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## On approximation by Lüroth Series

par KARMA DAJANI ET COR KRAAIKAMP

RÉSUMÉ. Pour  $x \in ]0, 1]$ , on note  $p_n/q_n$  la suite des convergents de la série de Lüroth associée, et on définit par  $\theta_n = q_n x - p_n$ ,  $n \geq 1$  ses coefficients d'approximation. Dans [BBDK], on détermine la fonction de répartition limite de la suite  $(\theta_n)$ , en utilisant l'extension naturelle du système ergodique sous-jacent au développement en série de Lüroth. Nous montrons ici que cela peut être fait sans cette considération. Plus précisément, nous démontrons que pour tout  $n$ , la répartition de  $\theta_n$  coïncide avec la répartition limite. On étudiera aussi la répartition pour presque tout  $x$  de la suite  $(\theta_n, \theta_{n+1})_{n \geq 1}$ , ainsi que celles issues de suites telles que  $(\theta_n + \theta_{n+1})_{n \geq 1}$ . On obtiendra que pour presque tout  $x$ , la suite  $(\theta_n, \theta_{n+1})$  possède une fonction de répartition continue et singulière. On observera de plus que  $\theta_n$  et  $\theta_{n+1}$  sont positivement corrélés.

ABSTRACT. Let  $x \in (0, 1]$  and  $p_n/q_n$ ,  $n \geq 1$  be its sequence of Lüroth Series convergents. Define the approximation coefficients  $\theta_n = \theta_n(x)$  by  $\theta_n = q_n x - p_n$ ,  $n \geq 1$ . In [BBDK] the limiting distribution of the sequence  $(\theta_n)_{n \geq 1}$  was obtained for a.e.  $x$  using the natural extension of the ergodic system underlying the Lüroth Series expansion. Here we show that this can be done without the natural extension. In fact we will prove that for each  $n$ ,  $\theta_n$  is already distributed according to the limiting distribution. Using the natural extension we will study the distribution for a.e.  $x$  of the sequence  $(\theta_n, \theta_{n+1})_{n \geq 1}$  and related sequences like  $(\theta_n + \theta_{n+1})_{n \geq 1}$ . It turns out that for a.e.  $x$  the sequence  $(\theta_n, \theta_{n+1})_{n \geq 1}$  is distributed according to a continuous singular distribution function  $G$ . Furthermore we will see that two consecutive  $\theta$ 's are positively correlated.

## 1. Introduction

Let  $x \in (0, 1]$ , then

$$(1) \quad x = \frac{1}{a_1} + \frac{1}{a_1(a_1-1)a_2} + \dots + \frac{1}{a_1(a_1-1)\dots a_{n-1}(a_{n-1}-1)a_n} + \dots,$$

where  $a_n \geq 2$ ,  $n \geq 1$ . J. Lüroth, who introduced the series expansion (1) in 1883, showed (among other things) that every irrational number  $x$  has a unique infinite expansion (1) and that each rational either has a finite or an infinite periodic expansion, see also [L] and [Pe]. The series expansion (1) of  $x$  is called the *Lüroth Series* of  $x$ .

Dynamically the Lüroth series expansion (1) of  $x$  is generated by the operator  $T : [0, 1] \rightarrow [0, 1]$ , defined by

$$(2) \quad Tx := \left\lfloor \frac{1}{x} \right\rfloor \left( \left\lfloor \frac{1}{x} \right\rfloor + 1 \right) x - \left\lfloor \frac{1}{x} \right\rfloor, \quad x \neq 0; \quad T0 := 0,$$

(see also figure 1), where  $\lfloor \xi \rfloor$  denotes the greatest integer not exceeding  $\xi$ . For  $x \in [0, 1]$  we define  $a(x) := \left\lfloor \frac{1}{x} \right\rfloor + 1$ ,  $x \neq 0$ ;  $a(0) := \infty$  and  $a_n(x) = a(T^{n-1}x)$  for  $n \geq 1$ . From (2) it follows that  $Tx = a_1(a_1-1)x - (a_1-1)$ , and therefore

$$x = \frac{1}{a_1} + \frac{1}{a_1(a_1-1)}Tx = \frac{1}{a_1} + \frac{1}{a_1(a_1-1)a_2} + \dots + \frac{T^n x}{a_1(a_1-1)\dots a_n(a_n-1)}.$$

Putting

$$(3) \quad \frac{p_n}{q_n} = \frac{1}{a_1} + \sum_{k=1}^{n-1} \frac{1}{a_1(a_1-1)\dots a_k(a_k-1)a_{k+1}}, \quad n \geq 1,$$

where  $q_1 := a_1$ ;  $q_n = a_1(a_1-1)\dots a_{n-1}(a_{n-1}-1)a_n$ ,  $n \geq 2$ , it follows from (3) that

$$(4) \quad x - \frac{p_n}{q_n} = \frac{T^n x}{q_n(a_n-1)}, \quad n \geq 1.$$

From  $a_n \geq 2$  and  $0 \leq T^n x \leq 1$  it follows that the series from (1) converges to  $x$ . We will write

$$(5) \quad x = \langle a_1, a_2, \dots, a_n, \dots \rangle \quad \text{and} \quad \frac{p_n}{q_n} = \langle a_1, a_2, \dots, a_n \rangle.$$

In [JdV], H. Jager and C. de Vroedt showed that the stochastic variables  $a_1(x), \dots, a_n(x), \dots$  are independent with  $\lambda_1(a_n = k) = \frac{1}{k(k-1)}$  for  $k \geq 2$ , and that  $T$  is measure preserving and ergodic with respect to Lebesgue measure. Here and in the following  $\lambda_n$  will denote Lebesgue measure on  $\mathbf{R}^n$ . From the ergodicity of  $T$  and Birkhoff's Individual Ergodic Theorem a

number of results were obtained, analogous to classical results on continued fractions, e.g.

$$\lim_{n \rightarrow \infty} (a_1 a_2 \cdots a_n)^{1/n} = e^c, \text{ a.e. where } c \approx 1.25,$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log(x - \frac{p_n}{q_n}) = -d, \text{ a.e., where } d \approx 2.03.$$

Here and in the following a.e. will be with respect to Lebesgue measure.

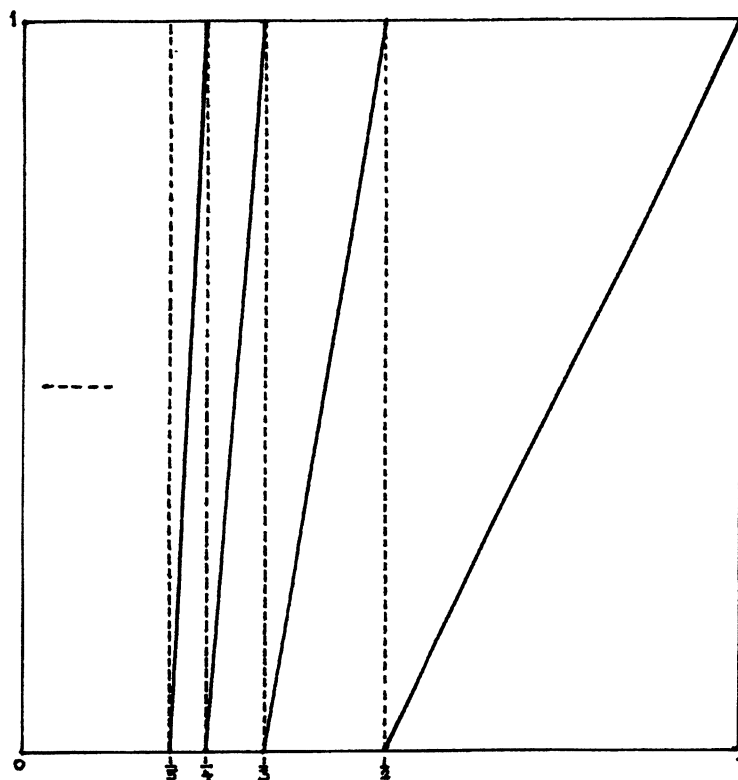


FIGURE 1. The map  $T$

In view of (4) it is natural to define and study the so-called *approximation coefficients*  $\theta_n = \theta_n(x)$ ,  $n \geq 1$ , defined by

$$\theta_n = \theta_n(x) := q_n \left| x - \frac{p_n}{q_n} \right|, \quad n \geq 1.$$

As in the case of the regular continued fraction these  $\theta$ 's give an indication of "the quality of approximation of  $x$  by its  $n$ -th convergent  $p_n/q_n$ ", see also [JK]. (In case of the regular continued fraction one defines  $\Theta_n :=$

$q_n|q_n x - p_n|$ ,  $n \geq 1$ , where  $p_n/q_n$  is the  $n$ -th regular convergent of  $x$ ). Note that the absolute value signs are in fact superfluous here. In view of (4) one has

$$(6) \quad \theta_n = \frac{T^n x}{a_n - 1}, \quad n \geq 1.$$

Putting  $T_n := T^n x$  it follows from (2) and (5) that

$$T_n = \langle a_{n+1}, a_{n+2}, \dots \rangle.$$

We say that  $T_n$  is the future of  $x$  at time  $n$ . Similarly is

$$\begin{aligned} V_n &= \langle a_n, a_{n-1}, \dots, a_1 \rangle \\ &= \frac{1}{a_n} + \frac{1}{a_n(a_n - 1)a_{n-1}} + \dots + \frac{1}{a_n(a_n - 1) \dots a_2(a_2 - 1)a_1} \end{aligned}$$

the past of  $x$  at time  $n$ . Putting  $V_0 := 0$ , from (6) one sees that  $\theta_n$  is expressed in terms of both the past (viz.  $a_n$ ) and the future. Therefore, in order to obtain the distribution of the sequence  $(\theta_n)_{n \geq 1}$  for a.e.  $x$  the natural extension of the ergodic system  $((0,1], \mathcal{B}_1, \lambda_1, T)$  (here  $\mathcal{B}_1$  is the collection of Borel sets of  $(0,1]$ ) was constructed in [BBDK].

**THEOREM 1.** ([BBDK]) Let  $\Omega := [0,1] \times [0,1]$  and  $\mathcal{B}_2$  be the collection of Borel sets of  $\Omega$ . Let  $\mathcal{T} : \Omega \rightarrow \Omega$  be defined by

$$\mathcal{T}(x, y) := \left( Tx, \frac{1}{a(x)} + \frac{y}{a(x)(a(x) - 1)} \right), \quad (x, y) \in \Omega,$$

then the system

$$([0,1] \times [0,1], \mathcal{B}_2, \lambda_2, \mathcal{T})$$

is the natural extension of  $([0,1], \mathcal{B}_1, \lambda_1, T)$ .

Moreover,  $([0,1] \times [0,1], \mathcal{B}_2, \lambda_2, \mathcal{T})$  is Bernoulli.

From this theorem we have the following lemma.

**LEMMA 1.** For almost all  $x$  the two-dimensional sequence

$$\mathcal{T}^n(x, 0) = (T_n, V_n), \quad n \geq 1,$$

is uniformly distributed over  $\Omega = [0,1] \times [0,1]$ .

The distribution of the sequence  $(\theta_n)_{n \geq 1}$  now follows from lemma 1.

**THEOREM 2.** For almost all  $x$  and for every  $z \in (0,1]$  the limit

$$\lim_{N \rightarrow \infty} \frac{1}{N} \# \{ 1 \leq j \leq N : \theta_j(x) < z \}$$

exists and equals  $F(z)$ , where

$$(7) \quad F(z) = \sum_{k=2}^{\lfloor \frac{1}{z} \rfloor + 1} \frac{z}{k} + \frac{1}{\lfloor \frac{1}{z} \rfloor + 1}, \quad 0 < z \leq 1.$$

Taking the first moment, theorem 2 yields that for a.e.  $x$

$$(8) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \theta_n = \frac{\zeta(2) - 1}{2} = 0.322467 \dots,$$

where  $\zeta(s)$  is the zeta-function.

*Remarks.* In fact one needs not use the natural extension to study the distribution of the sequence  $(\theta_n)_{n \geq 1}$ . Since

$$T_n = \frac{1}{a_{n+1}} + \frac{T_{n+1}}{a_{n+1}(a_{n+1} - 1)},$$

see also (2), it follows that

$$a_{n+1}T_n = 1 + \frac{T_{n+1}}{a_{n+1} - 1}$$

and therefore (6) yields that

$$(9) \quad \theta_{n+1} = a_{n+1}T_n - 1, \quad n \geq 1,$$

i.e. the distribution of the sequence  $(\theta_n)_{n \geq 1}$  can be obtained from  $([0, 1], \mathcal{B}, \lambda, T)$ .

It was pointed out by one of the referees that Lemma 1 and Theorem 2 above can also be obtained as simple Corollaries of a strong result by J. Galambos, see [G], Theorem 6.2. Furthermore, using (6) and Galambos' Theorem 6.2, one can calculate the exact distribution of each of the  $\theta_n$ 's, not only the limiting distribution.

In fact, from (9) and the fact that  $T_n$  is uniformly distributed on the unit interval for each  $n$ , one has

$$\begin{aligned} P(\theta_{n+1} < z) &= \sum_{a=2}^{\infty} P(T_n < \frac{z+1}{a}, T_n \in [\frac{1}{a}, \frac{1}{a-1})) \\ &= \sum_{k=2}^{\lfloor \frac{1}{z} \rfloor + 1} \frac{z}{k} + \frac{1}{\lfloor \frac{1}{z} \rfloor + 1}, \quad 0 < z \leq 1, \end{aligned}$$

thus we see that  $F$  from (7) is the distribution function for each  $\theta_n$ . As a corollary we also have that  $E(\theta_n) = \frac{1}{2}(\zeta(2) - 1) = 0.322467 \dots$ .

In this paper we will study the distribution for a.e.  $x$  of the sequence  $(\theta_n, \theta_{n+1})_{n \geq 1}$  and related sequences like  $(\theta_n + \theta_{n+1})_{n \geq 1}$ . We will show that two consecutive  $\theta$ 's are positively correlated.

## 2. On the relation between $\theta_n$ and $\theta_{n+1}$

From (6) and (9) it is natural to define the map  $\Psi : \Omega \rightarrow \Omega$ , given by

$$\Psi(x, y) := \left( \frac{x}{a(y) - 1}, a(x)x - 1 \right), \quad (x, y) \in \Omega.$$

Obviously one has

$$(10) \quad \Psi(T_n, V_n) = (\theta_n, \theta_{n+1}), \quad n \geq 1.$$

Putting

$$V_{A,B} := \{(x, y) \in \Omega : a(x) = A, a(y) = B\}, \quad A, B \geq 2,$$

one finds

$$V_{A,B} = \left( \frac{1}{A}, \frac{1}{A-1} \right] \times \left( \frac{1}{B}, \frac{1}{B-1} \right].$$

For  $(x, y) \in V_{A,B}$  one has  $\Psi(x, y) = (\frac{x}{B-1}, Ax - 1)$  (where  $1/A < x \leq 1/(A-1)$ ). Hence putting

$$\begin{cases} \alpha := \frac{x}{B-1} \Leftrightarrow x = (B-1)\alpha \\ \beta := Ax - 1 \end{cases}$$

yields

$$(11) \quad \beta = A(B-1)\alpha - 1, \quad \alpha \in \left( \frac{1}{A(B-1)}, \frac{1}{(A-1)(B-1)} \right].$$

Thus we see that  $\Psi$  maps the rectangle  $V_{A,B}$  onto the line segment  $L_{A,B}$ , which has endpoints  $(\frac{1}{A(B-1)}, 0)$  and  $(\frac{1}{(A-1)(B-1)}, \frac{1}{A-1})$ . Notice that from (10) and (11) one has

$$(12) \quad \theta_{n+1} = a_{n+1}(a_n - 1)\theta_n - 1, \quad n \geq 1,$$

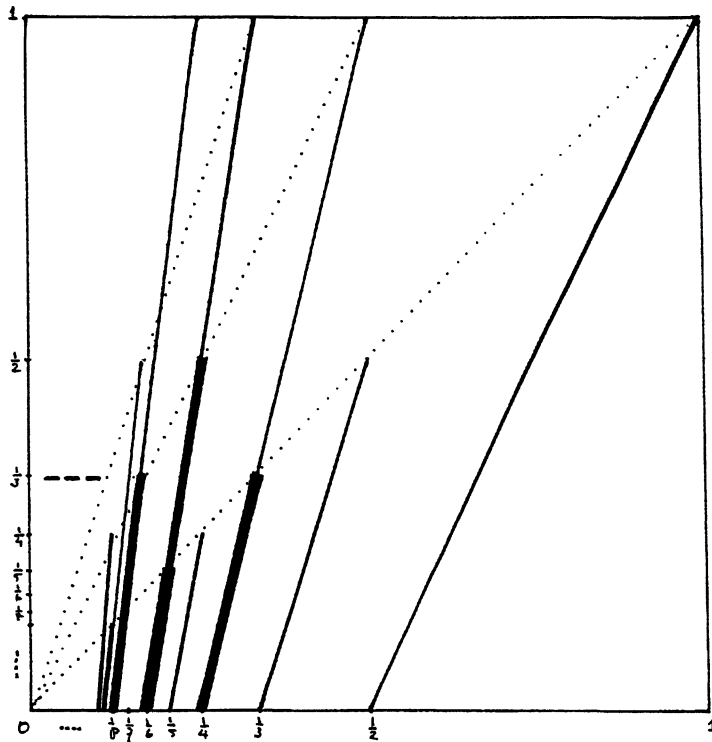
and  $(\theta_n, \theta_{n+1}) \in \Xi$ , where

$$\Xi := \bigcup_{A,B \geq 2} L_{A,B},$$

see also figure 2.

Notice, that from (6) it follows that always

$$0 \leq \theta_n < 1, \quad n \geq 1.$$

FIGURE 2. The set  $\Xi$ 

Note that figure 2 shows that a Vahlen-type theorem as one has for the continued fraction (see [JK]) is not possible for Lüroth Series. That is, there does not exist a constant  $c < 1$ , such that for every  $x$  one has

$$\min(\theta_n(x), \theta_{n+1}(x)) < c$$

(recall that for continued fractions always  $0 \leq \Theta_n(x) < 1$  and  $\min(\Theta_n(x), \Theta_{n+1}(x)) < 1/2$ ). However, it is also clear from figure 2, that

$$(T_n, V_n) \notin V_{2,2} \Leftrightarrow \theta_n < \frac{1}{2}$$

and

$$(T_n, V_n) \in \left(\frac{1}{2}, \frac{3}{4}\right) \times \left(\frac{1}{2}, 1\right] \Rightarrow \theta_{n+1} < \frac{1}{2}.$$

We have the following proposition, which follows directly from lemma 1 (see also theorem 2 with  $z = 1/2$ ).



**PROPOSITION 1.** *For almost all  $x$  one has with probability  $3/4$  that  $\theta_n < \frac{1}{2}$  and with probability  $7/8$  that*

$$\min(\theta_n(x), \theta_{n+1}(x)) < \frac{1}{2}.$$

*Furthermore, given that  $\theta_n < 1/2$  one has with probability  $5/6$  that  $\theta_{n+1} < 1/2$ . The same holds when  $\theta_n$  and  $\theta_{n+1}$  are interchanged.*

*Remarks.* In view of (12) it is obvious that for a.e.  $x$  two consecutive  $\theta$ 's are NOT independent. In fact proposition 1 suggests that two consecutive  $\theta$ 's are positively correlated. That this is the case almost surely is shown in section 3.2. The situation here is similar to that for the regular continued fraction; there Vahlen's theorem suggests that two consecutive  $\Theta$ 's are negatively correlated. This is indeed the case as was shown by Vincent Nolte in an unpublished document, see also [N].

### 3. On the distribution of $(\theta_n, \theta_{n+1})_{n \geq 1}$

In this section we will show in 3.1 that for almost all  $x$  the sequence  $(\theta_n, \theta_{n+1})_{n \geq 1}$  is distributed according to a continuous singular distribution function  $G$ . Before stating the result we first recall the definition of a continuous distribution function, see also [T], p. 20. In 3.2 we will study for a.e.  $x$  the distribution of the sequence  $(\theta_n + \theta_{n+1})_{n \geq 1}$ , which will then be used to show that two consecutive  $\theta$ 's are positively correlated.

#### 3.1. A continuous singular distribution function.

**DEFINITION 1.** *A distribution function  $G$  is said to be continuous singular if it is continuous and if there exists a Borel set  $S$  with Lebesgue measure zero such that  $\mu_G(S) = 1$ . Here  $\mu_G$  denotes the Lebesgue-Stieltjes measure determined by  $G$ .*

We have the following theorem.

**THEOREM 3.** *For almost all  $x$  and for all  $(z_1, z_2) \in [0, 1] \times [0, 1]$  the limit*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \# \{1 \leq j \leq N : \theta_j(x) < z_1, \theta_{j+1} < z_2\}$$

*exists and equals  $G(z_1, z_2)$ , where  $G$  is given by*

$$(13) \quad G(\xi, \eta) := \sum_{A, B \geq 2} \lambda_2(V_{A, B}^*(\xi, \eta)), \quad (\xi, \eta) \in \Omega,$$

*where*

$$(14) \quad V_{A, B}^*(\xi, \eta) := \{(\alpha, \beta) \in V_{A, B} : \alpha < \min((B-1)\xi, \frac{1+\eta}{A})\}.$$

*Finally,  $G$  is a continuous singular distribution function with support  $\Xi$ .*

*Proof.* The first assertion follows from (6), (9) and lemma 1. In order to show that  $G$  is a continuous distribution function we have to show, see also [T], section 2.2 :

- (i)  $G(x_1, x_2) \rightarrow 1$  as  $\min(x_1, x_2) \rightarrow \infty$ .
- (ii) For each  $i \in \{1, 2\}$ ,  $G(x_1, x_2) \rightarrow 0$  as  $x_i \rightarrow -\infty$ .
- (iii)  $G(x_1, x_2)$  is continuous.
- (iv) Let  $a = (a_1, a_2)$ ,  $b = (b_1, b_2)$ , where  $a_i < b_i$ ,  $i \in \{1, 2\}$  and put

$$(a, b] := \{x = (x_1, x_2) \in \mathbf{R}^2 : a_i < x_i \leq b_i, i \in \{1, 2\}\}.$$

Then for each cell  $(a, b] \subset \mathbf{R}^2$  we must have

$$\Delta_a^b G \geq 0,$$

where

$$\Delta_a^b G = G(b_1, b_2) - G(a_1, b_2) - G(b_1, a_2) + G(a_1, a_2).$$

Notice that (i) and (ii) follow from the definition of  $G$ ; clearly  $G$  is monotone in each of its coordinates, and in case  $x_i < 0$  (for  $i \in \{1, 2\}$ ) one has that  $G(x_1, x_2) = 0$ . In case  $\min(x_1, x_2) \geq 1$  it follows that  $G(x_1, x_2) = 1$ . That  $G$  is continuous clearly follows from (13). In order to prove (iv) we introduce for  $A, B \geq 2$  a function  $G_{A,B} : \Omega \rightarrow \mathbf{R}$ , given by

$$G_{A,B}(\xi, \eta) := \lambda_2(V_{A,B}^*(\xi, \eta)), \quad (\xi, \eta) \in \Omega,$$

where  $V_{A,B}^*(\xi, \eta)$  is as in (14). Notice that

$$G(\xi, \eta) := \sum_{A,B \geq 2} G_{A,B}(\xi, \eta), \quad (\xi, \eta) \in \Omega.$$

It is now sufficient to show that for all  $A, B \geq 2$  and each cell  $(a, b] \subset \Omega$  one has

$$\Delta_a^b G_{A,B} \geq 0.$$

Fix  $A, B \geq 2$  and let

$$m(\xi, \eta) = m_{A,B}(\xi, \eta) := \min \left( (B-1)\xi, \frac{1+\eta}{A} \right)$$

and  $\pi_1(\xi, \eta) = \pi_{(A,B),1} := (B-1)\xi$ ,  $\pi_2(\xi, \eta) = \pi_{(A,B),2} := \frac{1+\eta}{A}$ , one has the following, possibly overlapping, cases.

(I)  $m(a_1, b_2) < m(b_1, b_2)$  and

(Ia)  $m(a_1, a_2) < m(b_1, a_2)$ .

Notice that the monotonicity of  $\pi_2$  as a function of its first coordinate yields that

$$\pi_1(a_1, a_2) < \pi_2(a_1, a_2)$$

and therefore  $m(a_1, b_2) = m(a_1, a_2)$ , from which it follows, by definition of  $G_{A,B}$  :

$$\Delta_a^b G_{A,B} = G_{A,B}(b_1, b_2) - G_{A,B}(b_1, a_2) \geq 0.$$

(Ib)  $m(a_1, a_2) = m(b_1, a_2)$ .

In this case one has

$$\Delta_a^b G_{A,B} = G_{A,B}(b_1, b_2) - G_{A,B}(a_1, b_2) > 0.$$

(II)  $m(a_1, b_2) = m(b_1, b_2)$ , which implies that  $\pi_2(a_1, b_2) \leq \pi_1(a_1, b_2)$ , which in turn yields that

$$\pi_2(a_1, a_2) \leq \pi_1(a_1, a_2).$$

But then we only can have that

$$m(a_1, a_2) = m(b_1, a_2),$$

from which it at once follows that

$$\Delta_a^b G_{A,B} = 0.$$

(III)  $m(b_1, a_2) < m(b_1, b_2)$  : see case (I).

(IV)  $m(b_1, a_2) = m(b_1, b_2)$  : see case (II).

In order to show that  $\mu_G(\Xi) = 1$ , or equivalently that  $\mu_G(\Xi^c) = 0$ , it is sufficient to show that for each cell  $(a, b] \subset \Omega$ , for which

$$\text{card}((a, b] \cap \Xi) \leq 2,$$

one has that  $\mu_G((a, b]) = 0$ , which is equivalent with

$$\Delta_a^b G = 0.$$

Notice that we may assume that  $(a, b]$  is contained in  $\mathcal{S}_k$  for some  $k \geq 2$ , where

$$\mathcal{S}_k := \left(\frac{1}{k}, \frac{1}{k-1}\right] \times [0, 1], \quad \text{for } k \geq 2.$$

Obviously there are only finitely many values of  $A$  and  $B$  such that  $L_{A,B} \cap \mathcal{S}_k \neq \emptyset$ . Let  $A$  and  $B$  two such values, then  $(a, b]$  either "lies above"  $L_{A,B}$  or "below"  $L_{A,B}$ . Let  $\mathcal{U} = \mathcal{U}(a, b)$  be the collection of all pairs  $(A, B)$  for which  $(a, b]$  "lies above"  $L_{A,B}$ .

Clearly one has

$$\mu_G((a, b]) = \mu_G((a^*, b]) - \mu_G((a^*, b^*]) ,$$

where  $a^* := (a_1, 0)$  and  $b^* := (b_1, a_2)$ . For  $(A, B) \in \mathcal{U}$  we now define  $L_{A,B}^*$  by

$$L_{A,B}^* := \{(x, y) \in L_{A,B} : a_1 \leq x \leq b_1\} ,$$

then

$$\begin{aligned} \mu_G((a^*, b]) &= G(b_1, b_2) - G(a_1, b_2) = \lambda_2 \left( \bigcup_{(A,B) \in \mathcal{U}} \Psi^{-1} L_{A,B}^* \right) \\ &= G(b_1, a_2) - G(a_1, a_2) = \mu_G((a^*, b^*]) , \end{aligned}$$

from which the theorem follows.  $\square$

**3.2. On the correlation between  $\theta_n$  and  $\theta_{n+1}$ .** In section 2 we saw that it is likely that  $\theta_n$  and  $\theta_{n+1}$  are positively correlated. In order to show this, we first give some definitions.

**DEFINITION 2.** *The correlation-coefficient  $\rho(\theta_n, \theta_{n+1})$  of  $\theta_n$  and  $\theta_{n+1}$  is defined by*

$$\rho(\theta_n, \theta_{n+1}) := \frac{E(\theta_n \theta_{n+1}) - E(\theta_n)E(\theta_{n+1})}{\sqrt{V(\theta_n)}\sqrt{V(\theta_{n+1})}} ,$$

where  $E(\theta_n)$  is the expectation of  $\theta_n$ , as given in (8) and  $V(\theta_n)$  is the variance of  $\theta_n$ , defined by

$$V(\theta_n) := E(\theta_n^2) - (E(\theta_n))^2 .$$

The numerator of  $\rho(\theta_n, \theta_{n+1})$  equals the covariance  $C(\theta_n, \theta_{n+1})$  of  $\theta_n$  and  $\theta_{n+1}$ .

**DEFINITION 3.** *Let  $X$  and  $Y$  be two stochastic variables. Then the covariance  $C(X, Y)$  of  $X$  and  $Y$  is given by*

$$C(X, Y) := E((X - E(X))(Y - E(Y))) .$$

Notice that  $C(X, Y) > 0$  indicates that whenever  $X$  (resp.  $Y$ ) is bigger or smaller than its mean  $E(X)$  (resp.  $E(Y)$ ), the same is likely to hold for  $Y$  (resp.  $X$ ), i.e.  $X$  and  $Y$  are positively correlated. Similarly  $C(X, Y) < 0$  tells us that  $X$  and  $Y$  are negatively correlated. In case  $C(X, Y) = 0$  we say that  $X$  and  $Y$  are uncorrelated. One has that

$$X \text{ and } Y \text{ are independent} \Rightarrow X \text{ and } Y \text{ are uncorrelated},$$

but the converse does not hold in general.

THEOREM 4. *For almost all  $x$  one has that*

$$\rho(\theta_n, \theta_{n+1}) = \frac{(\zeta(2) - 1)((1 - 3\zeta(2) + 4\zeta(3))}{4\zeta(3) - (\zeta(2) - 1)(1 + 3\zeta(2))} = 0.5744202 \dots$$

*and therefore  $\theta_n$  and  $\theta_{n+1}$  are positively correlated a.s.*

Since  $E(\theta_n) = E(\theta_{n+1})$  and  $E(\theta_n^2) = E(\theta_{n+1}^2)$  one has

$$\rho(\theta_n, \theta_{n+1}) = \frac{E(\theta_n \theta_{n+1}) - E(\theta_n)^2}{V(\theta_n)}.$$

Notice, that from theorem 2 one has that  $F$  has density  $f$ , where

$$f(x) = \sum_{\ell=2}^k \frac{1}{\ell}, \quad \text{for } x \in \left(\frac{1}{k}, \frac{1}{k-1}\right], \quad k \geq 2.$$

Taking second moments thus yields

$$\begin{aligned} E(\theta_n^2) &= \sum_{k=2}^{\infty} \int_{\frac{1}{k}}^{\frac{1}{k-1}} z^2 f(z) dz = \sum_{k=2}^{\infty} \frac{1}{3k} \frac{1}{(k-1)^3} = \frac{1 - \zeta(2) + \zeta(3)}{3} \\ &= 0.185708 \dots \end{aligned}$$

But then it follows from (8) that

$$V(\theta_n) = \frac{4\zeta(3) - (\zeta(2) - 1)(1 + 3\zeta(2))}{12} = 0.0817226 \dots$$

In order to find  $E(\theta_n \theta_{n+1})$  we will determine  $E((\theta_n + \theta_{n+1})^2)$ , since

$$E((\theta_n + \theta_{n+1})^2) = 2E(\theta_n^2) + 2E(\theta_n \theta_{n+1}).$$

Hence we find for Lüroth series that

$$E(\theta_n \theta_{n+1}) = \frac{1}{2} E((\theta_n + \theta_{n+1})^2) - E(\theta_n^2).$$

From (6) and (9) one has

$$(15) \quad \theta_n + \theta_{n+1} = (a_{n+1} + \frac{1}{a_n - 1})T_n - 1, \quad n \geq 1,$$

and from the definition of  $L_{A,B}$  it follows that

$$(16) \quad \frac{1}{a_{n+1}(a_n - 1)} < \theta_n + \theta_{n+1} \leq \frac{a_n}{(a_{n+1} - 1)(a_n - 1)}, \quad n \geq 1.$$

From lemma 1, (16) and (17) we have the following theorem.

THEOREM 5. For almost all  $x$  and for every  $z \in (0, 2]$  the limit

$$\lim_{N \rightarrow \infty} \frac{1}{N} \# \{ 1 \leq j \leq N : \theta_j(x) + \theta_{j+1} < z \}$$

exists and equals  $S(z)$ , where  $S$  is a continuous distribution function with density  $s$ , given by

$$s(z) = \sum_{A, B \geq 2} \frac{B-1}{A(B-1)+1} 1_{(\frac{1}{A(B-1)}, \frac{B}{(A-1)(B-1)}]}(z).$$

Here  $1_{(\alpha, \beta]}(z)$  is the indicator function of the interval  $(\alpha, \beta]$ , i.e.

$$1_{(\alpha, \beta]}(z) := \begin{cases} 1 & , z \in (\alpha, \beta], \\ 0 & , z \notin (\alpha, \beta]. \end{cases}$$

Theorem 6 now at once yields that

$$\begin{aligned} E((\theta_n + \theta_{n+1})^2) &= \sum_{B=2}^{\infty} \sum_{A=2}^{\infty} \frac{1}{B} \int_{\frac{1}{A(B-1)}}^{\frac{B}{(A-1)(B-1)}} \frac{1}{A(B-1)+1} z^2 dz \\ &= \sum_{A, B \geq 2} \frac{A^2 B^2 + AB(A-1) + (A-1)^2}{3BA^3(A-1)^3(B-1)^3} = \frac{3 - 3\zeta(2) + 2\zeta(2)\zeta(3)}{3} \\ &= 0.6737 \dots \end{aligned}$$

But then

$$E(\theta_n \theta_{n+1}) - E(\theta_n)E(\theta_{n+1}) = \frac{(\zeta(2) - 1)((1 - 3\zeta(2) + 4\zeta(3))}{12},$$

and therefore the first assertion of theorem 5 is immediate. It follows that  $\theta_n$  and  $\theta_{n+1}$  are indeed positively correlated.  $\square$

Notice that the proof of theorem 6 can easily be adapted to derive the distribution for a.e.  $x$  of the sequence  $(\theta_n - \theta_{n+1})_{n \geq 1}$ . We leave this to the reader, but mention one - surprising - case : one has that the probability  $P(\theta_n < \theta_{n+1})$  that  $\theta_n$  is smaller than  $\theta_{n+1}$  is  $0.391 \dots$ , i.e.  $\theta_{n+1}$  has the tendency to be smaller than its predecessor. To be more precise, we have the following proposition.

PROPOSITION 2. For almost all  $x$  one has that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \# \{ 1 \leq j \leq N : \theta_j(x) < \theta_{j+1} \} = 0.391 \dots$$

*Proof.* From (6) and (9) it follows that  $\theta_n < \theta_{n+1}$  is equivalent with

$$T_n > \frac{a_n - 1}{a_{n+1}(a_n - 1) - 1} .$$

But then lemma 1 yields that  $\lim_{N \rightarrow \infty} \frac{1}{N} \# \{ 1 \leq j \leq N : \theta_j(x) < \theta_{j+1} \}$  exists for a.e.  $x$ , and equals  $\lambda(\mathcal{D})$ , where  $\mathcal{D}$  is given by

$$\mathcal{D} = \bigcup_{A,B \geq 2} \left[ \frac{B}{AB-1}, \frac{B}{A(B-1)-1} \right) \times \left[ 0, \frac{1}{B} \right) ,$$

see also figure 3. The proposition now follows from

$$\lambda(\mathcal{D}) = \frac{\gamma}{2} + \sum_{k \geq 2} \frac{\psi(2 - 1/k)}{k(k+1)} = 0.39234 \dots ,$$

where  $\gamma$  is Euler's constant and  $\psi(x) = \Gamma'(x)/\Gamma(x)$  . $\square$

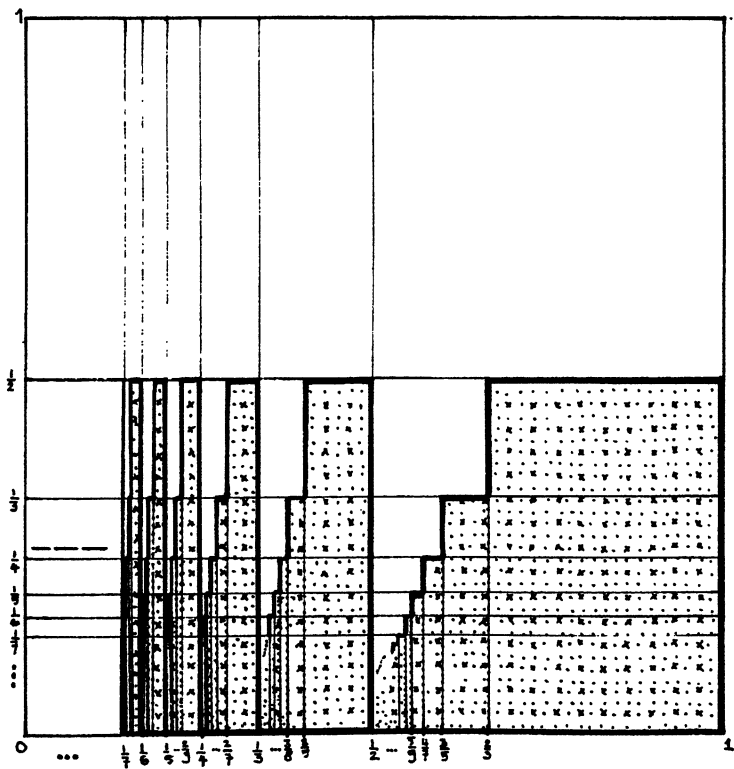


FIGURE 3. The set  $\mathcal{D}$

*Final remarks.* Recently Jose Barrionuevo, Bob Burton and the present authors generalized the whole concept of Lüroth Series, see also [BBDK]. A new class of series expansions, the so-called *Generalized Lüroth Series* (or GLS), was introduced and their ergodic properties were studied. Examples of these GLS are the recent *alternating Lüroth Series*, as introduced by S. Kalpazidou and A. and J. Knopfmacher, but also familiar expansions like  $r$ -adic expansions (for  $r \in \mathbf{Z}$ ,  $r \geq 2$ ). Although  $\beta$ -expansions are not in this class, it turned out that many important ergodic properties of these expansions can be obtained using the appropriate GLS-expansion, see also [DKS].

Here we only want to mention that the approach of this paper can be carried over to GLS-expansions. In order to keep the exposition clear and easy we only dealt with the "classical" Lüroth expansion. Details are left to the reader, see also section 3.1 of [BBDK].

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