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## Universal Codes and Unimodular Lattices

par ROBIN CHAPMAN ET PATRICK SOLÉ

RÉSUMÉ. Les codes résidus quadratiques binaires de longueur  $p + 1$  produisent par construction  $B$  et bourrage des réseaux de type II comme le réseau de Leech. Récemment, il a été prouvé que les codes résidus quadratiques quaternaires produisent les mêmes réseaux par construction  $A$  modulo 4. Nous montrons de manière directe l'équivalence des deux constructions pour  $p \leq 31$ . En dimension 32 nous obtenons un réseau extrémal de type II qui n'est pas isomètre au réseau de Barnes-Wall  $BW_{32}$ . On considère également l'équivalence entre construction  $B$  modulo 4 plus bourrage et construction  $A$  modulo 8. En dimension 48 elles conduisent toutes deux à une nouvelle description du réseau extrémal de type II appelé  $P_{48q}$ .

ABSTRACT. Binary quadratic residue codes of length  $p + 1$  produce via construction  $B$  and density doubling type II lattices like the Leech. Recently, quaternary quadratic residue codes have been shown to produce the same lattices by construction  $A$  modulo 4. We prove in a direct way the equivalence of these two constructions for  $p \leq 31$ . In dimension 32, we obtain an extremal lattice of type II not isometric to the Barnes-Wall lattice  $BW_{32}$ . The equivalence between construction  $B$  modulo 4 plus density doubling and construction  $A$  modulo 8 is also considered. In dimension 48 they both led to a new description of the extremal type II lattice  $P_{48q}$ .

### 1. Introduction

In [2], Bonnecaze, Solé and Calderbank introduce for primes  $p \equiv \pm 1 \pmod{8}$ , codes  $\widehat{Q}$  and  $\widehat{N}$ , the *universal extended quadratic residue codes*, of length  $p + 1$  over the 2-adic integers  $Z_{2^\infty}$ . For positive integers  $s$  they consider their reductions  $\widehat{Q}_s$  and  $\widehat{N}_s$  modulo  $2^s$ ;  $\widehat{Q}_2$  and  $\widehat{N}_2$  are just the standard binary extended quadratic residue codes, while  $\widehat{Q}_4$  and  $\widehat{N}_4$  are the *quaternary quadratic residue codes*. Given a code  $C$  of length  $n$  over  $Z_4$

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define  $\Lambda(C)$  as the set of vectors in  $Z^n$  which reduce modulo 4 to elements of  $C$ . If  $p \equiv -1 \pmod{8}$  the lattice  $\frac{1}{2}\Lambda(\widehat{Q}_4)$  is even and unimodular ([2] Corollary 4.1); if  $p = 7$  it is the  $E_8$  lattice, while if  $p = 23$  it is the Leech lattice.

Here we show, by means of an explicit isomorphism, that if  $p \equiv -1 \pmod{8}$  and  $p \leq 31$  then  $\frac{1}{2}\Lambda(\widehat{Q}_4)$  is isometric to a lattice  $L(\widehat{Q}_2)$  constructed from the binary quadratic residue code in a manner (construction  $B$  plus density doubling) generalizing the original construction of the Leech lattice. If  $p = 23$  this yields a short proof of what is perhaps the simplest construction of the Leech lattice [2]. If  $p = 31$  this, combined with results of Koch and Venkov, shows that  $BSBM_{32}$  introduced in [1] is not isometric to the Barnes-Wall lattice  $BW_{32}$ . In section §4 we consider a quaternary analogue of this situation, replacing construction  $B$  by construction  $B$  modulo 4, and construction  $A$  mod 4 by construction  $A$  mod 8. We show, inter alia, that  $P_{48q}$  can be obtained in the latter way from a quadratic residue code of length 48 over  $Z_8$ .

## 2. The main result

Throughout this section we assume that  $p$  is a prime satisfying  $p \equiv -1 \pmod{8}$ . We also fix an integer  $r$  such that  $r \equiv 1 \pmod{4}$ , and  $r^2 + p \equiv 0 \pmod{32}$ . In addition if  $p \leq 31$  we will assume that  $r^2 + p = 32$ . (If  $p = 7, 23$  or  $31$ , then  $r = 5, -3$  or  $1$  respectively.)

We first outline a construction of lattices from binary codes of length  $p + 1$ . Consider a self-orthogonal linear subcode  $C$  of  $Z_2^{p+1}$ , containing the all-ones word. Define  $L(C)$  to be the sublattice of  $Z^{p+1}$  generated by the following types of vectors:

- (1) all vectors of shape  $(8 \ 0^p)$ ,
- (2) all vectors of shape  $(4^2 \ 0^{p-1})$ ,
- (3) all vectors of shape  $(2^a \ 0^{p+1-a})$  whose support coincides with the support of an element of  $C$ ,
- (4) any vector of shape  $(r \ 1^p)$ .

This can be recast as the union of two cosets

$$2B(C) \cup ((r \ 1^p) + 2B(C)),$$

of the lattice  $2B(C)$  obtained, up to scaling, by construction  $B$  applied to  $C$  namely

$$B(C) := C + 2P_{p+1} + 4Z^{p+1},$$

with  $P_{p+1}$  denoting the parity-check code of length  $p + 1$ . It is clear that  $L(C)$  is a lattice, of index  $4^{p+1}/|C|$  in  $Z^{p+1}$ . If the code  $C$  is doubly even, then the norm of each vector in  $L(C)$  is divisible by 16. It follows that if  $C$  is

self-dual and doubly even then the lattice  $\frac{1}{\sqrt{8}}L(C)$  is even and unimodular. If  $C$  is  $\widehat{Q}_2$  or  $\widehat{N}_2$  then it has these properties. We give four examples of this construction.

$p$	lattice	reference
7	Gosset	[7]
23	Leech	[6, p.131]
31	$BW_{32}$	[7, 9]
31	$BSBM_{32}$	[1]

By the *norm* of an element in Euclidean space we mean the square of its length, and the *minimal norm* of a lattice is the least norm of a non-zero element of the lattice. It is easy to see that the minimum norm of  $\frac{1}{\sqrt{8}}L(\widehat{Q}_2)$  is  $\min(4, 2\lceil \frac{p+1}{16} \rceil, \frac{1}{2}mw(\widehat{Q}_2))$  where  $mw(C)$  is the minimum (Hamming) weight of the code  $C$ .

**THEOREM 1.** *The lattices  $\frac{1}{2}\Lambda(\widehat{Q}_4)$  and  $\frac{1}{\sqrt{8}}L(\widehat{Q}_2)$  are isometric for  $p \leq 31$ .*

*Proof.* Assume  $p \leq 31$ . We recall the definition of  $\widehat{Q}$  from §III of [2]. Let  $\delta$  be the square root of  $-p$  in  $Z_{2^\infty}$  with  $\delta \equiv -1 \pmod{4}$ . Note then that  $\delta \equiv -r \pmod{16}$ . The vectors  $m_\alpha$  ( $\alpha \in F_p \cup \{\infty\}$ ) are defined as the rows of the matrix

$$M = \begin{pmatrix} \delta & 1 & \dots & 1 \\ -1 & & & \\ \vdots & W + \delta I & & \\ -1 & & & \end{pmatrix}$$

where

$$W_{ij} = \binom{j-i}{p}.$$

(The rows and columns of this matrix are labelled in the order  $\infty, 0, 1, \dots, p-1$ .) The matrix  $W$  is called a Jacobsthal matrix, and is instrumental in building Hadamard matrices of Paley type [10, Chap. II]. We collect here the properties that we need

- (J1)  $JW = WJ = 0$
- (J2)  $WW^T = pI - J$
- (J3)  $A := \sum_{i=\square} W_{-i,1} = -1$
- (J4)  $B := \sum_{i=\square} W_{i,1} = 0$

where  $J$  stands for the all-one matrix. See [10, Chap. II, Lemma 7] for proofs of (J1) and (J2). To prove (J3), (J4) observe firstly that by (J1) we have, knowing that  $-1$  is not a quadratic residue, that  $A + B = -1$ .

Secondly, writing  $\chi$  for the Jacobi symbol we have

$$B = \frac{1}{2} \sum_{x \in F_p, x \neq 0} \chi(1 - x^2)$$

and by the character property of  $\chi$

$$B = \frac{1}{2} \sum_{x \in F_p, x \neq 0} \chi(1 - x)\chi(1 + x) = 0,$$

the last equality coming from (J2).

The coordinate positions in the code are labelled  $\infty, 0, 1, \dots, p - 1$ , regarded as elements of the projective line over  $F_p$ . The universal extended quadratic residue code is now defined as

$$\widehat{Q} = \left( \sum_{\alpha \in F_p \cup \{\infty\}} Q_{2^\infty} m_\alpha \right) \cap Z_{2^\infty}^{p+1},$$

where  $Q_{2^\infty}$  is the field of 2-adic numbers. A similar definition holds for  $\widehat{N}$  with  $W_{i,j}$  replaced by  $W_{i,-j}$ .

We can now describe  $\Lambda(\widehat{Q}_4)$  as the set of vectors in  $Z^{p+1}$  congruent modulo 4 to elements of  $\widehat{Q}$ . Let  $n_\alpha \in Z^{p+1}$  be the rows of the matrix

$$N = \begin{pmatrix} -r & 1 & \cdots & 1 \\ -1 & & & \\ \vdots & W - rI & & \\ -1 & & & \end{pmatrix}$$

so that for  $\alpha \in F_p \cup \{\infty\}$  we have  $n_\alpha \equiv m_\alpha \pmod{16}$ . Since by (J2) we have  $NN^t = 32I$  the matrix  $\frac{1}{\sqrt{32}}N$  is orthogonal. We claim that this matrix maps  $\frac{1}{\sqrt{8}}L(\widehat{N}_2)$  to  $\frac{1}{2}\Lambda(\widehat{Q}_4)$ . This is equivalent to saying that  $N$  maps  $\frac{1}{8}L(\widehat{N}_2)$  to  $\Lambda(\widehat{Q}_4)$ . Note that these codes and lattices are preserved by the automorphism  $\sigma$  coming from the permutation  $(0 \ 1 \ 2 \ \cdots \ p - 1)$  on  $F_p \cup \{\infty\}$ , and this automorphism maps  $m_\alpha$  to  $m_{\alpha+1}$  and  $n_\alpha$  to  $n_{\alpha+1}$ . This automorphism is the shift in the cyclic construction of the QR codes. We proceed to show that the images by  $N$  of the four types of vectors in construction  $L$  above lie in  $\Lambda(\widehat{Q}_4)$ .

Since the  $n_\alpha \in \Lambda(\widehat{Q}_4)$ , the matrix  $N$  takes the coordinate vectors, which lie in  $\frac{1}{8}L(\widehat{N}_2)$ , into  $\Lambda(\widehat{Q}_4)$ . For convenience let  $(a, b; c; d)$  denote the vector with  $\infty$ -coordinate  $a$ , 0-coordinate  $b$ , and generic  $\alpha$ -coordinate  $c$ , and

generic  $\beta$ -coordinate  $d$  where  $\alpha$  and  $\beta$  are any quadratic residue, and quadratic non-residue respectively. Now

$$\frac{1}{2}(n_\infty + n_0) = \left( \frac{r-1}{2}, \frac{-r-1}{2}; 0; -1 \right)$$

which lies in  $Z^{p+1}$  and is congruent to  $\frac{1}{2}(m_0 - m_\infty)$  modulo 8. Hence  $\frac{1}{2}(m_\infty + m_0) \in \Lambda(\widehat{Q}_4)$ . Applying  $\sigma$  it follows that  $\frac{1}{2}(m_\infty + m_\alpha) \in \Lambda(\widehat{Q}_4)$  for all  $\alpha \in F_p$ , and so  $\frac{1}{2}(m_\alpha + m_\beta) \in \Lambda(\widehat{Q}_4)$  for all  $\alpha, \beta \in F_p \cup \{\infty\}$ . Hence  $\frac{1}{8}vN \in \Lambda(\widehat{Q}_4)$  for all  $v$  of the shape  $(4^2 0^{p-1})$ . We next compute

$$\frac{1}{4} \left( n_\infty + \sum_{j \in Q'} n_j \right) = \left( -\frac{2r+p-1}{8}, -\frac{p+1}{8}; 0; -\frac{r-1}{4} \right)$$

where  $Q'$  is the set of quadratic non-residues modulo  $p$ . The last coordinates estimates come from (J3), (J4). Again this has integer coordinates, and is congruent modulo 4 to  $\frac{1}{4}(m_\infty + \sum_{j=\square} m_j)$ , so this vector lies in  $\Lambda(\widehat{Q}_4)$ . It follows that

$$\frac{1}{4} \left( n_\infty + \sum_{j \in Q'} n_{j+k} \right) \in \Lambda(\widehat{Q}_4)$$

for each  $k \in F_p$ . But  $\widehat{N}_2$  is generated by the vectors whose supports are the sets  $\{\infty\} \cup (k + Q')$ . ([2, p.370, III. A.]). It follows that if  $v$  has the shape  $(2^a 0^{p+1-a})$  and whose support is the same as that of an element of  $\widehat{N}_2$ , then  $\frac{1}{8}vN \in \Lambda(\widehat{Q}_4)$ . Finally

$$\frac{1}{8} \left( rn_\infty + \sum_{j \in F_p} n_j \right) = (-4, 0; 0; 0) \in \Lambda(\widehat{Q}_4)$$

and so  $\frac{1}{8}(r, 1; 1; 1)N \in \Lambda(\widehat{Q}_4)$ . Hence  $\frac{1}{8}L(\widehat{N}_2)N \subseteq \Lambda(\widehat{Q}_4)$ , and comparing determinants we see that  $\frac{1}{8}L(\widehat{N}_2)N = \Lambda(\widehat{Q}_4)$ . Since  $L(\widehat{Q}_2)$  and  $L(\widehat{N}_2)$  are isometric the Theorem follows.  $\square$

### 3. Application to the cases of $p = 23, 31$ .

If  $(a_1, \dots, a_n)$  is an element of a code over  $Z_4$ , then its *Euclidean weight* is  $w(a_1) + \dots + w(a_n)$  where

$$w(a) = \begin{cases} 0 & \text{if } a = 0, \\ 1 & \text{if } a = \pm 1, \\ 4 & \text{if } a = 2. \end{cases}$$

The *minimum Euclidean weight*  $mew(C)$  of a code  $C$  over  $Z_4$  is the least Euclidean weight of its non-zero elements. If  $C$  is a linear code then the

minimum norm of  $\Lambda(C)$  is  $\min(16, \text{mew}(C))$ . For  $p = 23$  and  $p = 31$ , the minimum norm of  $L(\widehat{N}_2)$  is 32, and so the minimum norm of  $\Lambda(\widehat{Q}_4)$  is 16. Hence  $\text{mew}(\widehat{Q}_4) \geq 16$ . In [2] this is proved in a more elaborate way for  $p = 23$ .

In [8] Koch and Venkov show that for the five non-isomorphic doubly even self-dual binary codes  $C_1, \dots, C_5$  of length 32, the lattices  $L(C_1), \dots, L(C_5)$  are all non-isometric. We can take  $C_1 = \widehat{Q}_2$ , and  $C_2$  to be the Reed-Muller code  $RM(2, 5)$ . Since  $L(RM(2, 5))$  is isometric to the Barnes-Wall lattice  $BW_{32}$  [9], it follows that  $\frac{1}{2}\Lambda(\widehat{Q}_4)$  for  $p = 31$  is not isometric to  $BW_{32}$ , confirming a conjecture of [1]. It is known that there are only two unimodular lattices in dimension 32 with minimal norm 4 and an automorphism of order 31 [12]. From the results of [1] and of the current paper we can infer that both can be constructed by construction  $A \bmod 4$  applied to an extended quaternary cyclic code: the quaternary Reed-Muller code  $QRM(2, 5)$  in the case of  $BW_{32}$  and the extended quadratic residue code  $\widehat{Q}_4$  in the case of  $BSBM_{32} := \frac{1}{2}\Lambda(\widehat{Q}_4)$ . Both lattices also appear in [11, 4].

#### 4. Quaternary Analogue

We assume in this § that  $p \geq 47$  is a prime  $\equiv -1 \pmod{8}$ , and that the integer  $r \equiv 1 \pmod{4}$  satisfies

$$r^2 + p = 96 = 16 \cdot 6,$$

if  $p = 47, 71$  and

$$r^2 + p = 128 = 16 \cdot 8.$$

if  $p = 79, 103, 127$ . The corresponding values of  $r$  are  $r = -7, 5$  in first case and  $r = -7, 5, 1$  in the second. For a quaternary code  $C$  of length  $p + 1$  we define

$$B_4(C) := C + 4P_{p+1} + 8Z^{p+1},$$

and

$$L_4(C) := 2B_4(C) \cup ((r \cdot 1^p) + 2B_4(C)).$$

For an octonary code  $C_8$  of length  $p + 1$ , we define

$$\Lambda_4(C_8) = C_8 + 8Z^{p+1}.$$

We have the following analogue of Theorem 1:

**THEOREM 2.** *The lattices  $\frac{1}{4}L_4(\widehat{Q}_4)$  and  $\frac{1}{\sqrt{8}}\Lambda_4(\widehat{Q}_8)$  are isometric for  $p = 47, 71, 79, 103, 127$ .*

The proof is analogous to the proof of Theorem 1 and is omitted.

**COROLLARY 1.** *For  $p = 47$  the lattice  $\frac{1}{\sqrt{8}}\Lambda(\widehat{Q}_8)$  has norm 6, and the code  $\widehat{Q}_8$  has euclidean minimum weight 48.*

*Proof.* Follows from the preceding theorem by noticing that  $\widehat{Q}_4$  has euclidean minimum weight 24 [1, 11, 5].  $\square$

The lattice  $L_4(\widehat{Q}_4)$  was considered in [3] and is isometric to  $P_{48q}$ . Adopting the definition of  $P_{48q}$  in §7.7 of [6], the orthogonal matrix

$$\frac{1}{\sqrt{96}} \begin{pmatrix} -7 & 1 & \cdots & 1 \\ -1 & & & \\ \vdots & W - 7I & & \\ -1 & & & \end{pmatrix}$$

takes  $P_{48q}$  to  $L_4(\widehat{N}_4)$  (which is isometric to  $L_4(\widehat{Q}_4)$ ) by a similar argument to Theorem 1. Similarly it is tantamount to conjecture that the conjectural extremal type II lattice of dimension 80 of example 3 of [13] is taken by

$$\frac{1}{\sqrt{128}} \begin{pmatrix} -7 & 1 & \cdots & 1 \\ -1 & & & \\ \vdots & W - 7I & & \\ -1 & & & \end{pmatrix}$$

into  $L_4(\widehat{N}_4)$ .

### 5. Conclusion

It would be interesting to lift the remaining three Conway-Pless codes over  $Z_4$  and obtain by construction  $A_4$  the three remaining zero-defect lattices of the Koch-Venkov classification. Similarly the construction of  $P_{48q}$  by construction  $B_3$  applied to ternary QR codes and density doubling [6, p.149] suggests a construction modulo 6. Eventually, quaternary double circulant codes which produce an even extremal unimodular lattice in dimension 40 [5] should be amenable to a similar analysis.

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## REFERENCES

- [1] A. Bonnetcaze, P. Solé, C. Bachoc, B. Mourrain, 'Type II Quaternary Codes' *IEEE Trans. Inform. Theory*, submitted (1995).
- [2] A. Bonnetcaze, P. Solé & A. R. Calderbank, 'Quaternary quadratic residue codes and unimodular lattices', *IEEE Trans. Inform. Theory*, vol. 41, pp. 366–377, March 1995.
- [3] A.R. Calderbank, private communication (1995).
- [4] A.R. Calderbank, G. MacGuire, P.V. Kumar, T. Helleseth, Cyclic Codes over  $Z_4$ , Locator polynomials, and Newton identities, preprint (1995).
- [5] A. R. Calderbank, N. J. A. Sloane, 'Double Circulant Codes over  $Z_4$  and Unimodular Lattices', *J. of Algebraic Combinatorics*, submitted.
- [6] J. H. Conway & N. J. A. Sloane, *Sphere Packings, Lattices and Groups*, Springer-Verlag, 1988.
- [7] G. D. Forney, 'Coset Codes II: Binary Lattices and related codes', *IEEE Trans. Information Th.* IT-34 (1988) 1152-1187.
- [8] H. Koch & B.B. Venkov, Ueber Ganzhalige Unimodulare Euklidische Gitter, *Crelle* 398 (1989) 144-168.
- [9] P. Loyer, P. Solé, 'Les Réseaux BW32 et U32 sont équivalents', *J. de Th. des Nombres de Bordeaux* 6 (1994) 359-362.
- [10] F. J. MacWilliams, N.J.A. Sloane, *The theory of error correcting codes* North-Holland (1977).
- [11] V. Pless, Z. Qian, 'Cyclic Codes and Quadratic Residue Codes over  $Z_4$  ', *IEEE Trans. Information Theory* submitted.
- [12] H-G. Quebbemann, 'Zur Klassifikation unimodularer Gitter mit Isometrie von Primzahlordnung', *Crelle* 326 (1981) 158-170.
- [13] R. Schulze-Pillot, 'Quadratic Residue Codes and Cyclotomic lattices', *Arch. Math.*, Vol. 60 (1993) 40-65.

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