexample to the conjecture of J.S. Lagarias [26]. It shows that the successive b.a. have no such an asymptotic property as a reader can see in Proposition 12. The idea of the proof was suggested to the author by N.P. Dolbilin.

1.3. Linear forms. Again, let \( \alpha_1, \ldots, \alpha_s \) be real numbers such that \( 1, \alpha_1, \ldots, \alpha_s \) are linearly independent over \( \mathbb{Z} \), and put \( \alpha = (\alpha_1, \ldots, \alpha_s) \).

For \( m = (m_0, m_1, \ldots, m_s) \in \mathbb{Z}^{s+1} \setminus \{0\} \) we define
\[
\zeta(m) = m_0 + m_1 \alpha_1 + \cdots + m_s \alpha_s, \quad M = \max_{j=0,1,\ldots,s} |m_j|.
\]

A vector \( m \in \mathbb{Z}^{s+1} \setminus \{0\} \) is a best approximation of \( \alpha \) in sense of linear form if
\[
\zeta(m) = \min_{n \in \mathbb{Z}^{s+1} \setminus \{0\}} |\zeta(n)|.
\]

All best approximations form sequences
\[
\zeta_1 > \zeta_2 > \cdots > \zeta_\nu > \zeta_{\nu+1} > \cdots,
\]
\[
M_1 < M_2 < \cdots < M_\nu < M_{\nu+1} < \cdots
\]

where \( m_\nu = (m_{0,\nu}, \ldots, m_{s,\nu}) \) is the vector of the \( \nu \)-th b.a., \( \zeta_\nu = \zeta(m_\nu) \) and \( M_\nu = \max_j |m_{j,\nu}| \).

By Minkowski’s Theorem we have \( \zeta_\nu M_\nu^{s+1} \leq 1 \).

1.3.1. Singular systems. The theorem on the order of approximations from §1.1.2 does not admit multidimensional generalization in the sense of linear form.

Theorem 15 (see [29, 35]). Let \( s \) be an integer \( \geq 1 \) and \( \psi \) a function such that \( \psi(y) \) decreases to zero when \( y \) tends to infinity. Then there exists an uncountable set of elements \( \alpha = (\alpha_1, \ldots, \alpha_s) \in \mathbb{R}^s \) such that

(i) \( 1, \alpha_1, \ldots, \alpha_s \) are linearly independent over \( \mathbb{Z} \),
(ii) the sequence of the best approximations of $\alpha$ satisfies

$$\zeta_\nu \leq \psi(M_{\nu+s-1}).$$

In the case $s = 1$ this theorem means that there are real numbers with any given order of the best approximations. In higher dimensions it gives something more.

Khinchin [15] defined a vector $\alpha = (\alpha_1, \ldots, \alpha_s) \in \mathbb{R}^s$ to be a $\psi$-singular system if for any $T > 0$ the system

$$||m_1\alpha_1 + \cdots + m_s\alpha_s|| < \psi(T), \quad M = \max_{1 \leq j \leq s} |m_j| < T$$

has a nontrivial solution $(m_1, m_2, \ldots, m_s) \in \mathbb{Z}^s$.

Proposition 16. System is $\psi$-singular $\iff \zeta_\nu < \psi(M_{\nu+1}), \; \forall \nu$.

1.3.2. Successive best approximations for linear form. Here we define $\Delta_\nu^s$ to be the determinant of the successive best approximations

$$\Delta_\nu^s = \begin{vmatrix} m_{0,\nu} & m_{1,\nu} & \cdots & m_{s,\nu} \\ \cdots & \cdots & \cdots & \cdots \\ m_{0,\nu+s} & m_{1,\nu+s} & \cdots & m_{s,\nu+s} \end{vmatrix}.$$

The proposition below follows from Minkowski theorem on convex body. It seems to me that it is a well-known fact, but I could not find the corresponding reference.

Proposition 17. Let $s = 2$. Then for infinitely many $\nu$ we have $\Delta_\nu^2 \neq 0$.

The theorem below was proved by the author in [35] by means of singular systems.

Theorem 18. Let $s \geq 3$. Then there exists an uncountable set of vectors $\alpha = (\alpha_1, \ldots, \alpha_s) \in \mathbb{R}^s$ such that

(i) $1, \alpha_1, \ldots, \alpha_s$ are linearly independent over $\mathbb{Z},$

and

(ii) there exists a linear subspace $\mathcal{L}_\alpha \subset \mathbb{R}^{s+1}$, $\dim \mathcal{L}_\alpha = 3$ satisfying the condition

$$m_\nu \in \mathcal{L}_\alpha, \; \forall \nu > \nu_0.$$ 

We see that for $s \geq 3$ almost all best approximations may asymptotically lie in a three-dimensional plane but they cannot lie in two-dimensional plane. Of course these examples are degenerated in sense of measure. For almost all vectors $\alpha \in \mathbb{R}^s$ (in the sense of Lebesgue) best approximations are asymptotically $(s + 1)$-dimensional.
2. INTEGRALS FROM QUASIPERIODIC FUNCTIONS

In the text below we discuss applications of diophantine results to certain problems of uniform distribution of irrational rotations on torus. A full review of the methods and results of the theory of uniform distribution is given in [25].

2.1. Uniform distribution. Let $T$ be the one-dimensional torus and $f : T^s \to \mathbb{R}$ defined by the series

$$f(x_1, \ldots, x_s) = \sum_{m \in \mathbb{Z}^s, m \neq 0} f_m \exp(2\pi i (m_1 x_1 + \cdots + m_s x_s)).$$

We also define the integral

$$I(T, \varphi) = I_{f, \omega}(T, \varphi) = \int_0^T f(\omega_1 t + \varphi_1, \ldots, \omega_s t + \varphi_s) \, dt,$$

where $\omega_1, \ldots, \omega_s \in \mathbb{R}$ are linearly independent over $\mathbb{Z}$, and $\varphi = (\varphi_1, \ldots, \varphi_s) \in \mathbb{R}^s$.

H. Weyl [51] proved that if $f$ is a continuous function, then for any $\varphi$ we have

$$I(T, \varphi) = o(T), \quad T \to \infty.$$

This equality holds uniformly in $\varphi$ if we suppose moreover that $f$ is smooth.

V.V. Kozlov conjectured that the integral $I(T, \varphi)$ is recurrent that is the following condition holds:

$$\forall \varepsilon > 0, \quad \forall T, \quad \exists T^* > T : |I(T^*, \varphi)| < \varepsilon.$$

This conjecture is true when $f$ is any trigonometric polynomial, and in this case (*) holds uniformly in $\varphi$. This implies that for any trigonometric polynomial $f$ of finite degree, $J^\infty$ defined by

$$J^\infty(T) = J^\infty_{f, \omega}(T) = \sup_{\varphi \in \mathbb{R}^s} |I(T, \varphi)|,$$

is itself recurrent, that is

$$\forall \varepsilon > 0, \quad \forall T, \quad \exists T^* > T : |J^\infty(T^*)| < \varepsilon.$$

2.2. Case $s = 2$. In the two-dimensional case, the conjecture above was proved by V.V. Kozlov himself for functions $f \in C^2(T^2)$ in [18] (see also [19]). It is also easy to see that when $f$ is a smooth function, then (*) holds uniformly in $\varphi$, that is ($\$) is true. E.A. Sidorov [44] obtained a similar result for “absolutely” continuous $f$. 
2.3. The general result. The author [34] proved the conjecture in the general case:

**Theorem 19.** Suppose that

\[ f(x_1, \ldots, x_s) = \sum_{m \in \mathbb{Z}^s \setminus \{0\}} f_m \exp(2\pi i (m_1 x_1 + \ldots + m_s x_s)) \]

belongs to the class \( C^d(T^s) \), where \( d > C_1 s^3 \) and \( \omega_1, \ldots, \omega_s \) are linearly independent over \( \mathbb{Z} \). Then for any \( \varphi \), the integral \( I(T, \varphi) \) satisfies (*).

The proof is based on consideration of best approximations in the sense of linear form (see §1.3.2.).

2.4. Metric results. It is known [49, 48] that for almost all (in the sense of Lebesgue) vectors \( \omega = (\omega_1, \ldots, \omega_s) \in \mathbb{R}^s \), if \( f \) is smooth enough, then the integral \( I(T, \varphi) \) is bounded when \( T \to \infty \) uniformly in \( \varphi \). Hence the integral \( I(T, \varphi) \) satisfies (*), uniformly in \( \varphi \). But even in the case \( s \geq 3 \), this result is not universal.

Let \( \Phi \) be a decreasing function and assume that the series \( \sum_{m \in \mathbb{Z}} \Phi(m) \) converges. We define a periodic function \( \Theta : T^s \to \mathbb{R} \) to be of the type \( \Phi \) if, the coefficients \( \Theta_{m_1, \ldots, m_s} \) in the expansion

\[ \Theta(x_1, \ldots, x_s) = \sum \Theta_{m_1, \ldots, m_s} e^{2\pi i m_1 x_1 + \ldots + m_s x_s}, \]

satisfy

\[ |\Theta_{m_1, \ldots, m_s}| \leq \Phi(M), \text{ where } M = \max_j |m_j|. \]

We consider

\[ J^\infty(T) = J^{2\infty}_{f, \omega}(T) = \max_{\varphi \in T^s} |I_{f, \omega}(T, \varphi)|; \]

\[ J^2(T) = J^2_{f, \omega}(T) = \left( \int_{T^s} |I_{f, \omega}(T, \varphi)|^2 \, d\varphi \right)^{1/2}. \]

The result below is proved in [29].

**Theorem 20.** Let \( s \geq 3 \). Then for any function \( \Phi \) which decreases to zero as \( y \to \infty \) and for any function \( \psi \) with \( \psi(y) = o(1) \) as \( y \to \infty \), there exist \( \omega_1, \omega_2, \ldots, \omega_s \) which are linearly independent over \( \mathbb{Z} \), and a function \( f \) of type \( \Phi \) such that \( \int_{T^s} f(x) \, dx = 0 \) and

\[ J^l(T) \gg T \psi(T) \quad \forall T, \quad l = 2, \infty. \]

We will reformulate Theorem 20 in the following way.
Theorem 21. Let $s \geq 3$, and $f : \mathbb{T}^s \to \mathbb{R}$ be smooth with zero mean value. Assume that in the expansion

$$f(x_1, \ldots, x_s) = \sum_{(m_1, \ldots, m_s) \neq 0} f_{m_1, \ldots, m_s} e^{2\pi i (m_1 x_1 + \cdots + m_s x_s)}$$

the coefficients $f_{m_1, \ldots, m_s}$, where $(m_1, \ldots, m_s) \neq 0$, are all different from zero.

Then there exist $\omega_1, \omega_2, \ldots, \omega_s$ which are linearly independent over $\mathbb{Z}$ such that

$$J^l(t) \gg t\psi(t) \quad \forall t; \quad l = 2, \infty.$$ 

An improvement of the latter result was obtained recently by E.V. Kolomeikina [20].

One can see that the behaviour of integrals $J_l$ in two-dimensional case radically differs from the case $s \geq 3$.

2.5. Odd functions. Sergei Konyagin's result. Recently, S. Konyagin [16] obtained the following result.

Theorem 22. The Kozlov's conjecture is true (that is $(\ast)$ holds) for arbitrary $s \geq 1$ and any function $f$ satisfying the condition

$$f(-x_1, \ldots, -x_s) = f(x_1, \ldots, x_s), \quad f \in C^r(\mathbb{T}^s), \quad r \asymp s^{2s}.$$ 

2.6. The smoothness. In [42],[41] it is shown that we need some kind of smoothness conditions on $f$ to insure that $(\ast)$ is true: indeed in the two-dimensional case ($s = 2$), there exists a function $f : \mathbb{T}^2 \to \mathbb{R}$ (with zero mean value) of the class $C^1(\mathbb{T}^2)$ such that $I(T, 0)$ tends to infinity when $T \to \infty$ (with the choice $\omega_1 = 1$ and $\omega_2 = \sqrt{2}$). On the other hand, in [44] it is shown that when $s = 2$, a sufficient condition on $f$ for having $(\ast)$, is $f$ to be absolutely continuous.

Developing an idea of D.V. Treshchev, the author, in [31], generalized Poincaré's example. He proved that for any real $\omega_1, \ldots, \omega_s$ which form a basis of a real algebraic field, there exists a function $f \in C^{s-2}(\mathbb{T}^s) \setminus C^{s-1}(\mathbb{T}^s)$ such that $I(., 0)$ does not satisfy the property $(\ast)$ with $\varphi = 0$.

One may find some results on algebraic numbers in [40] and [38]. Recently, S.V. Konyagin [16] proved that for some Liouville transcendental numbers, there exists $f \in C^d(\mathbb{T}^s)$ with $d \asymp 2^s/s$ such that $(\ast)$ is not satisfied.

Some early results are reviewed in [37].

2.7. Vector-functions: counterexample in dimension $s = 3$. Let $f^j : \mathbb{T}^s \to \mathbb{R}$, $j = 1, 2$ be defined by

$$f^j(x_1, \ldots, x_s) = \sum_{k \in \mathbb{R}^s \setminus \mathbb{Z}^s} f^j_k \exp(2\pi i (k_1 x_1 + \cdots + k_s x_s)).$$
For $\omega_1, \ldots, \omega_s \in \mathbb{R}$ be linearly independent over $\mathbb{Z}$, we put

$$I^j(T) = \int_0^T f^j(\omega_1 t, \ldots, \omega_s t) dt, \quad j = 1, 2.$$ 

The analogue of property (*) for the vector-integral $I = (I^1, I^2): \mathbb{R} \to \mathbb{R}^2$ becomes

\begin{equation}
\forall \varepsilon > 0 \quad \forall T \exists T^* > T : |I^1(T^*)| + |I^2(T^*)| < \varepsilon.
\end{equation}

**Proposition 23.** In the case when $s = 2$ and $f$ is a smooth vector-functions, then (7) holds.

**Proposition 24.** The analogue of Theorem 22 holds for vector-function, that is (7) is satisfied for any odd smooth vector-function $f$.

Let $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a positive function such that $\sum_{k \in \mathbb{Z}} \Phi(\max_{1 \leq j \leq s} |k_j|)$ converges. A vector-function $f = (f^1, f^2) : \mathbb{T} \to \mathbb{R}^2$ is defined to be a function of type $\Phi$ if we have

$$|f^j_k| \leq \Phi(\max_{1 \leq j \leq s} |k_j|) \quad \forall k, \quad j = 1, 2.$$ 

Recently, the author [34] constructed the following example.

**Theorem 25.** For any given positive function $\Phi$, there exist a vector-function $f = (f^1, f^2) : \mathbb{T} \to \mathbb{R}^2$ of the type $\Phi$ with zero mean value ($\int_{\mathbb{T}} f^j(x) dx = 0, j = 1, 2$) and numbers $\omega_1, \omega_2, \omega_3$, which are linearly independent over $\mathbb{Z}$ such that

$$|I^1(T)| + |I^2(T)| \to \infty, \quad \text{as} \ T \to +\infty.$$ 

**Acknowledgment.** The author thanks Prof. M. Waldschmidt and Prof. P. Voutier for their help in checking the language of the paper.

**REFERENCES**


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