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Substitution invariant Sturmian bisequences

par Bruno Parvaix

Résumé. Les suites sturmiennes indexées sur $\mathbb{Z}$, de pente $\alpha$ et d’intercept $\rho$, sont laissées fixes par une substitution non triviale si et seulement si $\alpha$ est un nombre de Sturm et $\rho$ appartient à $\mathbb{Q}(\alpha)$. On remarque aussi que les suites de Beatty permettent de définir des partitions de l’ensemble des entiers relatifs.

Abstract. We prove that a Sturmian bisequence, with slope $\alpha$ and intercept $\rho$, is fixed by some non-trivial substitution if and only if $\alpha$ is a Sturm number and $\rho$ belongs to $\mathbb{Q}(\alpha)$. We also detail a complementary system of integers connected with Beatty bisequences.

1. Introduction

Beatty sequences $([n\alpha + \rho])_{n \in \mathbb{N}}$ and $([n\alpha + \rho])_{n \in \mathbb{N}}$ have been studied extensively. Many papers deal with the case $\rho = 0$, see [1, 9, 10, 14, 15, 28, 29]. The inhomogeneous case is also discussed from several points of view [6, 7, 16, 20, 21, 22]. By the way, this Note provides a new contribution about complementary systems of integers. This problem arose, in various forms, in the works of A. S. Fraenkel [13], R. L. Graham [17] and R. Tijdeman [30, 31].

A natural way to examine Beatty sequences is to consider the class of Sturmian words defined by G. A. Hedlund and M. Morse in the context of topological dynamics, see [25, 26]. For further details, both [3] and [8] contain extensive lists of references. Here we are especially interested in substitution invariant Sturmian words. In [27] we elicited properties about some right-sided infinite Sturmian words the intercept of which is a particular homography of the slope. We therefore obtained a partial generalization of Crisp et al.’s main Theorem concerning cutting sequences [12]. The aim of this Note is the full characterization of Sturmian bisequences which are fixed by some non-trivial substitution.
2. Definitions and Notations

Let \( \mathbb{N} = \{0,1,2,\ldots\} \) and \( \mathbb{N}^- = \{-1,-2,\ldots\} \). Let \( \mathbb{Z} = \mathbb{N}^- \cup \mathbb{N} \) and \( \mathbb{N}^* = \mathbb{N} \setminus \{0\} \). We consider the sets \( \mathbb{Z}_{\beta, \delta} = \{[k\beta + \delta] \mid k \in \mathbb{Z}\} \) and \( \mathbb{Z}'_{\beta, \delta} = \{[k\beta + \delta] \mid k \in \mathbb{Z}\} \), with \( \beta \) irrational and \( \delta \) real. As usual \( \lfloor x \rfloor \) is the integer part and \( \lceil x \rceil \) the ceiling of any real number \( x \). Let \( r_{\beta, \delta} \) and \( r'_{\beta, \delta} \) be the generating bisequences of \( \mathbb{Z}_{\beta, \delta} \) and \( \mathbb{Z}'_{\beta, \delta} \); we set

\[
r_{\beta, \delta}(n) = \begin{cases} 1 & \text{if } n \in \mathbb{Z}_{\beta, \delta} \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad r'_{\beta, \delta}(n) = \begin{cases} 1 & \text{if } n \in \mathbb{Z}'_{\beta, \delta} \\ 0 & \text{otherwise} \end{cases}
\]

for each \( n \in \mathbb{Z} \). We say that two subsets of \( \mathbb{Z} \) are a complementary system if they form a partition of \( \mathbb{Z} \).

Let \( \mathcal{A}^* \) be the free monoid generated by the two-letter alphabet \( \mathcal{A} = \{0,1\} \). The set of right-sided infinite words is denoted by \( \mathcal{A}^\omega \) and \( \omega \mathcal{A} \) is the set of left-sided infinite words. A bisequence is a doubly infinite word and \( \omega \mathcal{A}^\omega \) is the set of bisequences over \( \mathcal{A} \). We say that the bisequences \( \ldots v_{-2}v_{-1}v_0v_1v_2 \ldots \) and \( \ldots v'_{-2}v'_{-1}v'_0v'_1v'_2 \ldots \) are equal if there exists an integer \( k \in \mathbb{Z} \) such that \( v_i = v'_{i+k} \) for each \( i \in \mathbb{Z} \). In this event, we note \( v_{-2}v_{-1}v_0v_1v_2 \ldots \simeq v'_{-2}v'_{-1}v'_0v'_1v'_2 \ldots \).

Let \( \alpha \) be irrational and \( \rho \) be real. Consider the bisequences \( z_{\alpha, \rho} \) and \( z'_{\alpha, \rho} \) defined by

\[
z_{\alpha, \rho}(n) = \lfloor (n + 1)\alpha + \rho \rfloor - \lfloor n\alpha + \rho \rfloor - \lfloor \alpha \rfloor
\]

and

\[
z'_{\alpha, \rho}(n) = \lfloor (n + 1)\alpha + \rho \rfloor - \lfloor n\alpha + \rho \rfloor - \lfloor \alpha \rfloor
\]

for each \( n \in \mathbb{Z} \). A bisequence \( x \) is said to be Sturmian if \( x \simeq z_{\alpha, \rho} \) or \( x \simeq z'_{\alpha, \rho} \) for a suitable choice of \( \alpha \) and \( \rho \). It is clear that \( z_{\alpha, \rho}(n) = z_{\alpha+1, \rho}(n) \) and \( z'_{\alpha, \rho}(n) = z'_{\alpha+1, \rho}(n) \) for each \( n \in \mathbb{Z} \), so without loss of generality, we may take \( 0 < \alpha < 1 \). Finally, a right-sided infinite word \( y \) is Sturmian if there exist a Sturmian bisequence \( x \) and a left-sided infinite word \( y' \) such that \( x \simeq y'y \). Noting that Sturmian words are intimately related to straight lines in the plane, the number \( \alpha \) is the slope and \( \rho \) the intercept.

A substitution \( f \) is a map from \( \mathcal{A}^* \) into itself such that \( f(uu') = f(u)f(u') \) for all finite words \( u \) and \( u' \). Let \( w = w_0w_1w_2 \ldots \) be a right-sided infinite word. Let \( \text{Inv} \) be the operator defined by \( \text{Inv}(w) = \ldots w_2w_1w_0 \) and \( \text{Inv}(\text{Inv}(w)) = w \). As usual, we set \( f(w) = f(w_0)f(w_1)f(w_2) \ldots \) and \( f(\text{Inv}(w)) = \ldots f(w_2)f(w_1)f(w_0) \). The image of \( \ldots v_{-2}v_{-1}v_0v_1v_2 \ldots \) under \( f \) is \( \ldots f(v_{-2})f(v_{-1})f(v_0)f(v_1)f(v_2) \ldots \). A one-sided infinite word \( y \) is fixed by \( f \) if \( f(y) = y \), and a bisequence \( x \) is fixed by \( f \) if \( f(x) \simeq x \).

Moreover a substitution \( f \) is Sturmian if \( f(w) \) is a right-sided infinite Sturmian word whenever \( w \) is. F. Mignosi and P. Séebold proved that a substitution \( f \) is Sturmian if and only if \( f \) is a composition of the three
substitutions \( E : \quad 0 \mapsto 1 , \quad \varphi : \quad 0 \mapsto 01 \) and \( \tilde{\varphi} : \quad 0 \mapsto 10 \) in any order and number [24]. A substitution \( f \) is \textit{locally Sturmian} if there exists a right-sided infinite Sturmian word \( w \) such that \( f(w) \) is Sturmian. J. Berstel and P. Séébold stated that any locally Sturmian substitution is actually Sturmian [4, 5].

Furthermore a substitution is non-trivial if it differs from the identical transformation over \( \mathcal{A} \). In [27] we proved that if a right-sided infinite Sturmian word is fixed by some non-trivial substitution then its slope \( \alpha \), with \( 0 < \alpha < 1 \), is a \textit{Sturm number}, that is, there exists an integer \( n \geq 2 \) such that:

\[
\alpha = [0, 1 + k_n, \ldots, k_2, k_1 + k_n] \text{ with } (k_1, k_n) \in \mathbb{N}^2 \setminus \{(0, 0)\}
\]
or

\[
\alpha = [0, 1, k_n, \ldots, k_2, k_1 + k_n] \text{ with } (k_1, k_n) \in \mathbb{N}^2
\]

where the partial quotients \( k_2, \ldots, k_{n-1} \) belong to \( \mathbb{N}^* \). Remark that these numbers were introduced, in a slightly different way, by Crisp \textit{et al.} [12].

3. RESULTS

As usual, for any quadratic irrational \( \alpha \), let \( \mathbb{Q}(\alpha) = \{ p + q\alpha \mid (p, q) \in \mathbb{Q}^2 \} \) be the splitting field of \( \alpha \) over \( \mathbb{Q} \). The main result of this Note is the full characterization of Sturmian bisequences which are invariant under some non-trivial substitution:

\textbf{Theorem 1.} Let \( x \) be a Sturmian bisequence with slope \( 0 < \alpha < 1 \). The word \( x \) is fixed by some non-trivial substitution if and only if \( \alpha \) is a Sturm number and \( \rho \) belongs to \( \mathbb{Q}(\alpha) \).

In [27], we computed the slope and the intercept of \( f(x) \) for any Sturmian substitution \( f \) and any right-sided infinite Sturmian word \( x \). Lemma 2 is a translation of these formulas for Sturmian bisequences:

\textbf{Lemma 2.} Let \( \alpha \) be irrational with \( 0 < \alpha < 1 \) and let \( \rho \) be real. Then

\[
E(z_{\alpha, \rho}) \simeq z'_{1-\alpha, 1-\rho} \text{ and } \varphi(z_{\alpha, \rho}) \simeq z'_{1-\alpha, 1-\rho} = \tilde{\varphi}(z_{\alpha, \rho}).
\]

Moreover

\[
E(z'_{\alpha, \rho}) \simeq z_{1-\alpha, 1-\rho} \text{ and } \varphi(z'_{\alpha, \rho}) \simeq z_{1-\alpha, 1-\rho} = \tilde{\varphi}(z'_{\alpha, \rho}).
\]

The proof of these properties requires a careful study of generating bisequences of Beatty bisequences:

\textbf{Lemma 3.} Let \( \beta > 1 \) be irrational and \( \delta \) be real. For each \( n \in \mathbb{Z} \), we have

\[
\tau_{\beta, \delta}(n) = z'_{\beta', -\delta}(n) \text{ and } \tau'_{\beta, \delta-1}(n) = z_{\beta', -\delta}(n).
\]
As an immediate corollary, we can characterize the occurrences of a letter in any Sturmian bisequence. More precisely we remark that

\[ \{ n \in \mathbb{Z} \mid z_{\gamma, \nu}(n) = 1 \} = \mathcal{Z}'_{\frac{1}{\gamma}, -\frac{\nu}{\gamma} - 1} \quad \text{and} \quad \{ n \in \mathbb{Z} \mid z'_{\gamma, \nu}(n) = 1 \} = \mathcal{Z}_{\frac{1}{\gamma}, -\frac{\nu}{\gamma}} \]

for each \( \gamma \) irrational with \( 0 < \gamma < 1 \) and \( \nu \) real. This result is a generalization of earlier work of A. S. Fraenkel, M. Mushkin and U. Tassa dealing with the homogeneous case [15]. From Lemma 3 we also obtain a property about complementary systems of integers:

**Proposition 4.** Let \( \beta > 1 \) be irrational and \( \delta \) be real. Then \( \mathcal{Z}_{\beta, \delta} \) and \( \mathcal{Z}'_{\frac{1}{\beta}, \frac{\delta}{\beta} - 1} \) as well as \( \mathcal{Z}'_{\beta, \delta} \) and \( \mathcal{Z}_{\frac{1}{\beta}, \frac{\delta}{\beta} + 1} \), are complementary systems of integers.

4. **Proofs**

First of all, we examine the generating bisequences of Beatty bisequences:

**Proof of Lemma 3.** Let \( n \in \mathbb{Z} \). If \( z'_{\frac{1}{\beta}, -\frac{\delta}{\beta}}(n) = 1 \) we state that

\[ \frac{n}{\beta} - \frac{\delta}{\beta} \leq \left\lfloor \frac{n + 1}{\beta} - \frac{\delta}{\beta} \right\rfloor - 1 < \frac{n + 1}{\beta} - \frac{\delta}{\beta} \]

thus \( n \leq \left\lfloor \frac{n}{\beta} - \frac{\delta}{\beta} \right\rfloor \beta + \delta < n + 1 \). Next comes \( \left\lfloor \frac{n}{\beta} - \frac{\delta}{\beta} \right\rfloor \beta + \delta = n \) and \( r_{\beta, \delta}(n) = 1 \).

Conversely, if \( r_{\beta, \delta}(n) = 1 \) there exists an integer \( k \in \mathbb{Z} \) such that \( |k\beta + \delta| = n \). We therefore observe that

\[ \left\lfloor \frac{n}{\beta} - \frac{\delta}{\beta} \right\rfloor - 1 < \frac{n}{\beta} - \frac{\delta}{\beta} \leq k < \frac{n + 1}{\beta} - \frac{\delta}{\beta} \leq \left\lfloor \frac{n + 1}{\beta} - \frac{\delta}{\beta} \right\rfloor . \]

It follows that \( \left\lfloor \frac{n}{\beta} - \frac{\delta}{\beta} \right\rfloor \leq k < \left\lfloor \frac{n + 1}{\beta} - \frac{\delta}{\beta} \right\rfloor \) and \( z'_{\frac{1}{\beta}, -\frac{\delta}{\beta}}(n) = 1 \).

The truth of the first statement is now clear, and we turn to the second part of the proof. Let \( n \in \mathbb{Z} \). If \( z_{\frac{1}{\beta}, -\frac{\delta}{\beta}}(n) = 1 \) then

\[ \frac{n}{\beta} - \frac{\delta}{\beta} < \left\lfloor \frac{n}{\beta} - \frac{\delta}{\beta} \right\rfloor + 1 = \left\lfloor \frac{n + 1}{\beta} - \frac{\delta}{\beta} \right\rfloor \leq \frac{n + 1}{\beta} - \frac{\delta}{\beta} \]

hence we have

\[ n < \left\lfloor \frac{n + 1}{\beta} - \frac{\delta}{\beta} \right\rfloor \beta + \delta \leq n + 1 \]

that is \( \left\lfloor \frac{n + 1}{\beta} - \frac{\delta}{\beta} \right\rfloor \beta + \delta - 1 = n \). This implies that \( r'_{\beta, \delta - 1}(n) = 1 \).
Conversely, if \( r'_{\beta-\delta-1}(n) = 1 \) then there exists \( k \in \mathbb{Z} \) such that \([k\beta+\delta-1] = n\). Thus we check \( z_{1-\delta,\delta}^1(n) = 1 \) since
\[
\left\lfloor \frac{n}{\beta} - \frac{\delta}{\beta} \right\rfloor \leq \frac{n}{\beta} - \frac{\delta}{\beta} < k \leq \left\lfloor \frac{n+1}{\beta} - \frac{\delta}{\beta} \right\rfloor.
\]

In order to describe the complementary system of integers, connected with a Beatty bisequence, we need to introduce the following Lemma:

**Lemma 5.** Let \( 0 < \alpha < 1 \) be irrational and \( \rho \) be real. Then \( E(z_{\alpha,\rho}) \simeq z_{1-\alpha,1-\rho} \) and \( E(z'_{\alpha,\rho}) \simeq z_{1-\alpha,1-\rho} \).

**Proof.** We only detail the proof concerning the first result. Let \( n \in \mathbb{Z} \). Since the relation \([a] = -\lceil-a\rceil\) holds for each real number \( a \), we verify
\[
z'_{1-\alpha,1-\rho}(n) = 1 - (-n\alpha - \rho) - (-\alpha - \rho)) = 1 - z_{\alpha,\rho}(n) = E(z_{\alpha,\rho}(n)).
\]

**Proof of Proposition 4.** Let \( n \in \mathbb{Z} \). From Lemma 3, we remark that
\[
n \in \mathbb{Z}_{\beta,\delta} \iff r_{\beta,\delta}(n) = 1 \iff z'_{1,\delta}(n) = 1.
\]

Then Lemma 5 implies that
\[
n \in \mathbb{Z}_{\beta,\delta} \iff z'_{1,\delta}(n) = 0 \iff r'_{\beta-1,\beta+\delta}(n) = 0 \iff r'_{\beta-1,\beta-\delta-1}(n) = 0.
\]

In other words, we get
\[
n \in \mathbb{Z}_{\beta,\delta} \iff n \notin \mathbb{Z}'_{\beta-1,\beta-\delta-1}.
\]

Furthermore, since \( \beta > 1 \) we can affirm that any integer occurs at most one time in \( \mathbb{Z}_{\beta,\delta} \). Clearly this property also holds for \( \mathbb{Z}'_{\beta-1,\beta-\delta-1} \). In short, the sets \( \mathbb{Z}_{\beta,\delta} \) and \( \mathbb{Z}'_{\beta-1,\beta-\delta-1} \) are a complementary system of integers. The part of proof concerning \( \mathbb{Z}'_{\beta,\delta} \) and \( \mathbb{Z}_{\beta-1,\beta-\delta+1} \) is similar in all respects.

From now on we study properties of substitution invariant Sturmian bisequences.

**Proof of Lemma 2.** Assume first that \( 0 \leq \rho < 1 \). We split the bisequence \( z_{\alpha,\rho} \) into the words
\[
w = z_{\alpha,\rho}(0) z_{\alpha,\rho}(1) \ldots z_{\alpha,\rho}(m) \ldots \in \mathcal{A}^\omega
\]
and
\[
w' = \ldots z_{\alpha,\rho}(-m) \ldots z_{\alpha,\rho}(-2) z_{\alpha,\rho}(-1) \in \omega \mathcal{A}.
\]
Let $\varphi(w) = y_0 y_1 \ldots$ with $y_j \in A$ for $j = 0, 1, \ldots$ We observe that $y_0 = 0 = z'_{1-\alpha, 1-\rho}(0)$. Let $n_{q+1}$ be the $(q+1)$-th occurrence of the letter 0 in the word $\varphi(w)$ for each $q \geq 1$. We easily check:

$$n_{q+1} = (q + \sum_{i=0}^{q-1} (1 - z_{\alpha, \rho}(i)) + 1) - 1 = 2q - [q\alpha + \rho] = [q(2 - \alpha) - \rho].$$

For each $n \geq 1$ we state that:

\[
\begin{align*}
y_n = 0 & \quad \iff \exists q \in \mathbb{N}^* \quad n = [q(2 - \alpha) - \rho] \\
& \quad \iff \exists q \in \mathbb{Z} \quad n = [q(2 - \alpha) - \rho] \\
& \quad \iff r'_{2-\alpha, -\rho}(n) = 1 \\
& \quad \iff z_{\frac{1}{2-\alpha}, \frac{\rho - 1}{2-\alpha}}(n) = 1.
\end{align*}
\]

From Lemma 5, we prove that $y_n = 0$ if and only if $z'_{1-\alpha, 1-\rho}(n) = 0$. In short we obtain $\varphi(w) = (z'_{1-\alpha, 1-\rho}(n))_{n \in \mathbb{N}}$. To compute $\varphi(w')$, we remark that

$$w' = \ldots z'_{\alpha, 1-\rho}(m) \ldots z'_{\alpha, 1-\rho}(1)z'_{\alpha, 1-\rho}(0).$$

Indeed, for each $n \in \mathbb{N}^*$ it is clear that

$$z_{\alpha, \rho}(-n) = [(-n + 1)\alpha + \rho] - [-n\alpha + \rho] - [\alpha] = -[(n - 1)\alpha - \rho] + [n\alpha - \rho] - [\alpha]$$

hence

$$z_{\alpha, \rho}(-n) = z'_{\alpha, -\rho}(n-1) = z'_{\alpha, 1-\rho}(n-1).$$

If we write $w' = \ldots a_m \ldots a_1 a_0$ over $\omega A$, we get

$$\varphi(w') = \ldots 01^{1-a_m} \ldots 01^{1-a_1} 01^{1-a_0}$$

because $\varphi(0) = 01$ and $\varphi(1) = 0$. We can deduce that

$$Inv(\varphi(w')) = 1^{1-a_0} 01^{1-a_1} 01^{1-a_m} 0 \ldots = \tilde{\varphi}(a_0 a_1 \ldots a_m \ldots)$$

and $\varphi(w') = Inv(\tilde{\varphi}((z'_{\alpha, 1-\rho}(n))_{n \in \mathbb{N}}))$. Much as above, we verify

$$\varphi((z'_{\alpha, 1-\rho}(n))_{n \in \mathbb{N}}) = (z_{\frac{1}{2-\alpha}, \frac{\rho - 1}{2-\alpha}}(n))_{n \in \mathbb{N}}.$$ 

Moreover we observe that $\tilde{\varphi}(a) = 1^{1-a} 0 \varphi(a) = 01^{1-a}$ for each $a \in \{0, 1\}$. Next comes $\varphi(u) = 0\tilde{\varphi}(u)$ for any $u \in A^\omega$, and consequently $\varphi(w') = Inv((z_{\frac{1}{2-\alpha}, \frac{1-a}{2-\alpha}}(n))_{n \in \mathbb{N}})$. Bearing in mind that $z_{\alpha, \rho} \simeq w'w$, and noting that $z_{\frac{1}{2-\alpha}, \frac{1-a}{2-\alpha}}(n) = z'_{\frac{1}{2-\alpha}, \frac{1-a}{2-\alpha}}(-n-1)$ for each $n \in \mathbb{N}$, we finally obtain $\varphi(z_{\alpha, \rho}) \simeq z'_{\frac{1}{2-\alpha}, \frac{1-a}{2-\alpha}}$. To conclude, we must prove that the relation
holds for each $k \in \mathbb{Z}$. Since $z'_{\beta, \delta+1} \simeq z'_{\beta, \delta} \simeq z'_{\beta, \delta+\beta}$ for arbitrarily $\beta$ irrational and $\delta$ real, we directly claim:

$$z'_{\frac{1}{2-\alpha}, \frac{1-(\rho+k)}{2-\alpha}} \simeq z'_{\frac{1}{2-\alpha}, \frac{1-(\rho+k)}{2-\alpha}} + \frac{k}{2-\alpha} \simeq z'_{\frac{1}{2-\alpha}, \frac{1-\rho}{2-\alpha}} \simeq \varphi(z_{\alpha, \rho}) \simeq \varphi(z_{\alpha, \rho+k})$$

The computation of $\tilde{\varphi}(z_{\alpha, \rho+k})$ becomes trivial because we have $\tilde{\varphi}(v) \simeq \varphi(v)$ for each $v \in \omega \mathcal{A}^\omega$. Finally, the part of proof concerning $z_{\alpha, \rho}$ is similar in all respects. 

For each Sturmian substitution $f$ it is therefore clear that $f(x)$ is a Sturmian bisequence whenever $x$ is. Now we turn to the proof of Theorem 1. Some preliminaries are required. Let $x$ and $y$ be two Sturmian bisequences. Let $f$ be a substitution such that $f(x) \simeq y$. There exist a word $x' \in \omega \mathcal{A}$ and a right-sided infinite Sturmian word $x''$ such that $x \simeq x'x''$. Since we have $y \simeq f(x')f(x'')$, the word $f(x'')$ is a right-sided infinite Sturmian word. Thus $f$ is locally Sturmian and consequently $f$ belongs to the monoid $\{E, \varphi, \tilde{\varphi}\}$.

Let us recall some basic properties about Sturmian bisequences. For any irrational $\alpha$ we set $\mathbb{Z} + \mathbb{Z} \alpha = \{a + b\alpha \mid (a, b) \in \mathbb{Z}^2\}$. Let $\Delta$ be the set of couples $(\beta, \delta)$ with $0 < \beta < 1$ irrational and $\delta$ real. We also set $\mathcal{U} = \{((\beta, \delta) \in \Delta \mid \forall k \in \mathbb{Z} \ k\beta + \delta \notin \mathbb{Z}\}$. Let $(\alpha, \rho) \in \Delta$ and $(\alpha', \rho') \in \Delta$. We have $z_{\alpha, \rho} \simeq z_{\alpha', \rho'}$ if and only if $\alpha = \alpha'$ and $\rho - \rho' \in \mathbb{Z} + \mathbb{Z} \alpha$, see [26]. A similar result can be stated from the relation $z'_{\alpha, \rho} \simeq z'_{\alpha', \rho'}$. Furthermore, if $z_{\alpha, \rho} \simeq z'_{\alpha', \rho'}$, then $(\alpha, \rho)$ belongs to $\mathcal{U}$ and $z_{\alpha, \rho} \simeq z'_{\alpha', \rho'}$. In short, if two Sturmian bisequences are equal then they have the same slope in $]0, 1[$.

Bearing these remarks in mind, we therefore obtain:

**Lemma 6.** Let $x$ be a Sturmian bisequence with slope $0 < \alpha < 1$. If $x$ is invariant under some non-trivial substitution then $\alpha$ is a Sturm number.

**Proof (Sketch).** Assume that there exists a non-trivial substitution $f$ such that $f(x) \simeq x$. Then $f$ belongs to $\{E, \varphi, \tilde{\varphi}\}$. Let $\beta \in ]0, 1[$ be the slope of $f(x)$ which is obtained by Lemma 2. Clearly this computation can be done regardless of intercepts, and there exists an homography $h$, with integer coefficients, such that $\beta = h(\alpha)$. Therefore it only remains to solve the equation $\alpha = h(\alpha)$. In this context, we have yet observed that $\alpha$ is a Sturm number: for a full characterization of the homographies connected with Sturmian substitutions, see the proof of Theorem 1 in [27].

In order to prove our main result, we add here a new necessary condition of invariance:

**Lemma 7.** Let $x$ be a Sturmian bisequence, with slope $\alpha$ and intercept $\rho$. If $x$ is invariant under some non-trivial substitution then $\rho$ belongs to $\mathbb{Q}(\alpha)$. 

Proof. Assume, without loss of generality, that $0 < \alpha < 1$. Let $f$ be a non-trivial substitution such that $f(x) \simeq x$. Lemma 6 implies that $\alpha$ is a Sturm number. Since $\alpha$ is a quadratic irrational, the image of $\alpha$ under any homography, with integer coefficients, belongs to $\mathbb{Q}(\alpha)$. Using Lemma 2, we compute the image of $x$ under $f$. Let $\beta$ be the slope and $\delta$ be the intercept we obtain. It is clear that $\beta \in \mathbb{Q}(\alpha)$ and $0 < \beta < 1$. We also remark that $\delta \in \mathbb{Q}(\alpha) + \rho \mathbb{Q}(\alpha)$. Since $f(x) \simeq x$, we must check $\beta = \alpha$ and $\delta - \rho \in \mathbb{Z} + \mathbb{Z} \alpha$. Now comes $\rho \in \mathbb{Q}(\alpha)$. □

Combining Lemmas 6 and 7, we establish the “only if part” of Theorem 1. Now we turn to the proof of the “if part”: the idea is to use some properties that we reported in [27]. First of all, a technical result concerning Sturmian continuations is required [26].

Definition 8 (cf. [26]). Let $y$ be a right-sided infinite Sturmian word. A Sturmian continuation of $y$ is a left-sided infinite word $y'$ such that $y'y$ is a Sturmian bisequence.

Lemma 9 (cf. [26]). Let $\alpha$ be irrational with $0 < \alpha < 1$ and $\rho$ be real. Each right-sided infinite Sturmian word $y$, with slope $\alpha$ and intercept $\rho$, admits at least one and at most two Sturmian continuations. In the case where $y$ admits different Sturmian continuations there exist two integers $k_1 \in \mathbb{Z}$ and $k_2 \in \mathbb{N}^*$ such that $\rho = k_1 + k_2 \alpha$.

Definition 10 (cf. [27]). For each $m \geq 1$, we set

$$C'(m) = \{(a, b) \in \mathbb{Z}^2 \mid 0 \leq a + b \leq m, \ 0 \leq a \leq m\} \setminus \{(m, 0)\}.$$

A right-sided infinite Sturmian word $y$ is said to be permitted if there exist an irrational $\alpha$ with $0 < \alpha < 1$, an integer $m \geq 1$ and a couple of integers $(a, b) \in C'(m)$ such that $y = (z_{\alpha, a/m + b/n}(n))_{n \in \mathbb{N}}$ or $y = (z'_{\alpha, a/m + b/n}(n))_{n \in \mathbb{N}}$.

Proposition 11 (cf. [27]). Let $\alpha$ be a Sturm number. Each permitted word $y$, with slope $\alpha$, is invariant under some non-trivial substitution.

Proof of Theorem 1. Let $\alpha$ be a Sturm number and $\rho \in \mathbb{Q}(\alpha)$. Let $x$ be a Sturmian bisequence such that $x \simeq z_{\alpha, \rho}$. Clearly there exists $(a, b, n) \in \mathbb{Z}^3$ with $n \geq 1$ such that $\rho = \frac{a + b \alpha}{n}$. Moreover, since $z_{\alpha, \delta + 1} \simeq z_{\alpha, \delta} \simeq z_{\alpha, \delta + \alpha}$ for each real $\delta$, we actually have $x \simeq z_{\alpha, a \ (mod \ n) + b \ (mod \ n)}$. As usual, the residue $i \ (mod \ n)$ is the integer $j$, with $0 \leq j < n$, such that there exists an integer $k \in \mathbb{Z}$ satisfying $j = i + kn$. For each real $\delta$ we set

$$z_{\alpha, \delta}^+ = z_{\alpha, \delta}(0) z_{\alpha, \delta}(1) \ldots \text{ and } \ldots z_{\alpha, \delta}^-(2) z_{\alpha, \delta}^-(1) = z_{\alpha, \delta}^-.$$

We first assume that $a \ (mod \ n) + b \ (mod \ n) \leq n$. Then

$$y = z_{\alpha, a \ (mod \ n) + b \ (mod \ n)}^+ \alpha.$$
is a right-sided infinite permitted word. From Proposition 11, it follows that there exists a non-trivial Sturmian substitution $f$ such that $f(y) = y$. Noting that

$$x \simeq z_{\alpha, a \pmod{n} + b \pmod{n} \alpha}$$

$$\simeq z_{\alpha, a \pmod{n} + b \pmod{n} \alpha} z^+_{\alpha, a \pmod{n} + b \pmod{n} \alpha}$$

we have

$$f(x) \simeq f(z^-_{\alpha, a \pmod{n} + b \pmod{n} \alpha}) z^+_{\alpha, a \pmod{n} + b \pmod{n} \alpha}.$$ 

Hence the word $y$ admits $z^-_{\alpha, a \pmod{n} + b \pmod{n} \alpha}$ and $f(z^-_{\alpha, a \pmod{n} + b \pmod{n} \alpha})$ as Sturmian continuations. If the relation

$$f(z^-_{\alpha, a \pmod{n} + b \pmod{n} \alpha}) = z^-_{\alpha, a \pmod{n} + b \pmod{n} \alpha}$$

is not valid then Lemma 9 implies that there exists $(k_1, k_2) \in \mathbb{Z}^2$ with $k_2 \geq 1$ such that

$$a \pmod{n} + b \pmod{n} \alpha = k_1 + k_2 \alpha.$$ 

In this event, since $\alpha$ is irrational we observe that $k_2 = 0$, which leads to a contradiction. We therefore obtain $f(x) \simeq x$.

If $n+1 \leq a \pmod{n} + b \pmod{n}$ we state that $(a \pmod{n}, (b \pmod{n}) - n)$ belongs to $C'(n)$. Since $x \simeq z_{\alpha, a \pmod{n} + b \pmod{n} \alpha}$ we easily verify that there exists a non-trivial substitution $g$ such that $g(x) \simeq x$.

There are no other possibilities and the truth of the claim is now clear for the word $z_{\alpha, \rho}$. The proof concerning $z'_{\alpha, \rho}$ is similar in all respects. 

\[\square\]

REFERENCES


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