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## Substitution invariant Sturmian bisequences

par BRUNO PARVAIX

RÉSUMÉ. Les suites sturmiennes indexées sur  $\mathbb{Z}$ , de pente  $\alpha$  et d'intercept  $\rho$ , sont laissées fixes par une substitution non triviale si et seulement si  $\alpha$  est un nombre de Sturm et  $\rho$  appartient à  $\mathbb{Q}(\alpha)$ . On remarque aussi que les suites de Beatty permettent de définir des partitions de l'ensemble des entiers relatifs.

ABSTRACT. We prove that a Sturmian bisequence, with slope  $\alpha$  and intercept  $\rho$ , is fixed by some non-trivial substitution if and only if  $\alpha$  is a Sturm number and  $\rho$  belongs to  $\mathbb{Q}(\alpha)$ . We also detail a complementary system of integers connected with Beatty bisequences.

### 1. INTRODUCTION

*Beatty sequences*  $(\lfloor n\alpha + \rho \rfloor)_{n \in \mathbb{N}}$  and  $(\lceil n\alpha + \rho \rceil)_{n \in \mathbb{N}}$  have been studied extensively. Many papers deal with the case  $\rho = 0$ , see [1, 9, 10, 14, 15, 28, 29]. The inhomogeneous case is also discussed from several points of view [6, 7, 16, 20, 21, 22]. By the way, this Note provides a new contribution about *complementary systems* of integers. This problem arose, in various forms, in the works of A. S. Fraenkel [13], R. L. Graham [17] and R. Tijdeman [30, 31].

A natural way to examine Beatty sequences is to consider the class of *Sturmian words* defined by G. A. Hedlund and M. Morse in the context of topological dynamics, see [25, 26]. For further details, both [3] and [8] contain extensive lists of references. Here we are especially interested in substitution invariant Sturmian words. In [27] we elicited properties about some right-sided infinite Sturmian words the *intercept* of which is a particular homography of the *slope*. We therefore obtained a partial generalization of Crisp *et al.*'s main Theorem concerning *cutting sequences* [12]. The aim of this Note is the full characterization of Sturmian bisequences which are fixed by some non-trivial substitution.

## 2. DEFINITIONS AND NOTATIONS

Let  $\mathbb{N} = \{0, 1, 2, \dots\}$  and  $\mathbb{N}^- = \{-1, -2, \dots\}$ . Let  $\mathbb{Z} = \mathbb{N}^- \cup \mathbb{N}$  and  $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ . We consider the sets  $\mathcal{Z}_{\beta, \delta} = \{[k\beta + \delta] \mid k \in \mathbb{Z}\}$  and  $\mathcal{Z}'_{\beta, \delta} = \{[k\beta + \delta] \mid k \in \mathbb{Z}\}$ , with  $\beta$  irrational and  $\delta$  real. As usual  $[x]$  is the integer part and  $\lceil x \rceil$  the ceiling of any real number  $x$ . Let  $r_{\beta, \delta}$  and  $r'_{\beta, \delta}$  be the generating bisequences of  $\mathcal{Z}_{\beta, \delta}$  and  $\mathcal{Z}'_{\beta, \delta}$ : we set

$$r_{\beta, \delta}(n) = \begin{cases} 1 & \text{if } n \in \mathcal{Z}_{\beta, \delta} \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad r'_{\beta, \delta}(n) = \begin{cases} 1 & \text{if } n \in \mathcal{Z}'_{\beta, \delta} \\ 0 & \text{otherwise} \end{cases}$$

for each  $n \in \mathbb{Z}$ . We say that two subsets of  $\mathbb{Z}$  are a complementary system if they form a partition of  $\mathbb{Z}$ .

Let  $\mathcal{A}^*$  be the free monoid generated by the two-letter alphabet  $\mathcal{A} = \{0, 1\}$ . The set of right-sided infinite words is denoted by  $\mathcal{A}^\omega$  and  ${}^\omega\mathcal{A}$  is the set of left-sided infinite words. A bisequence is a doubly infinite word and  ${}^\omega\mathcal{A}^\omega$  is the set of bisequences over  $\mathcal{A}$ . We say that the bisequences  $\dots v_{-2}v_{-1}v_0v_1v_2\dots$  and  $\dots v'_{-2}v'_{-1}v'_0v'_1v'_2\dots$  are equal if there exists an integer  $k \in \mathbb{Z}$  such that  $v_i = v'_{i+k}$  for each  $i \in \mathbb{Z}$ . In this event, we note  $\dots v_{-2}v_{-1}v_0v_1v_2\dots \simeq \dots v'_{-2}v'_{-1}v'_0v'_1v'_2\dots$ .

Let  $\alpha$  be irrational and  $\rho$  be real. Consider the bisequences  $z_{\alpha, \rho}$  and  $z'_{\alpha, \rho}$  defined by

$$z_{\alpha, \rho}(n) = [(n+1)\alpha + \rho] - [n\alpha + \rho] - [\alpha]$$

and

$$z'_{\alpha, \rho}(n) = \lceil (n+1)\alpha + \rho \rceil - \lceil n\alpha + \rho \rceil - [\alpha]$$

for each  $n \in \mathbb{Z}$ . A bisequence  $x$  is said to be Sturmian if  $x \simeq z_{\alpha, \rho}$  or  $x \simeq z'_{\alpha, \rho}$  for a suitable choice of  $\alpha$  and  $\rho$ . It is clear that  $z_{\alpha, \rho}(n) = z_{\alpha+1, \rho}(n)$  and  $z'_{\alpha, \rho}(n) = z'_{\alpha+1, \rho}(n)$  for each  $n \in \mathbb{Z}$ , so without loss of generality, we may take  $0 < \alpha < 1$ . Finally, a right-sided infinite word  $y$  is Sturmian if there exist a Sturmian bisequence  $x$  and a left-sided infinite word  $y'$  such that  $x \simeq y'y$ . Noting that Sturmian words are intimately related to straight lines in the plane, the number  $\alpha$  is the slope and  $\rho$  the intercept.

A substitution  $f$  is a map from  $\mathcal{A}^*$  into itself such that  $f(uu') = f(u)f(u')$  for all finite words  $u$  and  $u'$ . Let  $w = w_0w_1w_2\dots$  be a right-sided infinite word. Let  $Inv$  be the operator defined by  $Inv(w) = \dots w_2w_1w_0$  and  $Inv(Inv(w)) = w$ . As usual, we set  $f(w) = f(w_0)f(w_1)f(w_2)\dots$  and  $f(Inv(w)) = \dots f(w_2)f(w_1)f(w_0)$ . The image of  $\dots v_{-2}v_{-1}v_0v_1v_2\dots$  under  $f$  is  $\dots f(v_{-2})f(v_{-1})f(v_0)f(v_1)f(v_2)\dots$ . A one-sided infinite word  $y$  is fixed by  $f$  if  $f(y) = y$ , and a bisequence  $x$  is fixed by  $f$  if  $f(x) \simeq x$ .

Moreover a substitution  $f$  is *Sturmian* if  $f(w)$  is a right-sided infinite Sturmian word whenever  $w$  is. F. Mignosi and P. Séébold proved that a substitution  $f$  is Sturmian if and only if  $f$  is a composition of the three

substitutions  $E : \begin{array}{l} 0 \mapsto 1 \\ 1 \mapsto 0 \end{array}, \varphi : \begin{array}{l} 0 \mapsto 01 \\ 1 \mapsto 0 \end{array}$  and  $\tilde{\varphi} : \begin{array}{l} 0 \mapsto 10 \\ 1 \mapsto 0 \end{array}$  in any order and number [24]. A substitution  $f$  is *locally Sturmian* if there exists a right-sided infinite Sturmian word  $w$  such that  $f(w)$  is Sturmian. J. Berstel and P. Séébold stated that any locally Sturmian substitution is actually Sturmian [4, 5].

Furthermore a substitution is non-trivial if it differs from the identical transformation over  $\mathcal{A}$ . In [27] we proved that if a right-sided infinite Sturmian word is fixed by some non-trivial substitution then its slope  $\alpha$ , with  $0 < \alpha < 1$ , is a *Sturm number*, that is, there exists an integer  $n \geq 2$  such that:

$$\alpha = [0, 1 + k_n, \overline{k_{n-1}, \dots, k_2, k_1 + k_n}] \text{ with } (k_1, k_n) \in \mathbb{N}^2 \setminus \{(0, 0)\}$$

or

$$\alpha = [0, 1, k_n, \overline{k_{n-1}, \dots, k_2, k_1 + k_n}] \text{ with } (k_1, k_n) \in \mathbb{N}^{*2}$$

where the partial quotients  $k_2, \dots, k_{n-1}$  belong to  $\mathbb{N}^*$ . Remark that these numbers were introduced, in a slightly different way, by Crisp *et al.* [12].

### 3. RESULTS

As usual, for any quadratic irrational  $\alpha$ , let  $\mathbb{Q}(\alpha) = \{p + q\alpha \mid (p, q) \in \mathbb{Q}^2\}$  be the splitting field of  $\alpha$  over  $\mathbb{Q}$ . The main result of this Note is the full characterization of Sturmian bisequences which are invariant under some non-trivial substitution:

**Theorem 1.** *Let  $x$  be a Sturmian bisequence with slope  $0 < \alpha < 1$ . The word  $x$  is fixed by some non-trivial substitution if and only if  $\alpha$  is a Sturm number and  $\rho$  belongs to  $\mathbb{Q}(\alpha)$ .*

In [27], we computed the slope and the intercept of  $f(x)$  for any Sturmian substitution  $f$  and any right-sided infinite Sturmian word  $x$ . Lemma 2 is a translation of these formulas for Sturmian bisequences:

**Lemma 2.** *Let  $\alpha$  be irrational with  $0 < \alpha < 1$  and let  $\rho$  be real. Then*

$$E(z_{\alpha, \rho}) \simeq z'_{1-\alpha, 1-\rho} \text{ and } \varphi(z_{\alpha, \rho}) \simeq z'_{\frac{1-\alpha}{2-\alpha}, \frac{1-\rho}{2-\alpha}} \simeq \tilde{\varphi}(z_{\alpha, \rho}).$$

Moreover

$$E(z'_{\alpha, \rho}) \simeq z_{1-\alpha, 1-\rho} \text{ and } \varphi(z'_{\alpha, \rho}) \simeq z_{\frac{1-\alpha}{2-\alpha}, \frac{1-\rho}{2-\alpha}} \simeq \tilde{\varphi}(z'_{\alpha, \rho}).$$

The proof of these properties requires a careful study of generating bisequences of Beatty bisequences:

**Lemma 3.** *Let  $\beta > 1$  be irrational and  $\delta$  be real. For each  $n \in \mathbb{Z}$ , we have*

$$r_{\beta, \delta}(n) = z'_{\frac{1}{\beta}, \frac{-\delta}{\beta}}(n) \text{ and } r'_{\beta, \delta-1}(n) = z_{\frac{1}{\beta}, \frac{-\delta}{\beta}}(n).$$

As an immediate corollary, we can characterize the occurrences of a letter in any Sturmian bisequence. More precisely we remark that

$$\{n \in \mathbb{Z} \mid z_{\gamma, \nu}(n) = 1\} = \mathcal{Z}'_{\frac{1}{\gamma}, \frac{-\nu}{\gamma}-1} \text{ and } \{n \in \mathbb{Z} \mid z'_{\gamma, \nu}(n) = 1\} = \mathcal{Z}_{\frac{1}{\gamma}, \frac{-\nu}{\gamma}}$$

for each  $\gamma$  irrational with  $0 < \gamma < 1$  and  $\nu$  real. This result is a generalization of earlier work of A. S. Fraenkel, M. Mushkin and U. Tassa dealing with the homogeneous case [15]. From Lemma 3 we also obtain a property about complementary systems of integers:

**Proposition 4.** *Let  $\beta > 1$  be irrational and  $\delta$  be real. Then  $\mathcal{Z}_{\beta, \delta}$  and  $\mathcal{Z}'_{\frac{\beta}{\beta-1}, \frac{-\delta}{\beta-1}-1}$ , as well as  $\mathcal{Z}'_{\beta, \delta}$  and  $\mathcal{Z}_{\frac{\beta}{\beta-1}, \frac{-\delta}{\beta-1}+1}$ , are complementary systems of integers.*

#### 4. PROOFS

First of all, we examine the generating bisequences of Beatty bisequences:

*Proof of Lemma 3.* Let  $n \in \mathbb{Z}$ . If  $z'_{\frac{1}{\beta}, \frac{-\delta}{\beta}}(n) = 1$  we state that

$$\frac{n}{\beta} - \frac{\delta}{\beta} \leq \left\lceil \frac{n}{\beta} - \frac{\delta}{\beta} \right\rceil = \left\lceil \frac{n+1}{\beta} - \frac{\delta}{\beta} \right\rceil - 1 < \frac{n+1}{\beta} - \frac{\delta}{\beta}$$

thus  $n \leq \left\lceil \frac{n}{\beta} - \frac{\delta}{\beta} \right\rceil \beta + \delta < n+1$ . Next comes  $\left\lfloor \left\lceil \frac{n}{\beta} - \frac{\delta}{\beta} \right\rceil \beta + \delta \right\rfloor = n$  and  $r_{\beta, \delta}(n) = 1$ .

Conversely, if  $r_{\beta, \delta}(n) = 1$  there exists an integer  $k \in \mathbb{Z}$  such that  $\lfloor k\beta + \delta \rfloor = n$ . We therefore observe that

$$\left\lceil \frac{n}{\beta} - \frac{\delta}{\beta} \right\rceil - 1 < \frac{n}{\beta} - \frac{\delta}{\beta} \leq k < \frac{n+1}{\beta} - \frac{\delta}{\beta} \leq \left\lceil \frac{n+1}{\beta} - \frac{\delta}{\beta} \right\rceil.$$

It follows that  $\left\lceil \frac{n}{\beta} - \frac{\delta}{\beta} \right\rceil \leq k < \left\lceil \frac{n+1}{\beta} - \frac{\delta}{\beta} \right\rceil$  and  $z'_{\frac{1}{\beta}, \frac{-\delta}{\beta}}(n) = 1$ .

The truth of the first statement is now clear, and we turn to the second part of the proof. Let  $n \in \mathbb{Z}$ . If  $z_{\frac{1}{\beta}, \frac{-\delta}{\beta}}(n) = 1$  then

$$\frac{n}{\beta} - \frac{\delta}{\beta} < \left\lceil \frac{n}{\beta} - \frac{\delta}{\beta} \right\rceil + 1 = \left\lceil \frac{n+1}{\beta} - \frac{\delta}{\beta} \right\rceil \leq \frac{n+1}{\beta} - \frac{\delta}{\beta}$$

hence we have

$$n < \left\lceil \frac{n+1}{\beta} - \frac{\delta}{\beta} \right\rceil \beta + \delta \leq n+1$$

that is  $\left\lfloor \left\lceil \frac{n+1}{\beta} - \frac{\delta}{\beta} \right\rceil \beta + \delta - 1 \right\rfloor = n$ . This implies that  $r'_{\beta, \delta-1}(n) = 1$ .

Conversely, if  $r'_{\beta, \delta-1}(n) = 1$  then there exists  $k \in \mathbb{Z}$  such that  $\lceil k\beta + \delta - 1 \rceil = n$ . Thus we check  $z'_{\frac{1}{\beta}, \frac{-\delta}{\beta}}(n) = 1$  since

$$\left\lfloor \frac{n}{\beta} - \frac{\delta}{\beta} \right\rfloor \leq \frac{n}{\beta} - \frac{\delta}{\beta} < k \leq \left\lfloor \frac{n+1}{\beta} - \frac{\delta}{\beta} \right\rfloor.$$

□

In order to describe the complementary system of integers, connected with a Beatty bisequence, we need to introduce the following Lemma:

**Lemma 5.** *Let  $0 < \alpha < 1$  be irrational and  $\rho$  be real. Then  $E(z_{\alpha, \rho}) \simeq z'_{1-\alpha, 1-\rho}$  and  $E(z'_{\alpha, \rho}) \simeq z_{1-\alpha, 1-\rho}$ .*

*Proof.* We only detail the proof concerning the first result. Let  $n \in \mathbb{Z}$ . Since the relation  $\lfloor a \rfloor = -\lceil -a \rceil$  holds for each real number  $a$ , we verify

$$z'_{1-\alpha, 1-\rho}(n) = 1 - (\lceil -n\alpha - \rho \rceil - \lceil -(n+1)\alpha - \rho \rceil) = 1 - z_{\alpha, \rho}(n) = E(z_{\alpha, \rho}(n)).$$

□

*Proof of Proposition 4.* Let  $n \in \mathbb{Z}$ . From Lemma 3, we remark that

$$n \in \mathcal{Z}_{\beta, \delta} \Leftrightarrow r_{\beta, \delta}(n) = 1 \Leftrightarrow z'_{\frac{1}{\beta}, \frac{-\delta}{\beta}}(n) = 1.$$

Then Lemma 5 implies that

$$n \in \mathcal{Z}_{\beta, \delta} \Leftrightarrow z_{\frac{\beta-1}{\beta}, \frac{\beta+\delta}{\beta}}(n) = 0 \Leftrightarrow r'_{\frac{\beta}{\beta-1}, -\frac{\beta+\delta}{\beta-1}-1}(n) = 0 \Leftrightarrow r'_{\frac{\beta}{\beta-1}, -\frac{\delta}{\beta-1}-1}(n) = 0.$$

In other words, we get

$$n \in \mathcal{Z}_{\beta, \delta} \Leftrightarrow n \notin \mathcal{Z}'_{\frac{\beta}{\beta-1}, -\frac{\delta}{\beta-1}-1}.$$

Furthermore, since  $\beta > 1$  we can affirm that any integer occurs at most one time in  $\mathcal{Z}_{\beta, \delta}$ . Clearly this property also holds for  $\mathcal{Z}'_{\frac{\beta}{\beta-1}, -\frac{\delta}{\beta-1}-1}$ . In short, the sets  $\mathcal{Z}_{\beta, \delta}$  and  $\mathcal{Z}'_{\frac{\beta}{\beta-1}, -\frac{\delta}{\beta-1}-1}$  are a complementary system of integers. The part of proof concerning  $\mathcal{Z}'_{\beta, \delta}$  and  $\mathcal{Z}_{\frac{\beta}{\beta-1}, \frac{-\delta}{\beta-1}+1}$  is similar in all respects. □

From now on we study properties of substitution invariant Sturmian bisequences.

*Proof of Lemma 2.* Assume first that  $0 \leq \rho < 1$ . We split the bisequence  $z_{\alpha, \rho}$  into the words

$$w = z_{\alpha, \rho}(0)z_{\alpha, \rho}(1) \dots z_{\alpha, \rho}(m) \dots \in \mathcal{A}^\omega$$

and

$$w' = \dots z_{\alpha, \rho}(-m) \dots z_{\alpha, \rho}(-2)z_{\alpha, \rho}(-1) \in {}^\omega \mathcal{A}.$$

Let  $\varphi(w) = y_0 y_1 \dots$  with  $y_j \in \mathcal{A}$  for  $j = 0, 1, \dots$ . We observe that  $y_0 = 0 = z'_{\frac{1-\alpha}{2-\alpha}, \frac{1-\rho}{2-\alpha}}(0)$ . Let  $n_{q+1}$  be the  $(q+1)$ -th occurrence of the letter 0 in the word  $\varphi(w)$  for each  $q \geq 1$ . We easily check:

$$n_{q+1} = (q + \sum_{i=0}^{q-1} (1 - z_{\alpha, \rho}(i)) + 1) - 1 = 2q - \lfloor q\alpha + \rho \rfloor = \lceil q(2 - \alpha) - \rho \rceil.$$

For each  $n \geq 1$  we state that:

$$\begin{aligned} y_n = 0 &\Leftrightarrow \exists q \in \mathbb{N}^* \quad n = \lceil q(2 - \alpha) - \rho \rceil \\ &\Leftrightarrow \exists q \in \mathbb{Z} \quad n = \lceil q(2 - \alpha) - \rho \rceil \\ &\Leftrightarrow r'_{2-\alpha, -\rho}(n) = 1 \\ &\Leftrightarrow z_{\frac{1}{2-\alpha}, \frac{\rho-1}{2-\alpha}}(n) = 1. \end{aligned}$$

From Lemma 5, we prove that  $y_n = 0$  if and only if  $z'_{\frac{1-\alpha}{2-\alpha}, \frac{1-\rho}{2-\alpha}}(n) = 0$ . In short we obtain  $\varphi(w) = (z'_{\frac{1-\alpha}{2-\alpha}, \frac{1-\rho}{2-\alpha}}(n))_{n \in \mathbb{N}}$ . To compute  $\varphi(w')$ , we remark that

$$w' = \dots z'_{\alpha, 1-\rho}(m) \dots z'_{\alpha, 1-\rho}(1) z'_{\alpha, 1-\rho}(0).$$

Indeed, for each  $n \in \mathbb{N}^*$  it is clear that

$$\begin{aligned} z_{\alpha, \rho}(-n) &= \lfloor (-n+1)\alpha + \rho \rfloor - \lfloor -n\alpha + \rho \rfloor - \lfloor \alpha \rfloor \\ &= -\lceil (n-1)\alpha - \rho \rceil + \lceil n\alpha - \rho \rceil - \lfloor \alpha \rfloor \end{aligned}$$

hence

$$z_{\alpha, \rho}(-n) = z'_{\alpha, -\rho}(n-1) = z'_{\alpha, 1-\rho}(n-1).$$

If we write  $w' = \dots a_m \dots a_1 a_0$  over  ${}^\omega\mathcal{A}$ , we get

$$\varphi(w') = \dots 01^{1-a_m} \dots 01^{1-a_1} 01^{1-a_0}$$

because  $\varphi(0) = 01$  and  $\varphi(1) = 0$ . We can deduce that

$$\text{Inv}(\varphi(w')) = 1^{1-a_0} 01^{1-a_1} 0 \dots 1^{1-a_m} 0 \dots = \tilde{\varphi}(a_0 a_1 \dots a_m \dots)$$

and  $\varphi(w') = \text{Inv}(\tilde{\varphi}((z'_{\alpha, 1-\rho}(n))_{n \in \mathbb{N}}))$ . Much as above, we verify

$$\varphi((z'_{\alpha, 1-\rho}(n))_{n \in \mathbb{N}}) = (z_{\frac{1-\alpha}{2-\alpha}, \frac{\rho}{2-\alpha}}(n))_{n \in \mathbb{N}}.$$

Moreover we observe that  $\tilde{\varphi}(a) = 1^{1-a} 0$  and  $\varphi(a) = 01^{1-a}$  for each  $a \in \{0, 1\}$ . Next comes  $\varphi(u) = 0\tilde{\varphi}(u)$  for any  $u \in {}^\omega\mathcal{A}$ , and consequently  $\varphi(w') = \text{Inv}((z_{\frac{1-\alpha}{2-\alpha}, \frac{1-\alpha+\rho}{2-\alpha}}(n))_{n \in \mathbb{N}})$ . Bearing in mind that  $z_{\alpha, \rho} \simeq w'w$ , and noting that  $z_{\frac{1-\alpha}{2-\alpha}, \frac{1-\alpha+\rho}{2-\alpha}}(n) = z'_{\frac{1-\alpha}{2-\alpha}, \frac{1-\rho}{2-\alpha}}(-n-1)$  for each  $n \in \mathbb{N}$ , we finally obtain  $\varphi(z_{\alpha, \rho}) \simeq z'_{\frac{1-\alpha}{2-\alpha}, \frac{1-\rho}{2-\alpha}}$ . To conclude, we must prove that the relation

$\varphi(z_{\alpha, \rho+k}) \simeq z'_{\frac{1-\alpha}{2-\alpha}, \frac{1-(\rho+k)}{2-\alpha}}$  holds for each  $k \in \mathbb{Z}$ . Since  $z'_{\beta, \delta+1} \simeq z'_{\beta, \delta} \simeq z'_{\beta, \delta+\beta}$  for arbitrarily  $\beta$  irrational and  $\delta$  real, we directly claim:

$$z'_{\frac{1-\alpha}{2-\alpha}, \frac{1-(\rho+k)}{2-\alpha}} \simeq z'_{\frac{1-\alpha}{2-\alpha}, \frac{1-(\rho+k)}{2-\alpha} + k - k \frac{1-\alpha}{2-\alpha}} \simeq z'_{\frac{1-\alpha}{2-\alpha}, \frac{1-\rho}{2-\alpha}} \simeq \varphi(z_{\alpha, \rho}) \simeq \varphi(z_{\alpha, \rho+k}).$$

The computation of  $\tilde{\varphi}(z_{\alpha, \rho+k})$  becomes trivial because we have  $\tilde{\varphi}(v) \simeq \varphi(v)$  for each  $v \in {}^\omega\mathcal{A}^\omega$ . Finally, the part of proof concerning  $z'_{\alpha, \rho}$  is similar in all respects.  $\square$

For each Sturmian substitution  $f$  it is therefore clear that  $f(x)$  is a Sturmian bisequence whenever  $x$  is. Now we turn to the proof of Theorem 1. Some preliminaries are required. Let  $x$  and  $y$  be two Sturmian bisequences. Let  $f$  be a substitution such that  $f(x) \simeq y$ . There exist a word  $x' \in {}^\omega\mathcal{A}$  and a right-sided infinite Sturmian word  $x''$  such that  $x \simeq x'x''$ . Since we have  $y \simeq f(x')f(x'')$ , the word  $f(x'')$  is a right-sided infinite Sturmian word. Thus  $f$  is locally Sturmian and consequently  $f$  belongs to the monoid  $\{E, \varphi, \tilde{\varphi}\}^*$ .

Let us recall some basic properties about Sturmian bisequences. For any irrational  $\alpha$  we set  $\mathbb{Z} + \mathbb{Z}\alpha = \{a + b\alpha \mid (a, b) \in \mathbb{Z}^2\}$ . Let  $\Delta$  be the set of couples  $(\beta, \delta)$  with  $0 < \beta < 1$  irrational and  $\delta$  real. We also set  $\mathcal{U} = \{(\beta, \delta) \in \Delta \mid \forall k \in \mathbb{Z} \ k\beta + \delta \notin \mathbb{Z}\}$ . Let  $(\alpha, \rho) \in \Delta$  and  $(\alpha', \rho') \in \Delta$ . We have  $z_{\alpha, \rho} \simeq z_{\alpha', \rho'}$  if and only if  $\alpha = \alpha'$  and  $\rho - \rho' \in \mathbb{Z} + \mathbb{Z}\alpha$ , see [26]. A similar result can be stated from the relation  $z'_{\alpha, \rho} \simeq z'_{\alpha', \rho'}$ . Furthermore, if  $z_{\alpha, \rho} \simeq z'_{\alpha', \rho'}$  then  $(\alpha, \rho)$  belongs to  $\mathcal{U}$  and  $z_{\alpha, \rho} \simeq z'_{\alpha, \rho}$ . In short, if two Sturmian bisequences are equal then they have the same slope in  $]0, 1[$ . Bearing these remarks in mind, we therefore obtain:

**Lemma 6.** *Let  $x$  be a Sturmian bisequence with slope  $0 < \alpha < 1$ . If  $x$  is invariant under some non-trivial substitution then  $\alpha$  is a Sturm number.*

*Proof (Sketch).* Assume that there exists a non-trivial substitution  $f$  such that  $f(x) \simeq x$ . Then  $f$  belongs to  $\{E, \varphi, \tilde{\varphi}\}^*$ . Let  $\beta \in ]0, 1[$  be the slope of  $f(x)$  which is obtained by Lemma 2. Clearly this computation can be done regardless of intercepts, and there exists an homography  $h$ , with integer coefficients, such that  $\beta = h(\alpha)$ . Therefore it only remains to solve the equation  $\alpha = h(\alpha)$ . In this context, we have yet observed that  $\alpha$  is a Sturm number: for a full characterization of the homographies connected with Sturmian substitutions, see the proof of Theorem 1 in [27].  $\square$

In order to prove our main result, we add here a new necessary condition of invariance:

**Lemma 7.** *Let  $x$  be a Sturmian bisequence, with slope  $\alpha$  and intercept  $\rho$ . If  $x$  is invariant under some non-trivial substitution then  $\rho$  belongs to  $\mathbb{Q}(\alpha)$ .*



*Proof.* Assume, without loss of generality, that  $0 < \alpha < 1$ . Let  $f$  be a non-trivial substitution such that  $f(x) \simeq x$ . Lemma 6 implies that  $\alpha$  is a Sturm number. Since  $\alpha$  is a quadratic irrational, the image of  $\alpha$  under any homography, with integer coefficients, belongs to  $\mathbb{Q}(\alpha)$ . Using Lemma 2, we compute the image of  $x$  under  $f$ . Let  $\beta$  be the slope and  $\delta$  be the intercept we obtain. It is clear that  $\beta \in \mathbb{Q}(\alpha)$  and  $0 < \beta < 1$ . We also remark that  $\delta \in \mathbb{Q}(\alpha) + \rho\mathbb{Q}(\alpha)$ . Since  $f(x) \simeq x$ , we must check  $\beta = \alpha$  and  $\delta - \rho \in \mathbb{Z} + \mathbb{Z}\alpha$ . Next comes  $\rho \in \mathbb{Q}(\alpha)$ .  $\square$

Combining Lemmas 6 and 7, we establish the “only if part” of Theorem 1. Now we turn to the proof of the “if part”: the idea is to use some properties that we reported in [27]. First of all, a technical result concerning Sturmian continuations is required [26].

**Definition 8** (cf. [26]). Let  $y$  be a right-sided infinite Sturmian word. A Sturmian continuation of  $y$  is a left-sided infinite word  $y'$  such that  $y'y$  is a Sturmian bisequence.

**Lemma 9** (cf. [26]). *Let  $\alpha$  be irrational with  $0 < \alpha < 1$  and  $\rho$  be real. Each right-sided infinite Sturmian word  $y$ , with slope  $\alpha$  and intercept  $\rho$ , admits at least one and at most two Sturmian continuations. In the case where  $y$  admits different Sturmian continuations there exist two integers  $k_1 \in \mathbb{Z}$  and  $k_2 \in \mathbb{N}^*$  such that  $\rho = k_1 + k_2\alpha$ .*

**Definition 10** (cf. [27]). For each  $m \geq 1$ , we set

$$C'(m) = \{(a, b) \in \mathbb{Z}^2 \mid 0 \leq a + b \leq m, 0 \leq a \leq m\} \setminus \{(m, 0)\}.$$

A right-sided infinite Sturmian word  $y$  is said to be permitted if there exist an irrational  $\alpha$  with  $0 < \alpha < 1$ , an integer  $m \geq 1$  and a couple of integers  $(a, b) \in C'(m)$  such that  $y = (z_{\alpha, \frac{a}{m} + \frac{b}{m}\alpha}(n))_{n \in \mathbb{N}}$  or  $y = (z'_{\alpha, \frac{a}{m} + \frac{b}{m}\alpha}(n))_{n \in \mathbb{N}}$ .

**Proposition 11** (cf. [27]). *Let  $\alpha$  be a Sturm number. Each permitted word  $y$ , with slope  $\alpha$ , is invariant under some non-trivial substitution.*

*Proof of Theorem 1.* Let  $\alpha$  be a Sturm number and  $\rho \in \mathbb{Q}(\alpha)$ . Let  $x$  be a Sturmian bisequence such that  $x \simeq z_{\alpha, \rho}$ . Clearly there exists  $(a, b, n) \in \mathbb{Z}^3$  with  $n \geq 1$  such that  $\rho = \frac{a+b\alpha}{n}$ . Moreover, since  $z_{\alpha, \delta+1} \simeq z_{\alpha, \delta} \simeq z_{\alpha, \delta+\alpha}$  for each real  $\delta$ , we actually have  $x \simeq z_{\alpha, \frac{a \pmod n + (b \pmod n)\alpha}{n}}$ . As usual, the residue  $i \pmod n$  is the integer  $j$ , with  $0 \leq j < n$ , such that there exists an integer  $k \in \mathbb{Z}$  satisfying  $j = i + kn$ . For each real  $\delta$  we set

$$z_{\alpha, \delta}^+ = z_{\alpha, \delta}(0)z_{\alpha, \delta}(1) \dots \text{ and } \dots z_{\alpha, \delta}(-2)z_{\alpha, \delta}(-1) = z_{\alpha, \delta}^-.$$

We first assume that  $a \pmod n + b \pmod n \leq n$ . Then

$$y = z_{\alpha, \frac{a \pmod n + (b \pmod n)\alpha}{n}}^+$$

is a right-sided infinite permitted word. From Proposition 11, it follows that there exists a non-trivial Sturmian substitution  $f$  such that  $f(y) = y$ . Noting that

$$\begin{aligned} x &\simeq z_{\alpha, \frac{a \pmod n + (b \pmod n)\alpha}{n}} \\ &\simeq z_{\alpha, \frac{a \pmod n + (b \pmod n)\alpha}{n}}^- z_{\alpha, \frac{a \pmod n + (b \pmod n)\alpha}{n}}^+ \end{aligned}$$

we have

$$f(x) \simeq f(z_{\alpha, \frac{a \pmod n + (b \pmod n)\alpha}{n}}^-) z_{\alpha, \frac{a \pmod n + (b \pmod n)\alpha}{n}}^+.$$

Hence the word  $y$  admits  $z_{\alpha, \frac{a \pmod n + (b \pmod n)\alpha}{n}}^-$  and  $f(z_{\alpha, \frac{a \pmod n + (b \pmod n)\alpha}{n}}^-)$  as Sturmian continuations. If the relation

$$f(z_{\alpha, \frac{a \pmod n + (b \pmod n)\alpha}{n}}^-) = z_{\alpha, \frac{a \pmod n + (b \pmod n)\alpha}{n}}^-$$

is not valid then Lemma 9 implies that there exists  $(k_1, k_2) \in \mathbb{Z}^2$  with  $k_2 \geq 1$  such that

$$\frac{a \pmod n + (b \pmod n)\alpha}{n} = k_1 + k_2\alpha.$$

In this event, since  $\alpha$  is irrational we observe that  $k_2 = 0$ , which leads to a contradiction. We therefore obtain  $f(x) \simeq x$ .

If  $n+1 \leq a \pmod n + b \pmod n$  we state that  $(a \pmod n, (b \pmod n) - n)$  belongs to  $C'(n)$ . Since  $x \simeq z_{\alpha, \frac{a \pmod n + ((b \pmod n) - n)\alpha}{n}}$  we easily verify that there exists a non-trivial substitution  $g$  such that  $g(x) \simeq x$ .

There are no other possibilities and the truth of the claim is now clear for the word  $z_{\alpha, \rho}$ . The proof concerning  $z'_{\alpha, \rho}$  is similar in all respects.  $\square$

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