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## Irrationality of quick convergent series

par JAROSLAV HANČL

**RÉSUMÉ.** On démontre une généralisation d'un résultat dû à Badea concernant l'irrationalité de certaines séries à convergence rapide.

**ABSTRACT.** We generalize a previous result due to Badea relating to the irrationality of some quick convergent infinite series.

There are many papers concerning the irrationality of infinite series. Erdős [4] proved that if the sequence  $\{a_n\}_{n=1}^{\infty}$  of positive integers converges quickly to infinity, then the series  $\sum_{n=1}^{\infty} 1/a_n$  is an irrational number.

The author [7] defined the irrational sequences and proved criterion for them. Another result is due to Erdős and Strauss [5]. They proved that if  $\{a_k\}_{k=1}^{\infty}$  is a sequence of positive integers with  $\limsup_{n \rightarrow \infty} a_1 \cdots a_n / a_{n+1} < \infty$

and  $\limsup_{n \rightarrow \infty} a_n^2 / a_{n+1} \leq 1$ , then the number  $\sum_{n=1}^{\infty} 1/a_n$  is rational if and only

if  $a_{n+1} = a_n^2 - a_n + 1$  holds for every  $n > n_0$ . Sándor [8] proved that if  $\{a_n\}_{n=1}^{\infty}$  and  $\{b_n\}_{n=1}^{\infty}$  are two sequences of positive integers such that  $\limsup_{n \rightarrow \infty} a_n / (a_1 \cdots a_{n-1} b_n) = \infty$  and  $\liminf_{n \rightarrow \infty} a_n b_{n-1} / (a_{n-1} b_n) > 1$ , then the

number  $\sum_{n=1}^{\infty} b_n / a_n$  is irrational.

Finally Badea [1] proved that if  $\{a_n\}_{n=1}^{\infty}$  and  $\{b_n\}_{n=1}^{\infty}$  are two sequences of positive integers such that  $b_{n+1} > (b_n^2 - b_n) a_{n+1} / a_n + 1$ , then the sum  $\sum_{n=1}^{\infty} a_n / b_n$  is an irrational number. Later he generalized his result ([2]).

In this paper we will generalize Badea's result in another way and prove the following theorem.

THEOREM. Let  $\{a_n\}_{n=1}^{\infty}$  and  $\{b_n\}_{n=1}^{\infty}$  be two sequences of positive integers. If there is a natural number  $m$  such that the following three inequalities hold for every  $n > n_0$

$$(1) \quad b_n > m + 1$$

$$(2) \quad b_n \sum_{k=1}^m (-1)^k \binom{m}{k} \left( \prod_{j=n-m}^{n-k-1} b_j \right) \sum_{j=m-k}^{m-1} a_{n-m+j}/b_{n-m+j} >$$

$$> -a_n \sum_{k=0}^m (-1)^k \binom{m}{k} \prod_{j=n-m}^{n-1-k} b_j + \sum_{i=1}^m \sum_{k=0}^m (-1)^{i+k+1} \operatorname{sgn}(k+1-i) \binom{m}{i} \times$$

$$\times \binom{m}{k} \left( \prod_{s=n-m}^{n-i} b_s / \prod_{s=n-k}^{n-1} b_s \right) \sum_{j=\min(m-i, m-k-1)+1}^{j=\max(m-i, m-k-1)} a_{n-m+j}/b_{n-m+j}$$

and

$$(3) \quad b_n \sum_{k=1}^{m+1} (-1)^k \binom{m+1}{k} \left( \prod_{j=n-m-1}^{n-k-1} b_j \right) \sum_{j=m+1-k}^m a_{n-m+j-1}/b_{n-m+j-1} <$$

$$< -a_n \sum_{k=0}^{m+1} (-1)^k \binom{m+1}{k} \prod_{j=n-m-1}^{n-1-k} b_j + \sum_{i=1}^{m+1} \sum_{k=0}^{m+1} (-1)^{i+k+1} \operatorname{sgn}(k+1-i) \times$$

$$\times \binom{m+1}{i} \binom{m+1}{k} \left( \prod_{s=n-m-1}^{n-i} b_s / \prod_{s=n-k}^{n-1} b_s \right) \times$$

$$\times \sum_{j=\min(m+1-i, m-k)+1}^{\max(m+1-i, m-k)} a_{n-m+j-1}/b_{n-m+j-1}$$

then the number  $A = \sum_{n=1}^{\infty} a_n/b_n$  is irrational.

*Proof:* For the sake of simplicity we will suppose that (1) – (3) hold for every  $n$ . (If not, we define  $a'_n = a_{n+m+n_0}$ ,  $b'_n = b_{n+m+n_0}$  for every  $n =$

1, 2,  $\dots$  and these two sequences  $\{a'_n\}_{n=1}^\infty$  and  $\{b'_n\}_{n=1}^\infty$  satisfy then our above requirements.)

Let us denote

$$(4) \quad B_n = B_{n,0} = \prod_{i=1}^n b_i$$

$$(5) \quad A_n = A_{n,0} = B_n \sum_{i=1}^n a_i/b_i$$

$$B_{n,i} = B_{n,i-1} - B_{n-1,i-1} \quad i = 1, \dots, m+1$$

$$A_{n,i} = A_{n,i-1} - A_{n-1,i-1} \quad i = 1, \dots, m+1$$

One can prove by induction that

$$(6) \quad B_{n,i} = \sum_{j=0}^i \binom{i}{j} B_{n-j} (-1)^j$$

and

$$(7) \quad A_{n,i} = \sum_{j=0}^i \binom{i}{j} A_{n-j} (-1)^j$$

hold for  $i = 0, 1, \dots, m+1$ . (1) and (4) yield

$$(8) \quad \binom{i}{j} B_{n-j} - \binom{i}{j+1} B_{n-j-1} = \binom{i}{j} B_{n-j-1} (b_{n-j} - \frac{i-j}{j+1}) > 0$$

and

$$(9) \quad \begin{aligned} \binom{i}{j} A_{n-j} - \binom{i}{j+1} A_{n-j-1} &= \binom{i}{j} (A_{n-j} - \frac{i-j}{j+1} A_{n-j-1}) = \\ &= \binom{i}{j} B_{n-j-1} \left( a_{n-j} + (b_{n-j} - \frac{i-j}{j+1} \sum_{k=1}^{n-j-1} a_k/b_k) \right) > 0 \end{aligned}$$

for every natural number  $n$ . Then (4)–(9) imply that  $B_{n,i} > 0$  and  $A_{n,i} > 0$  for every positive integer  $n$  and  $i = 0, 1, \dots, m-1$ .

First we will prove that

$$(10) \quad A_{n,m}/B_{n,m} < A_{n+1,m}/B_{n+1,m} < \dots$$

and secondly

$$(11) \quad A_{n,m+1}/B_{n,m+1} > A_{n+1,m+1}/B_{n+1,m+1}$$

(11) implies that there is a number  $c \geq 0$  such that

$$c = \lim_{n \rightarrow \infty} A_{n,m+1}/B_{n,m+1}.$$

Using the famous theorem of Stolz (see e.g. [6]), we obtain

$$(12) \quad A = \lim_{n \rightarrow \infty} A_n/B_n = \dots = \lim_{n \rightarrow \infty} A_{n,m+1}/B_{n,m+1} = c.$$

On the other hand (10), (11), (12) and Brun's Theorem (see e.g. [3]) imply the irrationality of the number  $A$ .

Now we will prove (10) and (11). Using (4) and (5) we have

$$(13) \quad \begin{aligned} \frac{A_{n,m}}{B_{n,m}} - \frac{A_{n-1,m}}{B_{n-1,m}} &= \frac{\sum_{i=0}^m \binom{m}{i} A_{n-i} (-1)^i}{\sum_{i=0}^m \binom{m}{i} B_{n-i} (-1)^i} - \frac{\sum_{i=0}^m \binom{m}{i} A_{n-1-i} (-1)^i}{\sum_{i=0}^m \binom{m}{i} B_{n-1-i} (-1)^i} = \\ &= \frac{\sum_{i=0}^m (-1)^i \binom{m}{i} B_{n-i} (A_{n-m}/B_{n-m} + \sum_{j=1}^{m-i} a_{n-m+j}/b_{n-m+j})}{\sum_{i=0}^m \binom{m}{i} B_{n-i} (-1)^i} - \\ &= \frac{\sum_{i=0}^m (-1)^i \binom{m}{i} B_{n-i-1} (A_{n-m-1}/B_{n-m-1} + \sum_{j=1}^{m-i} a_{n-m+j-1}/b_{n-m+j-1})}{\sum_{i=0}^m (-1)^i \binom{m}{i} B_{n-i-1}} = \end{aligned}$$

$$\begin{aligned}
&= \frac{\sum_{i=0}^m (-1)^i \binom{m}{i} B_{n-i} \sum_{j=0}^{m-i} a_{n-m+j}/b_{n-m+j}}{\sum_{i=0}^m (-1)^i \binom{m}{i} B_{n-i}} - \\
&- \frac{\sum_{i=0}^m (-1)^i \binom{m}{i} B_{n-i-1} \sum_{j=1}^{m-i} a_{n-m+j-1}/b_{n-m+j-1}}{\sum_{i=0}^m (-1)^i \binom{m}{i} B_{n-i-1}} = \\
&= \sum_{i=0}^m \sum_{k=0}^m (-1)^{i+k} \binom{m}{i} \binom{m}{k} B_{n-i} B_{n-k-1} \left( \sum_{j=0}^{m-i} a_{n-m+j}/b_{n-m+j} - \right. \\
&\quad \left. - \sum_{s=1}^{m-k} a_{n-m-1+s}/b_{n-m-1+s} \right) / (B_{n,m} B_{n-1,m}) = \\
&= \left( B_n \sum_{k=0}^m (-1)^k \binom{m}{k} B_{n-1-k} \sum_{j=m-k}^m a_{n-m+j}/b_{n-m+j} - \right. \\
&\quad \left. - \sum_{i=1}^m \sum_{k=0}^m (-1)^{i+k+1} \binom{m}{i} \binom{m}{k} B_{n-i} B_{n-k-1} \left( \sum_{j=0}^{m-i} a_{n-m+j}/b_{n-m+j} - \right. \right. \\
&\quad \left. \left. - \sum_{s=1}^{m-k} a_{n-m+s-1}/b_{n-m+s-1} \right) \right) / (B_{n,m} B_{n-1,m}) = \\
&= \left( B_n \sum_{k=1}^m (-1)^k \binom{m}{k} B_{n-1-k} \sum_{j=m-k}^{m-1} a_{n-m+j}/b_{n-m+j} + \right. \\
&\quad \left. + B_{n-1} a_n \sum_{k=0}^m (-1)^k \binom{m}{k} B_{n-k-1} - \sum_{i=1}^m \sum_{k=0}^m (-1)^{i+k+1} \binom{m}{i} \binom{m}{k} B_{n-i} B_{n-k-1} \right. \\
&\quad \left. \times \sum_{j=\min(m-i, m-k-1)+1}^{\max(m-i, m-k-1)} \operatorname{sgn}(k+1-i) a_{n-m+j}/b_{n-m+j} \right) / (B_{n,m} B_{n-1,m}).
\end{aligned}$$

(13) and (2) yield (10). Similarly (13) (if we substitute  $m + 1$  instead of  $m$ ) and (3) yield (11).

*Remark:* If we put  $m = 0$  in the main theorem, then we receive  $b_n > 1$ ,  $0 > -a_n$  and  $-b_n a_{n-1}/b_{n-1} < -a_n(b_{n-1} - 1) - a_{n-1}/b_{n-1}$ . Thus  $b_n > (b_{n-1}^2 - b_{n-1})a_n/a_{n-1} + 1$  and this is the famous theorem due to Badea (see e.g. [1]).

Consequence 1: Let  $\{a_n\}_{n=1}^{\infty}$  and  $\{b_n\}_{n=1}^{\infty}$  be two sequences of positive integers. If

$$(14) \quad b_n > 2$$

$$(15) \quad b_n < (b_{n-1}^2 - b_{n-1})a_n/a_{n-1} + 1$$

$$(16) \quad \begin{aligned} b_n(-b_{n-1}a_{n-2} + 2b_{n-2}^2a_{n-1} - b_{n-2}a_{n-1}) > \\ > a_nb_{n-1}b_{n-2}(b_{n-1}b_{n-2} - 2b_{n-2} + 1) + 3a_{n-1}b_{n-2}^2 - 2b_{n-1}a_{n-2} \\ - 2b_{n-2}a_{n-1} + a_{n-2} \end{aligned}$$

hold for every  $n > n_0$ , then the number  $A = \sum_{n=1}^{\infty} a_n/b_n$  is irrational.

*Proof:* Let us put  $m = 1$  in the main theorem. Then (14) is (1), (15) is (2) and (16) is (3).

Consequence 2: Let  $\{b_n\}_{n=1}^{\infty}$  be a sequence of positive integers such that  $b_1 > 2$  and

$$(17) \quad kb_{n-1}^2 - (3k - 1)b_{n-1} < b_n < kb_{n-1}^2 - kb_{n-1}$$

hold for every  $n > n_0$  where  $k$  is a positive integer. Then the number  $A = \sum_{n=1}^{\infty} k^n/b_n$  is irrational.

*Proof:* Let us put  $a_n = k^n$  in consequence 1. Then (17) immediately implies (15) and

$$b_n > kb_{n-1}^2 - (3k - 1)b_{n-1} = kb_{n-1}(b_{n-1} - 3) + b_{n-1}.$$

This and  $b_1 > 2$  imply that the sequence  $\{b_n\}_{n=1}^{\infty}$  is increasing. Thus (15) is fulfilled too. Condition (16) can be rewritten in the following way

$$(18) \quad b_n(-b_{n-1} + 2kb_{n-2}^2 - kb_{n-2}) > \\ > k^2b_{n-1}b_{n-2}(b_{n-1}b_{n-2} - 2b_{n-2} + 1) + 3kb_{n-2}^2 - 2b_{n-1} - 2kb_{n-2} + 1.$$

Let us define the sequence  $\{s_n\}_{n=1}^\infty$  of nonnegative integers such that

$$(19) \quad s_n = kb_{n-1}^2 - kb_{n-1} - b_n.$$

(17) implies that

$$(20) \quad 0 < s_n < (2k-1)b_{n-1}.$$

Substituting (19) for (18) we obtain the equivalent inequality (21) with (18) :

$$(21) \quad (kb_{n-1}^2 - kb_{n-1} - s_n)(kb_{n-2}^2 + s_{n-1}) > \\ > k^2b_{n-1}b_{n-2}(b_{n-1}b_{n-2} - 2b_{n-2} + 1) + 3kb_{n-2}^2 - 2b_{n-1} - 2kb_{n-2} + 1.$$

Carrying out the equivalent calculations step by step, we receive

$$kb_{n-1}^2s_{n-1} - s_nkb_{n-2}^2 - s_ns_{n-1} - kb_{n-1}s_{n-1} + k^2b_{n-1}b_{n-2}^2 - k^2b_{n-1}b_{n-2} \\ - 3kb_{n-2}^2 + 2b_{n-1} + 2kb_{n-2} - 1 > 0.$$

Using (20) and the fact that  $\{b_n\}_{n=1}^\infty (b_1 > 2)$  is an increasing sequence, it is enough to prove that

$$(22) \quad kb_{n-1}^2 - (k-1)kb_{n-1}b_{n-2}^2 - Kb_{n-1}b_{n-2} > 0,$$

where  $K$  is a suitable constant. (22) is equivalent with

$$kb_{n-1}(b_{n-1} - kb_{n-2}^2) + kb_{n-1}b_{n-2}^2 - Kb_{n-1}b_{n-2} > 0.$$

(17) implies that

$$(23) \quad -(3k-1)b_{n-2} < b_{n-1} - kb_{n-2}^2.$$

Because of (23), it is enough to prove that

$$(24) \quad kb_{n-1}b_{n-2}^2 - K_1b_{n-1}b_{n-2} > 0,$$

where  $K_1$  is a suitable constant too. But (24) is true for every  $n > n_0$ . Thus (18) is right and the number  $A$  is irrational.

Examples: The numbers  $\sum_{n=1}^\infty 2^n/b_n$  and  $\sum_{n=1}^\infty 3^n/a_n$  where  $a_1 > 2$ ,  $b_1 > 2$ ,  $b_n = 2b_{n-1}^2 - 2b_{n-1} - 1$  and  $a_n = 3a_{n-1}^2 - 3a_{n-1} - 4$  are irrational.



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