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Journal de Théorie des Nombres de Bordeaux, tome 11, n° 1 (1999), p. 31-44

<http://www.numdam.org/item?id=JTNB_1999__11_1_31_0>
An Analogue of Pfister’s Local-Global Principle in the Burnside Ring

par MARTIN EPKENHANS

ABSTRACT. Let $N/K$ be a Galois extension with Galois group $\mathcal{G}$. We study the set $T(\mathcal{G})$ of $\mathbb{Z}$-linear combinations of characters in the Burnside ring $B(\mathcal{G})$ which give rise to $\mathbb{Z}$-linear combinations of trace forms of subextensions of $N/K$ which are trivial in the Witt ring $W(K)$ of $K$. In particular, we prove that the torsion subgroup of $B(\mathcal{G})/T(\mathcal{G})$ coincides with the kernel of the total signature homomorphism.

1. INTRODUCTION

Let $L/K$ be a finite, separable extension of fields of characteristic $\neq 2$. With it we associate the ‘trace form’ which is defined by $tr_{L/K} : L \to K : x \mapsto tr_{L/K}x^2$. P.E. Conner started to investigate the connection of the trace form of $L/K$ and the trace form of a normal closure $N/K$ of $L/K$. His work yields some polynomial vanishing theorems for trace forms (see [1]). These identities come from identities in the Burnside ring of the Galois group $\mathcal{G} = G(N/K)$ of $N/K$. We study the trace ideal $T(\mathcal{G})$ in $B(\mathcal{G})$, which is roughly speaking the set of $\mathbb{Z}$-linear combinations of trace forms of subextensions of $N/K$ which are trivial in the Witt ring $W(K)$ of $K$.

We first recall the definition of the Burnside ring $B(\mathcal{G})$ of a finite group $\mathcal{G}$. A theorem of Springer [6] gives rise to a homomorphism $h_{N/K} : B(\mathcal{G}) \to W(K)$. The trace ideal $T(\mathcal{G})$ is a finitely generated subgroup of the free abelian group $B(\mathcal{G})$. We introduce a signature homomorphism $\text{sign}_\sigma : B(\mathcal{G}) \to \mathbb{Z}$ for each element $\sigma \in \mathcal{G}$ of order $\leq 2$. These signature homomorphisms correspond to signatures of the Witt ring. We conclude that $T(\mathcal{G})$ is contained in the intersection $L(\mathcal{G})$ of all kernels of signatures. The
main theorem states that $T(G)$ and $L(G)$ are of equal rank. Hence the torsion subgroup of $B(G)/T(G)$ is given by the kernel of the total signature homomorphism. In section 7 we reduce our approach to 2-groups. The general case follows by induction via the Frattini subgroup of $G$.

2. Notation

We first fix our notations. Let $K$ be a field. Then $K^*$ denotes the multiplicative group of $K$, $K^{*2}$ is the group of squares in $K^*$. We write $K_s$ for a separable closure of $K$.

Let $N/K$ be a Galois extension, then $G(N/K)$ denotes the Galois group of $N/K$. If $\mathcal{H} < G(N/K)$ then $N^\mathcal{H}$ is the fixed field of $\mathcal{H}$ in $N$. Let $Aut(K)$ be the group of field automorphisms of $K$.

Now let $K$ be a field of characteristic $\neq 2$. Let $\psi, \varphi$ be non-degenerate quadratic forms over $K$. Then $\det_K \psi$ is the determinant of $\psi$. If $p$ is a real place of $K$ then $\operatorname{sign}_p \psi$ is the signature of $\psi$ with respect to $p$. $\psi \otimes \varphi$ is the product of $\psi$ and $\varphi$. For $m \in \mathbb{Z}$, $m \times \psi$ is the $m$-fold sum of $\psi$. $\psi \simeq \varphi$ indicates the isometry of $\psi$ and $\varphi$ over $K$. Let $L/K$ be a field extension. Then $\psi_L$ is the lifting of $\psi$ to a form over $L$ by scalar extension. $W(K)$ is the Witt ring of $K$. Let $a_1, \ldots, a_n \in K^*$. Then $\langle a_1, \ldots, a_n \rangle$ is the diagonal form $a_1X_1^2 + \ldots + a_nX_n^2$ over $K$. $\langle a_1, \ldots, a_n \rangle = \otimes_{i=1,\ldots,n} <1,-a_i>$ is the $n$-fold Pfister form defined by $a_1, \ldots, a_n$.

Let $L/K$ be a finite and separable field extension. The trace form of $L/K$ is the non-degenerate quadratic form $\operatorname{tr}_{L/K} \langle 1 \rangle : L \rightarrow K : x \mapsto \operatorname{tr}_{L/K}(x^2)$. We denote the trace form also by $\langle L/K \rangle$, resp $\langle L \rangle$ if no confusion can arise.

Let $M$ be a set. Then $\#M$ is the cardinality of $M$. $\operatorname{ord}(G)$, $\operatorname{ord}(\sigma)$ is the order of the finite group $G$, resp. of the element $\sigma \in G$.

3. The Burnside ring $B(G)$

Let $G$ be a finite group and let $\mathcal{H} < G$ be a subgroup of $G$. We denote the transitive action of $G$ on the set of left cosets $G/\mathcal{H} = \{a\mathcal{H}, a \in G\}$ by $(G, G/\mathcal{H})$. The transitive and faithful actions of $G$ on finite sets are in one-to-one correspondence with the set of conjugacy classes of subgroups of $G$. A subgroup $\mathcal{H}$ of $G$ induces a transitive action of degree $|G : \mathcal{H}|$, hence a representation of dimension $|G : \mathcal{H}|$. Let $\chi_\mathcal{H}$ denote the corresponding character. We sometimes write $\chi_\mathcal{H}^G$ to indicate that the character is defined on $G$.

Definition 1. Let $G$ be a finite group. The Burnside ring $B(G)$ of $G$ is the free abelian group freely generated by the set 
\{ $\chi_\mathcal{H}$ | $\mathcal{H}$ runs over representatives of conjugacy classes of subgroups of $G$ \}
and with multiplication given by
\[ \chi_{U_1} \cdot \chi_{U_2} = \bigoplus_{\sigma \in U_1 \backslash G/U_2} \chi_{U_1 \cap \sigma U_2 \sigma^{-1}}, \]
where the sum runs over a set of representatives of the double cosets in \( U_1 \backslash G/U_2 \).

**Remark 2.** \( \chi_G \) is the multiplicative identity, \( \chi_{\{e\}} =: \chi_1 \) is the regular character.

Another way of defining the multiplication is as follows. Let \( \rho_i : G \to GL(V_i), i = 1, 2 \) be representations of \( G \). Then \( \rho_1 \otimes \rho_2 : G \times G \to GL(V_1 \otimes V_2) \) is a representation of \( G \times G \) on \( V_1 \otimes V_2 \). According to the diagonal embedding \( G \to G \times G \) the representation \( \rho_1 \otimes \rho_2 \) restricts to a representation of \( G \) on \( V_1 \otimes V_2 \). For \( \rho_i = (G, G/U_i) \) we get \( \rho_1 \otimes \rho_2 |_G = \bigoplus_{\sigma \in U_1 \backslash G/U_2} (G, G/(U_1 \cap \sigma U_2 \sigma^{-1})). \)

### 4. The Homomorphism \( h_{N/K} : B(G(N/K)) \to W(K) \)

**Proposition 3** (T.A. Springer). Let \( N/K \) be a finite Galois extension with Galois group \( G(N/K) = G \). Then there is a well-defined ring homomorphism
\[ h_{N/K} : B(G) \to W(K) \]
with
\[ h_{N/K}(\chi_H) = \langle N^H \rangle \]
for all subgroups \( H \) of \( G \).

**Proof.** Let \( H < G \) be a subgroup of \( G \). Then \( h_{N/K} \) is well-defined as a group homomorphism since \( \langle N^\sigma H \sigma^{-1} \rangle = \langle \sigma(N^H) \rangle = \langle N^H \rangle \). Now the assertion follows from the next lemma. \( \square \)

**Lemma 4.** Let \( N/K \) be a finite Galois extension with Galois group \( G = G(N/K) \). Let \( U_1, U_2 \) be subgroups of \( G(N/K) \). Then
\[ \langle N^{U_1} \rangle \otimes \langle N^{U_2} \rangle = \bigoplus_{\sigma \in U_1 \backslash G/U_2} \langle N^{U_1 \cap \sigma U_2 \sigma^{-1}} \rangle, \]
where the sum runs over a set of representatives of the double cosets \( U_1 \backslash G/U_2 \).

**Proof.** (see [2], I.6.2) Let \( \alpha \in N \) with \( N^{U_1} = K(\alpha) \) and let \( f \in K[X] \) be the minimal polynomial of \( \alpha \) over \( K \). Set \( L := N^{U_2} \). From Frobenius reciprocity [5], 2.5.6 we get
\[ \langle N^{U_1} \rangle \otimes \langle N^{U_2} \rangle = \langle K(\alpha) \rangle \otimes \langle L \rangle = tr_{L/K}(\langle tr_{K(\alpha)/K}(\langle 1 \rangle_L) \rangle L) = tr_{L/K}(\langle L[X]/(f) \rangle)/L = \bigoplus_{i=1, \ldots, r} tr_{L/K}(\langle L[X]/(f_i) \rangle)/L, \]
where \( f = f_1 \cdots f_r \) is the decomposition of \( f \) into monic irreducible polynomials in \( L[X] \). Now consider \( tr_{L/K}(\langle L[X]/(g) \rangle)/L \) for some monic prime
divisor \( g \in L[X] \) of \( f \). Then \( g \) is the minimal polynomial of some conjugate \( \sigma(\alpha) \) of \( \alpha \) over \( L \). Hence

\[
\text{tr}_{L/K} <(L[X]/(g))/L> = \text{tr}_{L/K}(\text{tr}_{L(\sigma(\alpha))/L<1>}) = <L(\sigma(\alpha))>.
\]

Now

\[
L(\sigma(\alpha)) = L \cdot K(\sigma(\alpha)) = L \cdot \sigma(K(\alpha)) = N^{U_2} \cdot \sigma(N^{U_1}) = N^{U_1 \cap \sigma U_2 \sigma^{-1}}.
\]

The action of \( G \) on the roots of \( f \) induces an action of \( U_2 \) on the roots of \( f \), which is equivalent to the action of \( U_2 \) on \( G/U_1 \). Each orbit of this action corresponds to a monic irreducible factor \( g \in L[X] \) of \( f \). □

5. THE TRACE IDEAL IN \( B(G) \)

**Definition 5.** Let \( G \) be a finite group. Set

\[
T(G) := \cap \ker(h_{N/K}),
\]

where the intersection is taken over all Galois extensions \( N/K \) over all fields \( K \) of characteristic \( \neq 2 \) with Galois group \( G(N/K) \cong G \). We call \( T(G) \) the trace ideal of \( B(G) \).

6. THE MAIN RESULTS

**Theorem 6.** Let \( G \) be a finite group. Then the trace ideal \( T(G) \) of \( B(G) \) is a free abelian group of rank

\[
\text{rank}(T(G)) = \text{rank}(B(G)) - \#\{\text{conjugacy classes of elements } \sigma \in G \text{ of order } \leq 2\}.
\]

The proof of theorem 6 will be organized as follows. We start by defining in a rather canonical way signatures for elements in the Burnside ring. By lemma 8, the trace ideal is contained in the kernel \( L(G) \) of the total signature homomorphism. We compute the rank \( L(G) \) in lemma 14. Now the assertion follows from the equality of the ranks of \( T(G) \) and \( L(G) \), whose proof will be the subject of sections 7 and 8. In section 7 we reduce the proof of theorem 6 to 2-groups. Section 8 contains the proof of theorem 6 for 2-groups. It runs via induction over the Frattini subgroup of \( G \).

If \( G \) is a finite group then \( RC(G) \) denotes a set of representatives of the conjugacy classes of subgroups of \( G \). Further, \( RC_2(G) \) denotes a set of representatives of the conjugacy classes of elements of order 1 or 2 in \( G \). Let \( G_2 \) be a 2-Sylow subgroup of \( G \). Then we can choose \( RC_2(G) \subset G_2 \).

In the sequel we will use the following proposition of Sylvester.
Proposition 7. Let $K$ be field, $p$ be an ordering of $K$. Then for any separable polynomial $f(X) \in K[X]$ the signature of the trace form of $K[X]/(f(X))$ over $K$ equals the number of real roots of $f(X)$ with respect to the ordering $p$.

For a proof see [7].

Lemma 8. Let $G$ be a finite group and let $\sigma \in G$ be an element of order $\leq 2$. Then there is a Galois extension $N/K$ of algebraic number fields and an isomorphism $\iota : G \xrightarrow{\sim} G(N/K)$ such that

1. $K \subset \mathbb{R}$ and $N \subset \mathbb{C}$.
2. $\iota(\sigma)$ is induced by the complex conjugation.

Proof. Set $n := \text{ord}(G)$.

1. $\text{ord}(\sigma) = 2$. If $n = 2$, set $K = \mathbb{Q}, N = \mathbb{Q}(\sqrt{-1})$.

Now let $n = 2m \geq 4$. Consider the quadratic form $\psi = (m - 1) < 1, -1 > - 1, -2 >$ as a form over $\mathbb{Q}$. Then $\det\psi \not\in \mathbb{Q}^2$ and $\text{sign}_\psi = 0$. By theorems 1 and 3 of [4] there is a field extension $L/\mathbb{Q}$ with normal closure $N/\mathbb{Q}$ such that $N \subset \mathbb{C}, G(N/\mathbb{Q}) \simeq S_n$ and $L/\mathbb{Q}$ has trace form $\psi$. Here $S_n$ denotes the symmetric group on $n$ elements.

Let $\alpha \in L$ be a primitive element of $L/\mathbb{Q}$. Since $\text{sign}_\psi < L > = 0$ no conjugate of $\alpha$ is real (see proposition 7). Let $M := \{\alpha_1, \alpha_1, \ldots, \alpha_m, \alpha_m\}$ be the set of conjugates of $\alpha$. $\bar{\alpha}$ is the complex conjugate of $\alpha \in \mathbb{C}$. Let $\varphi : G \to M$ be a bijection such that for each $a \in G$ the set $\varphi(\{a, \sigma(a)\})$ consists of a pair of complex conjugate elements of $M$. Now according to the identification given by $\varphi$ we get a monomorphism $\iota : G \hookrightarrow S(G) \xrightarrow{\sim} S(M) \xrightarrow{\sim} G(N/\mathbb{Q})$. Then $\iota(\sigma)$ is given by the complex conjugation on $N$.

Set $K := N^{\iota(\sigma)}$. Since $\iota(\sigma) \in \iota(G)$ the field $K$ is real.

2. $\sigma = \text{id}$. Set $\psi = (n - 1) < 1 > - 1 < 2 >$.

Then $\det\psi \not\in \mathbb{Q}^2$ and $\text{sign}_\psi = n$. Now choose $L, N$ and $\alpha \in L$ as above. Since $\text{sign}_\psi = \text{sign}_L < L > = n$ all conjugates of $\alpha$ are real. Hence $L \subset N \subset \mathbb{R}$. Choose any injection $\iota : G \hookrightarrow G(N/\mathbb{Q})$ and set $K := N^{\iota(\sigma)} \subset \mathbb{R}$. □

Set

$$X = \sum_{\mathcal{H} \in \text{RC}(G)} m_{\mathcal{H}} \cdot \chi_{\mathcal{H}}, \quad m_{\mathcal{H}} \in \mathbb{Z}.$$ 

Let $N/K$ be a Galois extension with Galois group $G(N/K) = G$. Let $p$ be a real place of $K$. Then

$$h_{N/K}(X) = \sum_{\mathcal{H} \in \text{RC}(G)} m_{\mathcal{H}} < N^H > = 0$$
gives
\[
\text{sign}_p h_{N/K}(X) = 0 = \sum_{\mathcal{H} \in \text{RC}(\mathcal{G})} m_{\mathcal{H}} \cdot \text{sign}_p <N^\mathcal{H}>
\]

Let $\mathcal{H} \triangleleft \mathcal{G}$ and $N^\mathcal{H} = K(\alpha)$. By proposition 7, $\text{sign}_p <N^\mathcal{H}>$ equals the number of real conjugates of $\alpha$ with respect to the ordering $p$. Let $\sigma \in G(N/K)$ be the automorphism which is induced by the complex conjugation. Then $\text{sign}_p <N^\mathcal{H}>$ is the number of fixed points of the action of $<\sigma>$ on the set of conjugates of $\alpha$, which equals the number of fixed points of the action of $<\sigma>$ on $\mathcal{G}/\mathcal{H}$. Therefore the equation (I) is already determined by $\mathcal{G}$ and the conjugacy class of the complex conjugation in $\mathcal{G}$. This leads to the following definition.

**Definition 9.** Let $\sigma \in \mathcal{G}$ be an element of order $\leq 2$. Let $\mathcal{H}$ be a subgroup of $\mathcal{G}$ and let $\chi_{\mathcal{H}} \in B(\mathcal{G})$ be the corresponding character. Set

$$\text{sign}_\sigma \chi_{\mathcal{H}} = \#\{\text{fixed points of } <\sigma>, \mathcal{G}/\mathcal{H}\}.$$

Of course, $\text{sign}_\sigma \chi_{\mathcal{H}} = \chi_{\mathcal{H}}(\sigma)$. Since our approach is motivated by quadratic form considerations we feel it is more convenient to talk about signatures.

As usual $C_\mathcal{G}(\sigma)$ denotes the centralizer of $\sigma$ in $\mathcal{G}$. Let $\mathcal{G}\sigma = \{\rho^{-1}\sigma \rho \mid \rho \in \mathcal{G}\}$ be the set of conjugates of $\sigma$ in $\mathcal{G}$.

**Proposition 10.** Let $\mathcal{G}$ be a finite group, $\mathcal{H} \triangleleft \mathcal{G}$ a subgroup of $\mathcal{G}$. Let $\sigma \in \mathcal{G}$ be an element of order $\leq 2$. Then

$$\text{sign}_\sigma \chi_{\mathcal{H}} = \frac{\text{ord}(C_\mathcal{G}(\sigma)) \#(\mathcal{G}\sigma \cap \mathcal{H})}{\text{ord}(\mathcal{H})} = \frac{[\mathcal{G} : \mathcal{H}] \#(\mathcal{G}\sigma \cap \mathcal{H})}{\#\mathcal{G}\sigma}$$

**Proof.** Consider the action of $<\sigma>$ on $\mathcal{G}/\mathcal{H}$. Let $\rho \in \mathcal{G}$. Then $\rho \mathcal{H}$ is a fixed point if and only if $\rho^{-1}\sigma \rho \in \mathcal{H}$. Hence we can assume that

$$\mathcal{G}\sigma \cap \mathcal{H} = \{\sigma_1, \ldots, \sigma_r\}$$

is a set of $r > 0$ elements. Let

$$M = \{(\rho, \sigma_i) \mid \rho^{-1}\sigma \rho = \sigma_i\} \subset \mathcal{G} \times \{\sigma_1, \ldots, \sigma_r\}.$$ 

Obviously the cardinality of $M$ is the product of $\text{ord}(\mathcal{H})$ and the number of fixed points. Further, for $i = 1, \ldots, r$ we get

$$\#\{\rho \in \mathcal{G} \mid (\rho, \sigma_i) \in M\} = \text{ord}(C_\mathcal{G}(\sigma)).$$

Hence $\#M = \text{ord}(C_\mathcal{G}(\sigma)) \cdot \#\mathcal{G}\sigma \cap \mathcal{H}. \quad \square$

We abbreviate $\chi_{<\sigma>}$ to $\chi_r$.

**Corollary 11.** In the situation of proposition 10 we get

1. $\text{sign}_\sigma \chi_{\mathcal{H}} \equiv [\mathcal{G} : \mathcal{H}] \mod 2$.
2. $\text{sign}_{id} \chi_{\mathcal{H}} = [\mathcal{G} : \mathcal{H}].$
3. \( \text{sign}_\sigma \chi_H \neq 0 \) if and only if \( H \) contains some conjugate of \( \sigma \).

4. Let \( \tau \in G \) be an element of order \( \leq 2 \). Then \( \text{sign}_\sigma \chi_\tau \neq 0 \) if and only if \( \sigma \) and \( \tau \) are conjugate or \( \sigma = \text{id} \).

5. Let \( \tau \) and \( \sigma \) be two conjugate involutions. Then

\[
2 \cdot \|G\| \cdot \text{sign}_\sigma \chi_\tau = \text{ord}(G).
\]

6. If \( H \) is a normal subgroup of \( G \), then \( \text{sign}_\sigma \chi_H = 0 \) or \( = [G : H] \).

\( \text{sign}_\sigma \) extends to a homomorphism on \( B(G) \).

**Proposition 12.** Let \( G \) be a finite group and let \( \sigma \in G \) be an element of order \( \leq 2 \). Then there is a unique homomorphism

\[
\text{sign}_\sigma : B(G) \to \mathbb{Z}
\]

with \( \text{sign}_\sigma \chi_U = \# \{ \text{fixed points of } \langle \sigma \rangle, G/U \} \) for all subgroups \( U \) of \( G \).

**Proof.** We consider the representations and characters over fields of characteristic 0. Let \( \rho : G \to GL(V) \) be the underlying representation of \( \chi_U \). Hence we get \( \text{sign}_\sigma \chi_U = \text{trace}(\rho(\sigma)) = \chi_U(\sigma) \). Since \( \text{trace}(A \otimes B) = \text{trace}(A) \cdot \text{trace}(B) \), \( \text{sign}_\sigma \) is a ring homomorphism. \( \square \)

We conclude that \( T(G) \) is contained in the intersection of all kernels of signature homomorphisms.

**Definition 13.** Let \( G \) be a finite group. Set

\[
L(G) := \left\{ \sum_{H \in \text{RC}(G)} m_H \chi_H \mid \sum_{H \in \text{RC}(G)} m_H \cdot \text{sign}_\sigma \chi_H = 0 \right. \quad \text{for all } \sigma \in \text{RC}_2(G) \right\} \subset B(G).
\]

**Lemma 14.** Let \( G \) be a finite group of order \( n \). The system of linear equations given by

\[
\sum_{H \in \text{RC}(G)} \text{sign}_\sigma \chi_H \cdot x_H = 0, \quad \sigma \in \text{RC}_2(G)
\]

has rank \( \# \text{RC}_2(G) \).

**Proof.** Let \( \sigma_1 = \text{id}, \sigma_2, \ldots, \sigma_r \) be the \( r \) distinct elements of \( \text{RC}_2(G) \). Consider the coefficients \( \text{sign}_{\sigma_j} \chi_{<\sigma_i>} \) for \( i, j = 1, \ldots, r \). We get \( \text{sign}_{\text{id}} \chi_{<\sigma_i>} = \text{ord}(G) \text{ord}(\sigma_i) \in \{n, n/2\} \) for \( i = 1, \ldots, r \). For \( j = 2, \ldots, r \) we have \( \text{sign}_{\sigma_j} \chi_{<\sigma_i>} \neq 0 \) if and only if \( i = j \). \( \square \)
Remark 15. By lemma 14, \( L(\mathcal{G}) \) is a free abelian group of rank
\[
\text{rank}(B(\mathcal{G})) - \#RC_2(\mathcal{G}).
\]
Further, \( T(\mathcal{G}) \subset L(\mathcal{G}) \) by lemma 8 and the remarks following it. We get
\[
\text{rank}(T(\mathcal{G})) = \text{rank}(L(\mathcal{G}))
\]
if and only if there exists a positive integer \( a \in \mathbb{Z} \) with \( a \cdot L(\mathcal{G}) \subset T(\mathcal{G}) \).

By Pfisters local-global principle, \( L(\mathcal{G}) \) is the set of all \( X \in B(\mathcal{G}) \) such that
\( h_{N/K}(X) \) is a torsion form for any Galois extension \( N/K \) with \( G(N/K) \simeq \mathcal{G} \).
Hence the rank formula of theorem 6 is equivalent to the existence of an integer \( l \in \mathbb{Z}, l \geq 0 \) depending only on \( \mathcal{G} \) such that \( 2^l \) annihilates \( h_{N/K}(L(\mathcal{G})) \) for any Galois extension \( N/K \) with Galois group \( \mathcal{G} \).

Since \( T(\mathcal{G}) \subset L(\mathcal{G}) \) each signature homomorphism \( \text{sign}_\sigma \) induces a unique signature homomorphism \( \text{sign}_{T(\mathcal{G})} : B(\mathcal{G})/T(\mathcal{G}) \to \mathbb{Z} \). Hence we easily get from Theorem 6:

**Theorem 16 (Local-Global Principle).** An element \( X \in B(\mathcal{G}) \) is a torsion element in \( B(\mathcal{G})/T(\mathcal{G}) \)
if and only if \( \text{sign}_\sigma(X) = 0 \) for every \( \sigma \in \mathcal{G} \) of order \( \leq 2 \). Every torsion element of \( B(\mathcal{G})/T(\mathcal{G}) \) has 2-power order.

7. REDUCTION TO 2-GROUPS

**Proposition 17.** Let \( \mathcal{G} \) be a group of odd order. Then
\[
T(\mathcal{G}) = L(\mathcal{G}).
\]
Hence \( \text{rank}(T(\mathcal{G})) = \text{rank}(B(\mathcal{G})) - 1. \)

**Proof.** Let \( N/K \) be a Galois extension with Galois group \( G(N/K) \simeq \mathcal{G} \).
Let \( L \) be an intermediate field of \( N/K \). Then \( <L> = [L : K] \times <1> \) (see [2],
cor. I.6.5). Let \( X = \sum_{\mathcal{H} \in RC(\mathcal{G})} m_{\mathcal{H}} \cdot \chi_{\mathcal{H}} \).
Then \( h_{N/K}(X) = \sum_{\mathcal{H} \in RC(\mathcal{G})} m_{\mathcal{H}} \cdot [G : \mathcal{H}] \times <1> \).
Since ord(\( \mathcal{G} \)) is odd, \( L(\mathcal{G}) \) is defined by the equation
\[
\sum_{\mathcal{H} \in RC(\mathcal{G})} m_{\mathcal{H}} \cdot [G : \mathcal{H}] = 0
\]
(see corollary 11). Now the statement about the ranks follows from remark 15. \( \square \)

Let \( \mathcal{H}, \mathcal{U} \) be subgroups of \( \mathcal{G} \). Then the representation defined by the action
of \( \mathcal{G} \) on \( \mathcal{G}/\mathcal{U} \) restricts to a representation of \( \mathcal{H} \) on \( \mathcal{G}/\mathcal{U} \).
This defines a ring homomorphism
\[
\text{res}^\mathcal{G}_{\mathcal{H}} : B(\mathcal{G}) \to B(\mathcal{H}),
\]
the 'restriction map'. We get
\[
\text{res}^\mathcal{G}_{\mathcal{H}} \chi^\mathcal{G}_{\mathcal{U}} = \oplus_{\sigma \in \mathcal{H} \cap \mathcal{U}} \chi^\mathcal{H}_{\mathcal{H} \cap \mathcal{U} \sigma^{-1}} \in B(\mathcal{H}),
\]
where \( \chi^\mathcal{H}_{\mathcal{H} \cap \mathcal{U} \sigma^{-1}} \in B(\mathcal{H}) \) is a character of \( \mathcal{H} \).

**Proposition 18.** Let \( \mathcal{G} \) be a finite group and let \( \mathcal{H} \triangleleft \mathcal{G} \). Let \( \sigma \in \mathcal{H} \) be an element of order \( \leq 2 \). Then
There is an additive but not multiplicative corestriction map $\text{cor}^G_H : B(H) \to B(G)$ defined by $\text{cor}^G_H x^H = x^G_U$.

**Proposition 19.** Let $G$ be a finite group, $H < G$. Let $N/K$ be a Galois extension with $G(N/K) = G$. Let $s^* : W(K) \to W(N^H)$ be the lifting homomorphism. Then

\[
\begin{array}{ccc}
B(G) & \xrightarrow{h_{N/K}} & W(K) \\
\text{res}^G_H & \downarrow & s^* \\
B(H) & \xrightarrow{h_{N^H/K}} & W(N^H)
\end{array}
\]

and

\[
\begin{array}{ccc}
B(H) & \xrightarrow{h_{N^H/K}} & W(N^H) \\
\text{cor}^G_H & \downarrow & \text{tr}_{N^H/K} \\
B(G) & \xrightarrow{h_{N/K}} & W(K)
\end{array}
\]

commute.

**Proof.** We use the notation of lemma 4 and its proof. Set $L := N^H$. Then

\[
h_{N/L}(\text{res}^G_H(x^G_U)) = \prod_{\sigma \in U \setminus H} h_{N/L}(\chi^H_{U \cap \sigma U^{-1}}).
\]

\[
= \prod_{\sigma \in U \setminus H} \langle N^H \cap \sigma U^{-1} \rangle / L = \prod_{i=1, \ldots, r} \langle L[X]/(f_i) \rangle / L
\]

\[
= \langle (L[X]/(f_1 \cdots f_r)) \rangle / L = \langle (K[X]/(f)) \otimes L \rangle
\]

\[
= s^* \langle N^U / K \rangle = s^* \circ h_{N/K}(\chi^G_U).
\]
Lemma 20. Let $\mathcal{H} < G$ be finite groups.

1. Then $\text{res}^G_{\mathcal{H}}(L(G)) \subseteq L(\mathcal{H})$.
2. Let $[G : \mathcal{H}]$ be odd.
   
   (a) Then $\text{res}^G_{\mathcal{H}}(X) \in L(\mathcal{H})$ if and only if $X \in L(G)$.
   
   (b) $\text{res}^G_{\mathcal{H}}(X) \in T(\mathcal{H})$ implies $X \in T(G)$.

Proof. 1. follows from proposition 18.
2. Choose $RC_2(\mathcal{G}) \subset RC_2(\mathcal{H})$ and apply proposition 18.
(b) Let $N/K$ be a Galois extension with $G(N/K) = \mathcal{G}$ and let $X \in B(\mathcal{G})$ with $\text{res}^G_{\mathcal{H}}(X) \in T(\mathcal{H})$. Now $h_{N/K} \circ \text{res}^G_{\mathcal{H}}(X) = 0 = s^* \circ h_{N/K}(X)$ by proposition 19. By a theorem of Springer $s^*$ is injective (see [5], 2.5.3). Thus $h_{N/K}(X) = 0$.

From $X \in T(G)$ we get $X \in \ker(h_{N/K})$, hence $\text{res}^G_{\mathcal{H}}(X) \in \ker(h_{N/N^\mathcal{H}})$. But we do not get $\text{res}^G_{\mathcal{H}}(X) \in T(\mathcal{H})$. We only get $\text{res}^G_{\mathcal{H}}(X) \in \cap \ker(h_{N/K})$, where the intersection runs over all Galois extensions $N/K$ with Galois group $G$ and such that $G < \text{Aut}(N)$.

Let $\exp(G)$ denote the exponent of $G$.

Proposition 21. Let $G$ be a finite group and let $G_2$ be a 2-Sylow subgroup of $G$.

1. Then the rank formula of theorem 6 holds for $G$ if it holds for any 2-Sylow subgroup of $G$, in which case $\exp(L(G)/T(G))$ divides the exponent of $L(G_2)/T(G_2)$.
2. Suppose there is a set $X$ of fields such that $G < \text{Aut}(N)$ for any $N \in X$ and such that

$$
T(G_2) = \bigcap_{N \in X} \bigcap_{U < \text{Aut}(N), U \sim G_2} \ker(h_{N/N^U}).
$$

Then $X \in T(G)$ if and only if $\text{res}^{G_2}_{G}(X) \in T(G_2)$. Hence $L(G)/T(G)$ is isomorphic to a subgroup of $L(G_2)/T(G_2)$.

Proof. 1. If the rank formula holds for $G_2$, then by remark 15 there is a positive integer $a$ with $a \cdot L(G_2) \subseteq T(G_2)$. Let $X \in L(G)$. Then $\text{res}^{G}_{G_2}(X) \in L(G_2)$ and $\text{res}^{G}_{G_2}(aX) = a \cdot \text{res}^{G}_{G_2}(X) \in a \cdot L(G_2) \subseteq T(G_2)$. Hence $aX \in T(G)$ by
lemma 20(2)(b). The proof of (2) is left to the reader.

8. PROOF OF THEOREM 6

Let \( J_2(G) \) be the set of involutions of the 2-group \( G \). For a subgroup \( H \) of \( G \) define
\[
X^G_H := X_H := \text{ord}(H) \cdot \chi^G_H - \chi^G_H + \sum_{\tau \in \text{RC}_2(G), \tau \neq 1} \#(G_\tau \cap H) \cdot (\chi^G_1 - 2 \cdot \chi^G_\tau)
\]
and let
\[
M_G := \{X_H \mid H \in \text{RC}(G) - \text{RC}_2(G)\}.
\]
By proposition 10 and corollary 11, \( M \) is a free subset of \( \mathcal{L}(G) \) which consists of \( \text{rank}(\mathcal{L}(G)) \) elements. We will prove by induction that \( M_G \) is contained in \( \mathcal{T}(G) \).

**Lemma 22.** Let \( G \) be a 2-group. Then \( M_G \) is a free subset of \( \mathcal{T}(G) \) consisting of \( \text{rank}(\mathcal{L}(G)) \) elements.

**Proof.** Observe that \( \mathcal{T}(\mathbb{Z}_2) = 0 \). Let \( G \) be a group of order \( 2^l \geq 4 \) and let \( N/K \) be a Galois extension with Galois group \( G \). Now we proceed by induction.

1. Let \( H \) be a subgroup with \( H \neq G \). Let \( \tau, \tau' \in G \) be involutions. Then \( \chi_\tau = \chi_{\tau'} \) if and only if \( \tau' \in G_\tau \). Since \( J_2(G) \) is the disjoint union of the conjugacy classes of the involutions of \( G \) we get
\[
J_2(H) = J_2(G) \cap H = \bigcup_{\tau \in \text{RC}_2(G), \tau \neq 1} G_\tau \cap H.
\]
Let \( U \subset G \) be a maximal subgroup of \( G \) which contains \( H \). Then
\[
X^U_H = \text{ord}(H) \cdot \chi^U_H - \chi^U_H + \sum_{\tau \in \text{RC}_2(U), \tau \neq 1} \#(U_\tau \cap H)(\chi^U_1 - 2 \cdot \chi^U_\tau)
\]
\[
= \text{ord}(H) \cdot \chi^U_H - \chi^U_H + \sum_{\tau \in J_2(H)} (\chi^U_1 - 2 \cdot \chi^U_\tau).
\]
Now \( X^U_H \in \mathcal{T}(U) \) by induction hypothesis. Hence \( h_{N/K}(X^U_H) = 0 \), which gives \( h_{N/K}(X^U_H) = \text{tr}_{N/K}(h_{N/K}(X^U_H)) = 0 \) (see proposition 19). Hence \( X^G_H \in \mathcal{T}(G) \) if \( H \neq G \).

2. Next we have to prove \( X^G_G \in \mathcal{T}(G) \). First we consider an elementary abelian group. Then
\[
X^G_G = 2^l \cdot \chi^G_G + (2^l - 2) \cdot \chi^G_1 - 2 \cdot \sum_{\tau \in G, \tau \neq 1} \chi^G_\tau.
\]
Let $N = K(\sqrt{a_1}, \ldots, \sqrt{a_l})$. We know that $< N > = 2^l > \otimes < -a_1, \ldots, -a_l >$ (see [3], prop. 1).

Now expand the Pfister form $< -a_1, \ldots, -a_l > = < 1, b_2, \ldots, b_2l >$. Then the entries $b_2, \ldots, b_{2l}$ are in one-to-one correspondence with the quadratic subextensions of $N/K$. There are exactly $2^{l-1} - 1$ elements $\tau \in G$, $\tau \neq id$ such that $K(\sqrt{b_i}) \subset N^\tau$. Hence

$$h_{N/K}(X_G^G) = 2^l < 1 > \perp (2^l - 2) < N > - 2 \sum_{\tau \in G, \tau \neq id} < N^\tau > = 0.$$

Now we can assume that $G$ is not an elementary abelian group. Let $U_1, \ldots, U_m$ be the maximal subgroups of $G$. Since $G$ is not a group of order 2, we get $J_2(C) \subset \bigcup_{i=1}^m U_i$. This gives

$$\sum_{\tau \in J_2(G)} (\chi_1 - 2 \cdot \chi_\tau) = \sum_{U = U_1 \cap \ldots \cap U_r} (-1)^{r+1} \sum_{\tau \in J_2(U)} (\chi_1 - 2 \cdot \chi_\tau),$$

where the sum runs over the set of all non-empty subsets of $\{1, \ldots, m\}$. Let $\Phi(G)$ denote the Frattini subgroup of $G$. Let $2^k$ be its order and set $V = G/\Phi(G)$. Let $F$ be the fixed field of $\Phi(G)$. Then $F/K$ is an elementary abelian extension. Let $\{i_1, \ldots, i_r\} \subset \{1, \ldots, m\}$ be a set of $r$ different indices. Set $H = U_{i_1} \cap \ldots \cap U_{i_r}$. Then $X_H^H \in T(H)$ by induction hypothesis. We get $h_{N/N_H}(X_H^H) = 0$, which implies

$$\sum_{\tau \in J_2(H)} \left( < N/N_H > - 2 \times < N^\tau/N_H > \right) = < N/N_H > - \text{ord}(H) \times < 1 >.$$

Set $V' = H/\Phi(G)$ and suppose $H \neq \Phi(G)$. By (1) we know that $X_{V'}^V \in T(V)$ for all subgroups $V'$ of $V$ with $V' \neq 1$. This gives

$$\text{ord}(H/\Phi(G)) \times < 1 > = < F/N_H > - \sum_{\tau \in J_2(V')} < F^\tau/N_H > - 2 \times < F^\tau/N_H >.$$
We further get
\[
h_{N/K}(\sum_{\tau \in \mathcal{J}_2(\mathcal{H})} (\chi^G_1 - 2 \cdot \chi^G_T)) = \sum_{\tau \in \mathcal{J}_2(\mathcal{H})} (\langle N \rangle - 2x \langle N^T \rangle)
\]
\[
= \text{tr}_{N^{K/H}}[\sum_{\tau \in \mathcal{J}_2(\mathcal{H})} (\langle N/N^{H} \rangle - 2x \langle N^T/N^{H} \rangle)]
\]
\[
= \text{tr}_{N^{H/K}}(\langle N/N^{H} \rangle - \text{ord}(\mathcal{H}) \times \langle 1 \rangle)
\]
\[
= \langle N \rangle - 2^k \times \text{tr}_{N^{H/K}}(\text{ord}(\mathcal{H}/\Phi(\mathcal{G})) \times \langle 1 \rangle)
\]
\[
= \langle N \rangle - 2^k \times \text{tr}_{N^{H/K}}(\langle F/N^{H} \rangle)
\]
\[
- \sum_{\tau \in \mathcal{J}_2(\mathcal{H}/\Phi(\mathcal{G}))} (\langle F/N^{H} \rangle - 2x \langle F^T/N^{H} \rangle))
\]
\[
= \langle N \rangle - 2^k \times \langle F \rangle + 2^k \times \sum_{\tau \in \mathcal{J}_2(\mathcal{H}/\Phi(\mathcal{G}))} (\langle F \rangle - 2x \langle F^T \rangle)
\]

If \( \mathcal{H} = \Phi(\mathcal{G}) \), then \( \mathcal{J}_2(\mathcal{H}/\Phi(\mathcal{G})) \) is empty and \( N^{H} = F \). Hence the formula also holds in this situation.

Now \( \sum_{\tau=0}^{n} (-1)^{\tau} \binom{n}{\tau} = 0 \) implies
\[
h_{N/K}(X^G_\mathcal{G}) = 2^l \times \langle 1 \rangle - \langle N \rangle + \sum_{\mathcal{H}} (-1)^{r+1} \sum_{\tau \in \mathcal{J}_2(\mathcal{H})} (\langle N \rangle - 2x \langle N^T \rangle)
\]
\[
= 2^l \times \langle 1 \rangle - 2^k \times \langle F \rangle + 2^k \times \sum_{\mathcal{H}} (-1)^{r+1} \sum_{\tau \in \mathcal{J}_2(\mathcal{H}/\Phi(\mathcal{G}))} (\langle F \rangle - 2x \langle F^T \rangle)
\]
\[
= 2^k \times [\text{ord}(\mathcal{V}) \times \langle 1 \rangle - \langle F \rangle + h_{F/K}(\sum_{\tau \in \mathcal{J}_2(\mathcal{V})} (\chi^\mathcal{V}_1 - 2 \cdot \chi^\mathcal{V}_T))]
\]
\[
= 2^k h_{F/K}(X^\mathcal{V}_\mathcal{G}) = 0
\]
by the above. \( \square \)

9. Open Questions

We conclude with some open questions. How does the exponent of \( B(\mathcal{G})/T(\mathcal{G}) \) depend on \( \mathcal{G} \)?

**Proposition 23.** Let \( \mathcal{G} \) be a finite group. If a 2-Sylow subgroup \( \mathcal{G}_2 \) of \( \mathcal{G} \) is a normal subgroup of \( \mathcal{G} \), then the restriction homomorphism induces an epimorphism
\[
\text{res} : L(\mathcal{G}) \rightarrow L(\mathcal{G}_2)/T(\mathcal{G}_2)
\]
**Proof.** Let \( \text{cor} : \mathcal{B}(G_2) \to \mathcal{B}(G) \) be the corestriction. This is an additive homomorphism. Since \( G_2 \) is normal in \( G \) we get \( \text{res} \circ \text{cor} = [G : G_2] \cdot \text{id} \).

By Theorem 6 there is an integer \( l \in \mathbb{N} \) such that \( 2^l \cdot L(G_2) \subset T(G_2) \). Let \( k, t \in \mathbb{Z}, k > 0 \) with \( k \cdot [G : G_2] = 1 + t \cdot 2^l \). Then \( \text{res} \circ \text{cor}(kX) = X + t \cdot 2^l X \equiv X \mod T(G_2) \).

This leads to the following question: Does the restriction homomorphism induces an isomorphism \( \text{res} : L(G)/T(G) \to L(G_2)/T(G_2) \)?

We know that the answer is affirmative if \( G \) is an abelian group whose 2-Sylow subgroup is cyclic or elementary abelian. In these cases \( L(G)/T(G) \) has exponent 2. If \( G \) is the dihedral group of order 8, then the exponent is 2. In the case of the quaternion group \( Q_8 \) of order 8 we get \( \exp(L(Q_8)/T(Q_8)) = 4 \).

**REFERENCES**


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