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Some problems on mean values of the Riemann zeta-function

par ALEKSANDAR IVIĆ*

RÉSUMÉ. On s'intéresse à des problèmes et des résultats relatifs aux valeurs moyennes de la fonction $\zeta(s)$. On étudie en particulier des valeurs moyennes de $|\zeta(\frac{1}{2} + it)|$, ainsi que le moment d'ordre 4 de $|\zeta(\sigma + it)|$ pour $1/2 < \sigma < 1$.

ABSTRACT. Several problems and results on mean values of $\zeta(s)$ are discussed. These include mean values of $|\zeta(\frac{1}{2} + it)|$ and the fourth moment of $|\zeta(\sigma + it)|$ for $1/2 < \sigma < 1$.

1. Introduction

One of the fundamental problems in the theory of the Riemann zeta-function $\zeta(s)$ is the evaluation of power moments, namely integrals of $|\zeta(\sigma + it)|^k$, where $k > 0$ and σ are fixed real numbers. This topic is extensively discussed in [4], [5] and [16], where additional references to other works may be found. Of particular interest are the values of σ in the so-called "critical" strip $0 < \sigma < 1$, while the case $\sigma = 1$ is treated in [2] and [6]. In view of the functional equation $\zeta(s) = \chi(s)\zeta(1-s)$, where

$$\chi(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) = \left(\frac{2\pi}{t}\right)^{\sigma+it-1/2} e^{i(t+\pi/4)} \left(1 + O\left(\frac{1}{t}\right)\right)$$

for $s = \sigma + it$, $t \geq t_0 > 0$, it transpires that the relevant range for σ in the evaluation of power moments of $\zeta(s)$ is $1/2 \leq \sigma < 1$.

The aim of this paper is to discuss several problems and results involving power moments of $|\zeta(\sigma + it)|$. Some of the problems that I have in mind are quite deep, and even partial solutions would be significant. In section 2 problems concerning mean values on the "critical line" $\sigma = 1/2$ are discussed. Section 3 is devoted to problems connected with the evaluation of

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$\int_0^T |\zeta(\sigma + it)|^4 dt$ for $1/2 < \sigma < 1$ fixed. This topic is a natural one, since problems involving the fourth moment on $\sigma = 1/2$ are extensively treated in several works of Y. Motohashi and the author (see Ch. 5 of [5] and [8], where additional references may be found). Motohashi found a way to apply the powerful methods of spectral theory to this problem, thereby opening the path to a thorough analysis of this topic. Thus it seems appropriate to complete the knowledge on the fourth moment of $\zeta(s)$ by considering the range $1/2 < \sigma < 1$ as well.

The notation used in the text is standard, whenever this is possible. ε denotes positive constants which may be arbitrarily small, but are not necessarily the same ones at each occurrence. $f(x) \ll g(x)$ and $f(x) = O(g(x))$ both mean that $|f(x)| \leq cg(x)$ for $x \geq x_0$, some $c > 0$ and $g(x) > 0$. $f(x) = o(g(x))$ as $x \rightarrow \infty$ means that $\lim_{x \rightarrow \infty} f(x)/g(x) = 0$, while the Perelli symbol $f(x) = \infty(g(x))$ means that $\lim_{x \rightarrow \infty} f(x)/g(x) = +\infty$. $f(x) = \Omega(g(x))$ means that $\limsup_{x \rightarrow \infty} |f(x)|/g(x) > 0$, $f(x) = \Omega_+(g(x))$ (resp. $f(x) = \Omega_-(g(x))$) that $\limsup_{x \rightarrow \infty} f(x)/g(x) > 0$ (resp. $\limsup_{x \rightarrow \infty} f(x)/g(x) < 0$), provided that $g(x) > 0$ for $x \geq x_0$. Finally $f(x) = \Omega_{\pm}(g(x))$ means that both $f(x) = \Omega_+(g(x))$ and $f(x) = \Omega_-(g(x))$ hold.

2. Problems on the critical line $\sigma = 1/2$

In this section we shall investigate some mean value problems on the critical line $\sigma = 1/2$. The first problem is as follows. Let $0 \leq H \leq T$, $0 \leq \alpha < \beta$ and $T \rightarrow \infty$. For which values of α, β and $H = o(T)$ does one have

$$(1) \quad \int_T^{T+H} |\zeta(\tfrac{1}{2} + it)|^{\alpha} dt \leq \int_T^{T+H} |\zeta(\tfrac{1}{2} + it)|^{\beta} dt ?$$

Furthermore, can one find specific values of α and β (they may be constants or even functions of T) and $H = H(T)$ such that (1) fails to hold?

It turns out that this is not an easy problem, and what I can prove is certainly not the complete solution. It is contained in

THEOREM 1. *Let $\beta_0 > 0$ be any fixed constant. If $0 \leq \alpha < \beta$, then (1) holds for $\beta \geq \beta_0$ and $\log \log T \ll H \leq T$. If $0 \leq \alpha < \beta < \beta_0$ and*

$H = \infty(\log T)$, then

$$(2) \quad (1 + o(1)) \int_T^{T+H} |\zeta(\tfrac{1}{2} + it)|^\alpha dt \leq \int_T^{T+H} |\zeta(\tfrac{1}{2} + it)|^\beta dt.$$

Proof. What Theorem 1 roughly says is that (1) holds for β not too small, while for small β only the weaker asymptotic inequality (2) can be established.

Assume first that $0 \leq \alpha < \beta$ and $\beta \geq \beta_0 > 0$. By Hölder's inequality for integrals we have

$$\int_T^{T+H} |\zeta(\tfrac{1}{2} + it)|^\alpha dt \leq \left(\int_T^{T+H} |\zeta(\tfrac{1}{2} + it)|^\beta dt \right)^{\alpha/\beta} H^{(\beta-\alpha)/\beta} \leq \int_T^{T+H} |\zeta(\tfrac{1}{2} + it)|^\beta dt$$

provided that

$$(3) \quad \int_T^{T+H} |\zeta(\tfrac{1}{2} + it)|^\beta dt \geq H.$$

It is enough to prove (3) for $\beta = \beta_0$, since for $\beta > \beta_0$ the inequality again follows by Hölder's inequality. Now (3) follows from the results on mean values of K. Ramachandra (see his monograph [14] for an extensive account). For example, by Corollary 1 to Th. 1 of Ramachandra [13] we have

$$(4) \quad \int_T^{T+H} |\zeta(\tfrac{1}{2} + it)|^{\beta_0} dt \gg H(\log H)^{\beta_0^2/4} \quad (\log \log T \ll H \leq T),$$

if we assume that β_0 is rational, which we may since β_0 is arbitrary (but fixed). Since

$$\lim_{T \rightarrow \infty} (\log H)^{\beta_0^2/4} = +\infty,$$

we obtain (3) with $\beta = \beta_0$ from (4).

We suppose now that $0 \leq \alpha \leq \beta \leq \beta_0$ and $H = \infty(\log T)$. From Ivić-Perelli [7] (or (6.38) of [5]) one has

$$0 \leq \int_{\frac{1}{2}}^1 (N(\sigma, T+H) - N(\sigma, T)) d\sigma = \int_T^{T+H} \log |\zeta(\tfrac{1}{2} + it)| dt + O(\log T),$$

which implies with a suitable $C > 0$ that

$$(5) \quad \int_T^{T+H} \log |\zeta(\tfrac{1}{2} + it)| dt \geq -C \log T.$$

We obtain

$$\begin{aligned} \int_T^{T+H} \log |\zeta(\tfrac{1}{2} + it)| dt &= \frac{1}{\beta} \int_T^{T+H} \log |\zeta(\tfrac{1}{2} + it)|^\beta dt \\ &\leq \frac{H}{\beta} \log \left(\frac{1}{H} \int_T^{T+H} |\zeta(\tfrac{1}{2} + it)|^\beta dt \right) \end{aligned}$$

similarly as in [2]. By using (5) it follows that

$$(6) \quad \frac{1}{H} \int_T^{T+H} |\zeta(\tfrac{1}{2} + it)|^\beta dt \geq e^{-\beta C H^{-1} \log T} \geq e^{-\beta_0 C H^{-1} \log T} \geq 1 - \frac{\beta_0 C \log T}{H}.$$

Thus we have from (6) (with $D = \beta_0 C$) and Hölder's inequality

$$\begin{aligned} \int_T^{T+H} |\zeta(\tfrac{1}{2} + it)|^\beta dt &= \left(\int_T^{T+H} |\zeta(\tfrac{1}{2} + it)|^\beta dt \right)^{\alpha/\beta} \left(\int_T^{T+H} |\zeta(\tfrac{1}{2} + it)|^\beta dt \right)^{1-\alpha/\beta} \\ &\geq \left(\int_T^{T+H} |\zeta(\tfrac{1}{2} + it)|^\beta dt \right)^{\alpha/\beta} H^{1-\alpha/\beta} \left(1 - \frac{D \log T}{H} \right)^{1-\alpha/\beta} \\ &\geq \int_T^{T+H} |\zeta(\tfrac{1}{2} + it)|^\alpha dt \left(1 - \frac{D \log T}{H} \right)^{1-\alpha/\beta} \\ &\geq \left(1 - \frac{D \log T}{H} \right) \int_T^{T+H} |\zeta(\tfrac{1}{2} + it)|^\alpha dt = (1 + o(1)) \int_T^{T+H} |\zeta(\tfrac{1}{2} + it)|^\alpha dt \end{aligned}$$

since $H = \infty(\log T)$. This completes the proof of Theorem 1.

Concerning the values of α, β and H for which (1) fails to hold I wish to make the following remark: For many $0 \leq \alpha < \beta$ there exists arbitrarily large values of T such that

$$(7) \quad \int_T^{T+T^{-1/6}} |\zeta(\tfrac{1}{2} + it)|^\alpha dt > \int_T^{T+T^{-1/6}} |\zeta(\tfrac{1}{2} + it)|^\beta dt.$$

For T we may simply take the points for which $\zeta(\frac{1}{2} + iT) = 0$, and there are $\gg \tau$ such points in $[0, \tau]$. If, as usual, for any real σ we define

$$\mu(\sigma) = \limsup_{t \rightarrow \infty} \frac{\log |\zeta(\sigma + it)|}{\log t},$$

then

$$\zeta'(\tfrac{1}{2} + it) \ll_\varepsilon |t|^{\mu(\frac{1}{2}) + \varepsilon},$$

which follows by applying Cauchy's theorem to a circle of radius $1/\log t$ with center at $\frac{1}{2} + it$. Since $\mu(\frac{1}{2}) < \frac{1}{6}$, it follows that for $T \leq t \leq T+H$, $H = T^{-1/6}$,

$$\zeta(\tfrac{1}{2} + it) \ll H \max_{T \leq t \leq T+H} |\zeta'(\tfrac{1}{2} + it)| \ll HT^{\mu(\frac{1}{2}) + \varepsilon} \leq \frac{1}{2}$$

for sufficiently small $\varepsilon > 0$ and $T \geq T_0(\varepsilon)$. Hence

$$|\zeta(\tfrac{1}{2} + it)|^{\beta - \alpha} \leq 2^{\alpha - \beta} < 1,$$

$$|\zeta(\tfrac{1}{2} + it)|^\alpha > |\zeta(\tfrac{1}{2} + it)|^\beta,$$

and (7) readily follows. Under the Riemann hypothesis one can in (7) replace $H = T^{-1/6}$ by $H = \exp(-\frac{A \log T}{\log \log T})$ with some $A > 0$. However, the largest such H is difficult to determine. Perhaps $H = \exp(-A \sqrt{\log \log T})$ can be taken unconditionally, or even larger H is permissible? This is certainly an open and difficult question.

The construction leading to (7) was basically simple: one finds an interval which is not too small, and where $|\zeta(\frac{1}{2} + it)| \leq \frac{1}{2}$ holds. Points around zeros of $\zeta(\frac{1}{2} + iT)$ are of course likely candidates for such intervals, only we can estimate (unconditionally) $\zeta(\frac{1}{2} + it)$ rather crudely near these zeros. This accounts for the rather poor value $H = T^{-1/6}$ in (7). The following problems then naturally may be posed: What is the measure $\mu(A_T)$ of the set

$$A_T = \{t : T \leq t \leq 2T, |\zeta(\tfrac{1}{2} + it)| \leq \tfrac{1}{2}\} ?$$

Clearly A_T consists of disjoint intervals $[T_1, T_1 + H_1], \dots, [T_R, T_R + H_R]$ where $R = R(T)$ and $H_r > 0$ for $r = 1, \dots, R$. What is the order of magnitude of the function

$$H(T) := \max_{1 \leq j \leq R(T)} H_j ?$$

Many results are known on the problems involving *large* values of $|\zeta(\frac{1}{2} + it)|$, but here is a problem involving *small* values of $|\zeta(\frac{1}{2} + it)|$. The significance of $H(T)$ is that obviously

$$\int_T^{T+H(T)} |\zeta(\tfrac{1}{2} + it)|^\alpha dt > \int_T^{T+H(T)} |\zeta(\tfrac{1}{2} + it)|^\beta dt \quad (0 \leq \alpha < \beta).$$

I recall that, by results of A. Selberg (see D. Joyner [9]) and A. Laurinćikas [10], for a given real y one has ($\mu(\cdot)$ again denotes the measure of a set)

$$\lim_{T \rightarrow \infty} \frac{1}{T} \mu(0 \leq t \leq T : |\zeta(\tfrac{1}{2} + it)| \leq e^{y\sqrt{\frac{1}{2} \log \log T}}) = (2\pi)^{-1/2} \int_{-\infty}^y e^{-u^2/2} du,$$

but determining the true order of magnitude of $\mu(A_T)$ and $H(T)$ is a different (and perhaps even harder) problem. Presumably $R(T) \ll T \log T$, so that in view of

$$\mu(A_T) \leq R(T)H(T) \ll H(T)T \log T$$

we would need a lower bound for $\mu(A_T)$ in order to improve (7).

Let $\delta > 0$ be a given constant and define

$$K(T) = \{\inf k : k = k(T) \text{ and } \int_0^T |\zeta(\tfrac{1}{2} + it)|^{2k} dt > T^{1+\delta} \text{ for } T \geq T_0(\delta)\}.$$

The problem is to bound, as accurately as possible, the function $K(T)$. Certainly we have

$$(9) \quad K(T) \ll_\delta \frac{\log T}{\sqrt{\log \log T}},$$

which easily follows from the limit law (8). The significance of $K(T)$ comes from the fact that from

$$T^{1+\delta} < \int_0^T |\zeta(\tfrac{1}{2} + it)|^{2k} dt \leq T \max_{0 \leq t \leq T} |\zeta(\tfrac{1}{2} + it)|^{3K(T)}$$

one obtains

$$(10) \quad \max_{0 \leq t \leq T} |\zeta(\tfrac{1}{2} + it)| \geq \exp\left(\frac{\delta \log T}{3K(T)}\right).$$

One can substantially improve (9) by using R. Balasubramanian's bound [1]

$$(11) \quad \max_{0 \leq t \leq T} |\zeta(\tfrac{1}{2} + it)| \geq \exp\left(\frac{3}{4} \left(\frac{\log H}{\log \log H}\right)^{1/2}\right),$$

which is valid for $100 \log \log T \leq H \leq T, T \geq T_0$. Actually (11) is proved with $3/4 + \eta$ for some $\eta > 0$ as the constant in the exponential. Hence with $H = T$ it follows that

$$|\zeta(\tfrac{1}{2} + iT')| = \max_{\frac{1}{4}T \leq t \leq \frac{1}{2}T} |\zeta(\tfrac{1}{2} + it)| \geq \exp\left(\left(\frac{3}{4} + \frac{\eta}{2}\right) \left(\frac{\log T}{\log \log T}\right)^{1/2}\right).$$

Thus if $|t - T'| \leq T^{-1/6}$, then

$$\begin{aligned} |\zeta(\tfrac{1}{2} + it)| &= |\zeta(\tfrac{1}{2} + iT')| + O(|T' - t| \max_{|v - T'| \leq T^{-1/6}} |\zeta'(\tfrac{1}{2} + iv)|) \\ &\geq \exp\left(\left(\frac{3}{4} + \frac{\eta}{3}\right) \left(\frac{\log T}{\log \log T}\right)^{1/2}\right). \end{aligned}$$

This gives

$$\begin{aligned} \int_0^T |\zeta(\tfrac{1}{2} + it)|^{2k} dt &\geq \int_{T' - T^{-1/6}}^{T' + T^{-1/6}} |\zeta(\tfrac{1}{2} + it)|^{2k} dt \\ &\geq 2T^{-1/6} \exp\left(\frac{3k\sqrt{\log T}}{2\sqrt{\log \log T}}\right) > T^{1+\delta} \end{aligned}$$

certainly for

$$(12) \quad k = (2 + \delta) \sqrt{\log T \log \log T}.$$

Hence (12) gives trivially

$$(13) \quad K(T) \leq (2 + \delta) \sqrt{\log T \log \log T}.$$

Naturally, any improvement of (13) would be of great interest, since in view of (10) it would mean, improvement of (11) in the most interesting case when $H = T$. Perhaps even

$$K(T) \ll_{\delta} \log \log T$$

holds. In the other direction

$$K(T) = o(\log \log T) \quad (T \rightarrow \infty)$$

would, in view of (10), contradict the Riemann hypothesis which gives (see Ch. XIV of [16]), for some $A > 0$,

$$\zeta\left(\frac{1}{2} + it\right) \ll \exp\left(\frac{A \log t}{\log \log t}\right).$$

Hence it is reasonable to conjecture that

$$K(T) = \Omega(\log \log T)$$

holds unconditionally.

3. The fourth moment for $1/2 < \sigma < 1$

In this section problems involving the fourth moment of $|\zeta(\sigma + it)|$ ($1/2 < \sigma < 1$) will be discussed. To this end we define, for fixed σ satisfying $1/2 < \sigma < 1$,

$$(14) \quad E_1(T, \sigma) = \int_0^T |\zeta(\sigma + it)|^2 dt - \zeta(2\sigma)T - (2\pi)^{2\sigma-1} \frac{\zeta(2-2\sigma)}{2-2\sigma} T^{2-2\sigma}$$

and

$$(15) \quad E_2(T, \sigma) = \int_0^T |\zeta(\sigma + it)|^4 dt - \frac{\zeta^4(2\sigma)}{\zeta(4\sigma)} T - (A_1(\sigma) \log T + A_2(\sigma)) T^{2-2\sigma} - A_3(\sigma) T^{3-4\sigma}$$

as the error terms for the second and fourth moment in the critical strip, respectively. The constants $A_j(\sigma)$ ($j = 1, 2, 3$) are such that both

$$\lim_{\sigma \rightarrow 1/2+0} E_1(T, \sigma) = E_1(T) \equiv E(T), \quad \lim_{\sigma \rightarrow 1/2+0} E_2(T, \sigma) = E_2(T)$$

hold, where

$$E(T) = \int_0^T |\zeta(\tfrac{1}{2} + it)|^2 dt - T(\log \frac{T}{2\pi} + 2\gamma - 1),$$

$$E_2(T) = \int_0^T |\zeta(\tfrac{1}{2} + it)|^4 dt - TP_4(\log T).$$

Here γ is Euler's constant and $P_4(y)$ is a polynomial of degree four in y with suitable coefficients, of which the leading one equal $1/(2\pi^2)$. A detailed account on $E(T)$ and $E_2(T)$ is to be found in [5].

Prof. Y. Motohashi kindly informed me in correspondence (Jan. 7, 1991) that he evaluated, for $0 < \Delta \leq T/\log T$ and $1/2 < \sigma < 1$,

$$(16) \quad I_4(T, \sigma; \Delta) := (\Delta\sqrt{\pi})^{-1} \int_{-\infty}^{\infty} |\zeta(\sigma + iT + it)|^4 e^{-(t/\Delta)^2} dt$$

by means of spectral theory of automorphic forms. The method of proof is similar to the one that he used for evaluating $I_4(T, \frac{1}{2}; \Delta)$ (see e.g. Ch. 5 of [5]). Motohashi notes that the expressions for $A_j(\sigma)$ in (15) turn out to be quite complicated, and in particular $A_1(\sigma) = 0$ cannot be ruled out. He also stated that he can obtain

$$(17) \quad E_2(T, \sigma) \ll T^{2/(1+4\sigma)} \log^C T \quad (\tfrac{1}{2} < \sigma < 1).$$

It will be sketched a little later how by taking $\Delta = T^{2/(1+4\sigma)}$ in the integrated version of (16) one can obtain (17) for $1/2 < \sigma < 3/4$. This is because $\Delta > T^{1/2}$ has to be observed, and $2/(1+4\sigma) \geq 1/2$ for $\sigma \geq 3/4$. But for $\sigma \geq 3/4$ we have $2/(1+4\sigma) \geq 2-2\sigma$, so that the right-hand side of (17) is larger than the second main term in (15) for $\int_0^T |\zeta(\sigma + it)|^4 dt$.

Therefore for $3/4 < \sigma < 1$ (17) is superseded by

THEOREM 2. For fixed σ satisfying $1/2 < \sigma < 1$ we have

$$(18) \quad \int_0^T |\zeta(\sigma + it)|^4 dt = \frac{\zeta^4(2\sigma)}{\zeta(4\sigma)} T + O(T^{2-2\sigma} \log^3 T).$$

Proof. Note first that the only published result heretofore on the integral in (18) is contained in Th. 8.5 of [4]. This is

$$(19) \quad \int_0^T |\zeta(\sigma + it)|^4 dt = \frac{\zeta^4(2\sigma)}{\zeta(4\sigma)} T + O(T^{\frac{1}{2}(3-2\sigma)+\varepsilon}) \quad (1/2 < \sigma < 1),$$

so that (18) sharpens (19).

The proof of (18) follows the method of §4.3 of my book [5], with $k = 2$, and with some modifications that will be now indicated. All the notation will be as in Ch. 4 of [5]. From Theorem 4.2 with

$$\Sigma_1(t) = \sum_{n=1}^{\infty} d(n) \nu\left(\frac{t}{2\pi n}\right) n^{-\sigma-it}, \quad \Sigma_2(t) = \sum_{n=1}^{\infty} d(n) \nu\left(\frac{t}{2\pi n}\right) n^{\sigma-1+it},$$

where $d(n)$ is the number of divisors of n and $\nu(\cdot)$ is the smoothing function, we have

$$\zeta^2(\sigma + it) = \Sigma_1(t) + \chi^2(\sigma + it) \Sigma_2(t) + O(R_2(t)),$$

hence

$$\zeta^2(\sigma - it) = \overline{\Sigma_1}(t) + \chi^2(\sigma - it) \overline{\Sigma_2}(t) + O(R_2(t)).$$

Here $R_2(t)$ is the error term in the smoothed approximate functional equation for $\zeta^2(s)$. By (4.39) and the bound at the bottom of p. 179 of [5] we have, for $T \ll t \ll T$,

$$(20) \quad R_2(t) = t^{\varepsilon-1-2\sigma} \log t + t^{2\varepsilon-1} \int_{-T^\varepsilon}^{T^\varepsilon} |\zeta(\sigma + it + \delta + iv)|^2 dv,$$

where δ is any constant such that $0 < \delta < 1$. If $f(t)$ is the appropriate smoothing function that majorizes or minorizes the characteristic function

of $[T, 2T]$, then from the expressions for $\zeta^2(\sigma \pm it)$ we obtain

$$\begin{aligned} \int_0^\infty f(t) |\zeta(\sigma + it)|^4 dt &= \int_0^\infty f(t) |\Sigma_1(t)|^2 dt \\ &+ 2 \operatorname{Re} \left\{ \int_0^\infty f(t) \chi^2(\sigma - it) \Sigma_1(t) \overline{\Sigma_2(t)} dt \right\} + \int_0^\infty f(t) |\chi^2(\sigma + it) \Sigma_2(t)|^2 dt \\ &+ O \left(\int_{T/2}^{5T/2} R_2(t) (|\Sigma_1(t)| + T^{1-2\sigma} |\Sigma_2(t)|) dt \right) + O \left(\int_{T/2}^{5T/2} R_2^2(t) dt \right). \end{aligned}$$

By using (20) and taking δ sufficiently small it follows that

$$(21) \quad \int_{T/2}^{5T/2} R_2^2(t) dt \ll T^{\varepsilon-1}.$$

The mean value theorem for Dirichlet polynomials (see Th. 5.2 of [4]) cannot be used directly for the evaluation of mean values of $\Sigma_1(t)$ and $\Sigma_2(t)$, because the sums in question contain the ν -function. However, this is not an essential difficulty, since this function is smooth. Thus we can square out the sums, perform integration by parts on non-diagonal terms, use inequality (5.5) of [4] and the first derivative test (Lemma 2.1 of [4]). In this way we obtain

$$\int_0^\infty f(t) |\Sigma_1(t)|^2 dt \leq \int_{T/2}^{5T/2} |\Sigma_1(t)|^2 dt \ll T$$

and

$$\begin{aligned} \int_0^\infty f(t) |\chi^2(\sigma + it) \Sigma_2(t)|^2 dt &\ll T^{2-4\sigma} \int_{T/2}^{5T/2} |\Sigma_2(t)|^2 dt \\ &\ll T^{2-4\sigma} \left(T \sum_{n \leq T} d^2(n) n^{2\sigma-2} + \sum_{n \leq T} d^2(n) n^{2\sigma-1} \right. \\ &\left. + \sum_{m \neq n \leq T} d(m) d(n) (mn)^{\sigma-1} \frac{1}{m \log^2(\frac{m}{n})} \right) \ll T^{2-2\sigma} \log^3 T. \end{aligned}$$

To estimate the last double sum above one sets $|m - n| = r$ and uses the bound

$$(22) \quad \sum_{n \leq x} d(n)d(n+r) \ll \left(\sum_{d|r} \frac{1}{d} \right) x \log^2 x.$$

This is uniform for $1 \leq r \leq x$ and follows from the work of P. Shiu [16] on multiplicative functions. Hence from the above estimates we obtain

$$\begin{aligned} & \int_0^\infty f(t) |\zeta(\sigma + it)|^4 dt = \\ & \int_0^\infty f(t) |\Sigma_1(t)|^2 dt + 2 \operatorname{Re} \left\{ \int_0^\infty f(t) \chi^2(\sigma - it) \Sigma_1(t) \overline{\Sigma_2(t)} dt \right\} + O(T^{2-2\sigma} \log^3 T). \end{aligned}$$

Note that the argument in [5] that precedes (4.58) gives

$$\begin{aligned} \int_0^\infty f(t) \chi^2(\sigma - it) \Sigma_1(t) \overline{\Sigma_2(t)} dt & \ll T_0^{-1} T^{1-2\sigma} \sum_{m \ll T} d(m) m^{-\sigma} \sum_{n \ll T} d(n) n^{\sigma-1} \\ & \ll T_0^{-1} T^{1-2\sigma} T^{1-\sigma} \log T \cdot T^\sigma \log T = T^{2-2\sigma} T_0^{-1} \log^2 T \end{aligned}$$

for a parameter T_0 satisfying $T^\epsilon \ll T_0 \ll T^{\epsilon-1}$, so that we further have

$$\begin{aligned} (23) \quad & \int_0^\infty f(t) |\zeta(\sigma + it)|^4 dt = \sum_{n=1}^\infty d^2(n) n^{-2\sigma} \operatorname{Re} \left\{ \int_0^\infty f(t) \nu\left(\frac{t}{2\pi n}\right) dt \right\} \\ & + \sum_{m, n=1; m \neq n, 1-\delta \leq m/n \leq 1+\delta}^\infty d(m)d(n) (mn)^{-\sigma} \operatorname{Re} \left\{ \int_0^\infty f(t) \nu\left(\frac{t}{2\pi m}\right) \left(\frac{m}{n}\right)^{it} dt \right\} \\ & + O(T^{2-2\sigma} \log^3 T), \end{aligned}$$

by the argument leading to Theorem 4.3 of [5]. In fact, (23) is a weak analogue of Th. 4.3, since the error term in (23) actually contributes to the second main term in the asymptotic formula for $\int_0^T |\zeta(\sigma + it)|^4 dt$, but for our purposes (23) is sufficient. By the first derivative test we have

$$\sum_{m, n=1; m \neq n, 1-\delta \leq m/n \leq 1+\delta}^\infty d(m)d(n) (mn)^{-\sigma} \operatorname{Re} \left\{ \int_0^\infty f(t) \nu\left(\frac{t}{2\pi m}\right) \left(\frac{m}{n}\right)^{it} dt \right\}$$

$$\begin{aligned}
&\ll \sum_{m \neq n \leq 3T, 1-\delta \leq m/n \leq 1+\delta} d(m)d(n)(mn)^{-\sigma} \left| \log \frac{m}{n} \right|^{-1} \quad (\delta = 1/2) \\
&\ll \sum_{n \leq 3T} d(n)n^{1-2\sigma} \sum_{n/2 \leq m \leq 3n/2, m \neq n} d(m)|m-n|^{-1} \\
&\ll \sum_{1 \leq r \leq 3T} \frac{1}{r} \sum_{n \leq 3T} d(n)d(n+r)n^{1-2\sigma} \ll T^{2-2\sigma} \log^3 T,
\end{aligned}$$

where we used partial summation and (22). Finally with

$$\hat{f}(s) = \int_0^\infty f(x)x^{s-1}dx, \quad P(s) = \int_0^\infty \nu(x)x^{-s}dx$$

we have, similarly as in the proof of (4.62) in [5],

$$\begin{aligned}
&\sum_{n=1}^\infty d^2(n)n^{-2\sigma} \operatorname{Re} \left\{ \int_0^\infty f(t)\nu\left(\frac{t}{2\pi n}\right) dt \right\} \\
&= \operatorname{Re} \left\{ \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \hat{f}(s)(2\pi)^{1-s} \frac{\zeta^4(2\sigma+s-1)}{\zeta(4\sigma+2s-2)} P(s) ds \right\},
\end{aligned}$$

where $c > 0$. For $c > \frac{3}{2} - 2\sigma$ the poles of the integrand are $s = 1$ with residue

$$\hat{f}(1) \frac{\zeta^4(2\sigma)}{\zeta(4\sigma)} = \left(\int_T^{2T} dx + O(T_0) \right) \frac{\zeta^4(2\sigma)}{\zeta(4\sigma)} = (T + O(T_0)) \frac{\zeta^4(2\sigma)}{\zeta(4\sigma)},$$

and at $s = 2 - 2\sigma$ with residue

$$T^{2-2\sigma} (D_1(\sigma) \log^3 T + D_2(\sigma) \log^2 T + D_3(\sigma) \log T + D_4(\sigma)) + O(T_0 T^\epsilon).$$

Hence shifting the line of integration to $\operatorname{Re} s = 2 - 2\sigma - \delta$ for small $\delta > 0$ we obtain, in view of

$$\hat{f}(s) \ll T^{\operatorname{Re} s}, \quad P(s) \ll_A |\operatorname{Im} s|^{-A} \quad (A > 0 \text{ fixed}),$$

that

$$\sum_{n=1}^\infty d^2(n)n^{-2\sigma} \operatorname{Re} \left\{ \int_0^\infty f(t)\nu\left(\frac{t}{2\pi n}\right) dt \right\} = \frac{\zeta^4(2\sigma)}{\zeta(4\sigma)} T + O(T^{2-2\sigma} \log^3 T)$$

with a suitable choice of T_0 . This completes the proof of (18), with the remark that the above proof clearly shows that by further elaboration one could obtain a more exact estimation of $E_2(T, \sigma)$. However, any improvements of (18) that could be obtained in this way would not improve (17) for σ close to $1/2$.

We note that (17) is analogous to

$$(24) \quad E_1(T, \sigma) \ll T^{1/(1+4\sigma)} \log^c T \quad (c > 0, 1/2 < \sigma < 3/4),$$

proved by K. Matsumoto [11]. Bounds for $E_1(T, \sigma)$ when $3/4 \leq \sigma < 1$ are given in Ch. 2 of [5] by using the theory of exponent pairs. In particular, it is proved that $E_1(T, \sigma) \ll T^{1-\sigma}$ holds for $1/2 < \sigma < 1$, which supersedes (24) for $3/4 \leq \sigma < 1$, so that the analogy with (17) is complete.

The evaluation of (16) may be obtained by going carefully through Motohashi's evaluation of $I_4(T, \frac{1}{2}; \Delta)$ with the appropriate modifications. The latter is extensively expounded in Ch. 5 of [5]. In particular, in (5.90) of [5] the expressions over the discrete and continuous spectrum provide analytic continuation for $u = v = w = z = \sigma$. Hence eventually one obtains

$$(25) \quad \begin{aligned} I_4(T, \sigma; \Delta) &= F_0(T, \sigma; \Delta) + \\ &+ \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{|\zeta(\frac{1}{2} + i\xi)|^4 |\zeta(2\sigma - \frac{1}{2} + i\xi)|^2}{|\zeta(1 + 2i\xi)|^2} \Theta(\xi; T, \sigma, \Delta) d\xi \\ &+ \sum_{j=1}^{\infty} \alpha_j H_j^2(1/2) H_j(2\sigma - \frac{1}{2}) \Theta(\kappa_j; T, \sigma, \Delta) + \\ &+ \sum_{k=6}^{\infty} \sum_{j=1}^{d_{2k}} \alpha_{j,2k} H_{j,2k}^2(2\sigma - \frac{1}{2}) \Lambda(k; T, \sigma, \Delta) + O\left(\frac{\log^2 T}{T}\right), \end{aligned}$$

where the functions F_0, Θ, Λ appearing in (25) are the appropriate modifications of the functions $F_0(T, \Delta), \Theta(\xi; T, \Delta), \Lambda(k; T, \Delta)$ appearing in (5.10) of [5], and the remaining notation from spectral theory is the same as in [5]. To see quickly what will be the shape of the asymptotic expression for $I_4(T, \sigma; \Delta)$ when $T^{1/2} \leq \Delta \leq T^{1-\epsilon}$, note first that the contribution of everything except the discrete spectrum over non-holomorphic cusp forms will be $O(\log^C T)$. This is the same as for the case $\sigma = 1/2$, and moreover the discrete spectrum may be truncated at $\kappa_j = T\Delta^{-1} \log^2 T$ with

negligible error. This is essentially due to the presence of the exponential factor $\exp(-(t/\Delta)^2)$ in the definition of I_4 , which is in one way or another reproduced through all the transformations leading to (25). Further note that the function Θ in (25) will itself contain the function

$$M(s, w; \Delta) := \int_0^\infty y^{s-1} (1+y)^{-w} \exp\left(-\frac{\Delta^2}{4} \log^2(1+y)\right) dy \quad (\text{Res} > 0).$$

In the case of $I_4(T, \frac{1}{2}; \Delta)$ that above function essentially had to be evaluated at $s = 1/2 \pm i\kappa_j, w = 1/2 \pm iT$, but in the case of the general $I_4(T, \sigma; \Delta)$ it has to be evaluated at $s = 2\sigma - 1/2 \pm i\kappa_j, w = \sigma \pm iT$. In both cases this may be achieved by the saddle-point method (this is where the condition $\Delta > T^{1/2}$ becomes useful) and, for $1 \ll \kappa_j \leq T\Delta^{-1} \log^2 T$, the functions

$$M\left(\frac{1}{2} - i\kappa_j, \frac{1}{2} + iT; \Delta\right), \quad M\left(2\sigma - \frac{1}{2} - i\kappa_j, \sigma + iT; \Delta\right)$$

have the same saddle point $y_0 \sim \kappa_j/T$. Since

$$y_0^{2\sigma-1/2-1} \sim y_0^{-1/2} (\kappa_j T^{-1})^{2\sigma-1} \quad (T \rightarrow \infty),$$

one will obtain in $I_4(T, \sigma; \Delta)$ essentially the same expression as for $\sigma = 1/2$, only each term will be multiplied by a factor which is asymptotic to $(\kappa_j/T)^{2\sigma-1}$, and $H_j(2\sigma - 1/2)$ will appear instead of $H_j(1/2)$ at one place. Therefore one should obtain, for $T^{1/2} \leq \Delta \leq T^{1-\varepsilon}$ and a suitable constant $C(\sigma)$,

$$\begin{aligned} I_4(T, \sigma, \Delta) &\sim C(\sigma) T^{1/2-2\sigma} \times \\ (26) \quad &\times \sum_{\kappa_j \leq \Delta^{-1} \log^2 T} \alpha_j \kappa_j^{2\sigma-3/2} H_j^2\left(\frac{1}{2}\right) H_j\left(2\sigma - \frac{1}{2}\right) e^{-\left(\frac{\Delta \kappa_j}{2T}\right)^2} \sin\left(\kappa_j \log \frac{\kappa_j}{4eT}\right) \\ &+ O(\log^C T). \end{aligned}$$

By the same principles the integrated version of (25) should read, for $V^{1/2} \leq \Delta \leq V^{1-\varepsilon}$,

$$\begin{aligned} (27) \quad &\int_V^{2V} I_4(T, \sigma; \Delta) dT \sim O(\Delta) + O(V^{1/2} \log V) + C(\sigma) V^{(3-4\sigma)/2} \times \\ &\times \sum_{\kappa_j \leq T\Delta^{-1} \log^2 T} \alpha_j \kappa_j^{2\sigma-5/2} H_j^2\left(\frac{1}{2}\right) H_j\left(2\sigma - \frac{1}{2}\right) e^{-\left(\frac{\Delta \kappa_j}{2V}\right)^2} \cos\left(\kappa_j \log \frac{\kappa_j}{4eV}\right). \end{aligned}$$

One can obtain without difficulty an analogue of Lemma 5.1 of [5] for $E_2(T, \sigma)$, which enables one to obtain upper bounds for $E_2(T, \sigma)$. In conjunction with (27) we shall therefore obtain, for $T^{1/2} \leq \Delta \leq T^{-\epsilon}$ and $1/2 < \sigma < 3/4$,

$$\begin{aligned} E_2(2T, \sigma) - E_2(T, \sigma) &\ll \Delta \log T + T^{1/2} \log^C T + \\ &\quad \max_{T/3 \leq t \leq 3T} t^{\frac{1}{2}(3-4\sigma)} \times \\ &\quad \times \sum_{\kappa_j \leq T\Delta^{-1} \log^2 T} \alpha_j \kappa_j^{2\sigma-5/2} H^2\left(\frac{1}{2}\right) |H_j(2\sigma - \frac{1}{2}) e^{-(\frac{\Delta \kappa_j}{2t})^2} \cos(\kappa_j \log \frac{\kappa_j}{4et})| \\ &\ll \Delta \log T + T^{1/2} \log^C T + T^{(3-4\sigma)/2} (T\Delta^{-1})^{2\sigma-1/2} \log^C T \\ &\ll T^{2/(1+4\sigma)} \log^C T \end{aligned}$$

for $\Delta = T^{2/(1+4\sigma)}$, thereby establishing Motohashi's result (17). Here we used the bound

$$\begin{aligned} \sum_{\kappa_j \leq K} H_j^2\left(\frac{1}{2}\right) |H_j(2\sigma - \frac{1}{2})| &\leq \left(\sum_{\kappa_j \leq K} H_j^4\left(\frac{1}{2}\right) \right)^{1/2} \left(\sum_{\kappa_j \leq K} H_j^2(2\sigma - \frac{1}{2}) \right)^{1/2} \\ &\ll K^2 \log^C K, \end{aligned}$$

since $\sum_{\kappa_j \leq K} H_j^4\left(\frac{1}{2}\right) \ll K^2 \log^C K$, and also

$$(28) \quad \sum_{\kappa_j \leq K} H_j^2(2\sigma - \frac{1}{2}) \ll K^2 \log^C K \quad (1/2 \leq \sigma \leq 1).$$

The bound in (28) is proved analogously as in the well-known case $\sigma = 1/2$, only it is less difficult since $2\sigma - 1/2 \geq 1/2$ for $\sigma \geq 1/2$, and in general $H_j(s)$ (like many other functions defined by analytic continuation of Dirichlet series) is less difficult to handle as $\text{Re } s$ increases.

The first open problem I have in mind concerning $E_2(T, \sigma)$ is the conjecture pertaining to its true order of magnitude, namely

(29)

$$E_2(T, \sigma) = O(T^{\frac{1}{2}(3-4\sigma)+\epsilon}), E_2(T, \sigma) = \Omega_{\pm}(T^{1/2(3-4\sigma)}) \quad (1/2 < \sigma < 3/4).$$

Since $3/2 - 2\sigma > 0$ only for $\sigma < 3/4$, the line $\sigma = 3/4$ appears to be a sort of a boundary both for $E_2(T, \sigma)$ and $E_1(T, \sigma)$. For the latter function this

phenomenon was mentioned already by K. Matsumoto [11]. The O -bound in (29) is certainly very difficult, while the omega-results may be within reach. For $\sigma = 1/2$ it is known that $E_2(T) = E_2(T, \frac{1}{2}) = \Omega(T^{1/2})$ (see Ch. 5 of [5]), although I am certain that the sharper result

$$E_2(T) = \Omega_{\pm}(T^{1/2})$$

must hold. Another reason for the fact that very likely "something" happens with $E_2(T, \sigma)$ at $\sigma = 3/4$ is that, for $\sigma > 3/4$, we have (see (26) and (27))

$$H_j(2\sigma - \tfrac{1}{2}) = \sum_{n=1}^{\infty} t_j(n) n^{1/2-2\sigma} \ll_j 1,$$

while for $\sigma \leq 3/4$ the above series representation is not valid.

For $1/2 \leq \sigma < 3/4$ fixed I also conjecture that

$$(30) \quad \int_0^T E_2^2(t, \sigma) dt \sim C_2(\sigma) T^{4-4\sigma} \quad (T \rightarrow \infty)$$

holds with a suitable $C_2(\sigma) > 0$. However, I have no ideas what the explicit value of $C_2(\sigma)$ ought to be. For the less difficult problem of the mean square of $E_1(t, \sigma)$ the situation is different. Namely K. Matsumoto and T. Meurman [12] proved

$$(31) \quad \int_0^T E_1^2(t, \sigma) dt = C_1(\sigma) T^{(5-4\sigma)/2} + O(T) \quad (1/2 < \sigma < 3/4)$$

with

$$C_1(\sigma) = \frac{2}{5-4\sigma} (2\pi)^{(4\sigma-3)/2} \frac{\zeta^2(3/2)}{\zeta(3)} \zeta(\tfrac{5}{2} - 2\sigma) \zeta(\tfrac{1}{2} + 2\sigma)$$

and

$$(32) \quad \int_0^T E_1^2(t, \tfrac{3}{4}) dt = \frac{\zeta^2(3/2)\zeta(2)}{\zeta(3)} T \log T + O(T \log^{1/2} T).$$

Matsumoto and Meurman also proved

$$(33) \quad E_1(T, \sigma) = \Omega_+(T^{3/4-\sigma} (\log T)^{\sigma-1/4}) \quad (1/2 < \sigma < 3/4),$$

while (32) yields $E_1(T, \frac{3}{4}) = \Omega(\log^{1/2} T)$. Note that (33) is the (strong) analogue of the conjectural Ω_+ -result (29) for $E_2(T, \sigma)$. I am convinced that the technique used for proving the Ω_- -result for $E(T) = E_1(T, \frac{1}{2})$ (see Th. 3.4 of [5]) can be used to obtain $E_1(T, \sigma) = \Omega_-(T^{3/4-\sigma})$ for $1/2 < \sigma < 3/4$, and maybe even a slightly stronger result (i.e. $T^{3/4-\sigma}$ multiplied by a log log-factor, or even by a log-factor).

In view of (29), its analogue (unproved yet)

$$E_1(T, \sigma) = O(T^{3/4-\sigma+\varepsilon}) \quad (1/2 < \sigma < 3/4),$$

and

$$|\zeta(\sigma + iT)|^{2k} \ll \log T \left(\int_{T-1}^{T+1} |\zeta(\sigma + it)|^{2k} dt + 1 \right) \quad (k \in \mathbb{N}),$$

an unsettling possibility comes to my mind. It involves the function

$$\mu(\sigma) = \limsup_{t \rightarrow \infty} \frac{\log |\zeta(\sigma + it)|}{\log t},$$

for which one has trivially $\mu(\sigma) = \frac{1}{2} - \sigma$ for $\sigma \leq 0$ and $\mu(\sigma) = 0$ for $\sigma \geq 1$. If the Lindelöf hypothesis that $\zeta(\frac{1}{2} + it) \ll t^\varepsilon$ is true, then the graph of $\mu(\sigma)$ consists of the line segments $1/2 - \sigma$ for $\sigma \leq 1/2$ and 0 for $\sigma \geq 1/2$. But the above discussion prompts me to think that it is not unlikely that perhaps one has even

$$\mu(\sigma) = \begin{cases} 1/2 - \sigma & \sigma \leq 1/4, \\ 3/8 - \sigma/2 & 1/4 \leq \sigma \leq 3/4, \\ 0 & \sigma \geq 3/4. \end{cases}$$

This is much weaker than the Lindelöf hypothesis, as it implies $\zeta(\frac{1}{2} + it) = \Omega(t^{1/8-\delta})$ for any given $\delta > 0$. Thus it would a fortiori contradict the Riemann hypothesis, since it is well-known that the Riemann hypothesis implies the Lindelöf hypothesis.

If the problem of establishing (30) is perhaps intractable, perhaps it is possible to prove

$$(34) \quad \int_0^T E_2^2(t, \sigma) dt \ll T^{4-4\sigma} \log^C T \quad (C > 0, 1/2 < \sigma < 3/4),$$

since (34) for $E_2(T) = E_2(T, \frac{1}{2})$ was proved in [8]. By the method used there it follows that

$$(35) \quad \int_{t_r}^{t_r+\Delta} |\zeta(\sigma + it)|^4 dt \ll \Delta + \Delta^{-1} \int_{t_r-2\Delta}^{t_r+2\Delta} |E_2(t, \sigma)| dt,$$

and we are going to impose the spacing condition

(36)

$$T < t_1 < \dots < t_R \leq 2T, t_{r+1} - t_r \geq \Delta (r = 1, \dots, R-1), T^\epsilon \leq \Delta \leq T^{1-\epsilon}.$$

Then if (34) holds we obtain from (35) that

$$(37) \quad \sum_{r \leq R} \left(\int_{t_r}^{t_r+\Delta} |\zeta(\sigma + it)|^4 dt \right)^2 \ll R\Delta^2 + \Delta^{-1} \sum_{r \leq R} \int_{t_r-2\Delta}^{t_r+2\Delta} E_2^2(t, \sigma) dt$$

$$\ll R\Delta^2 + \Delta^{-1} \int_{T/3}^{3T} E_2^2(t, \sigma) dt \ll R\Delta^2 + T^{4-4\sigma} \Delta^{-1} \log^C T.$$

Hence by the Cauchy-Schwarz inequality one obtains from (37), for $1/2 < \sigma < 3/4$,

$$(38) \quad \sum_{r \leq R} \int_{t_r}^{t_r+\Delta} |\zeta(\sigma + it)|^4 dt \ll R\Delta + R^{1/2} T^{2-2\sigma} \Delta^{-1/2} \log^C T$$

provided that (36) holds. I note that Motohashi and I proved in [8] that, for $1/2 < \sigma < 3/4$,

$$(39) \quad \sum_{r \leq R} \int_{t_r}^{t_r+\Delta} |\zeta(\sigma + it)|^4 dt \ll R\Delta + R^\sigma T^{2-2\sigma} \Delta^{\sigma-1} \log^C T$$

again if (36) holds. Since

$$R^{1/2} T^{2-2\sigma} \Delta^{-1/2} < R^\sigma T^{2-2\sigma} \Delta^{\sigma-1} \quad (\sigma > 1/2),$$

it follows that (38) improves (39) for $1/2 < \sigma < 3/4$. This shows the importance of establishing (34).

The discussion leading to (26) indicates that, crudely speaking, the spectral part appearing in the expression for $I_4(T, \sigma; \Delta)$ is by a factor of $\Delta^{1-2\sigma}$ smaller than the corresponding sum in the expression for $I_4(T, \frac{1}{2}; \Delta)$. If one follows the proof of

$$(40) \quad \sum_{r \leq R} \int_{t_r}^{t_r + \Delta} |\zeta(\tfrac{1}{2} + it)|^4 dt \ll (R\Delta + R^{1/2}T\Delta^{-1/2}) \log^C T$$

(see Ch. 5 of [5]) with the appropriate modifications, one obtains

$$(41) \quad \sum_{r \leq R} \int_{t_r}^{t_r + \Delta} |\zeta(\sigma + it)|^4 dt \ll (R\Delta + R^{1/2}T\Delta^{1/2-2\sigma}) \log^C T$$

for $1/2 < \sigma < 3/4$ and $T^{1/2} \leq \Delta \leq T$. However, once (40) is known for $\Delta \geq T^{1/2}$, it can be easily established for $\Delta < T^{1/2}$, since $R\Delta \ll R^{1/2}T\Delta^{-1/2}$ for $\Delta \leq T^{1/2}$ and each interval $[t_r, t_r + \Delta]$ lies in at most two intervals of the form $[T + (n-1)T^{1/2}, T + nT^{1/2}]$, ($n = 1, 2, \dots$). This procedure does not carry over to (41), in the sense that we cannot deduce in an obvious way the validity of (41) for $\Delta < T^{1/2}$ once it is known for $\Delta \geq T^{1/2}$. Note that (39) improves (41) for $R \leq T^2\Delta^{-3}$, while (38) improves (41) in the whole range $1/2 < \sigma < 3/4$.

Another problem involving $E_2(T, \sigma)$ is to prove that $E_2(T, \sigma)$ has arbitrarily large zeros for a fixed σ satisfying $1/2 < \sigma < 3/4$. This is a trivial consequence (since $E_2(T, \sigma)$ is a continuous function of T) of the conjectural Ω_{\pm} -result in (29), but perhaps a direct proof of this result might be within reach. The corresponding problem for $E_2(T) = E_2(T, \frac{1}{2})$ was mentioned in Ch. 5 of [5], where it was also noted how one obtains

$$\limsup_{T \rightarrow \infty} |E_2(T)|T^{-1/2} = +\infty$$

if a certain linear independence of spectral values can be established. The problem of the existence of arbitrarily large zeros of $E_2(T)$ still remains open. Closely related to the above topic is problem of sign changes of $E_2(T, \sigma)$. For $E_1(T, \sigma)$ I have proved (see Th. 3.3 of [5]) that every interval $[T, T + DT^{1/2}]$ for suitable $D > 0$ and $T \geq T_0$ contains points τ_1, τ_2 such that

$$E_1(\tau_1, \sigma) > B\tau_1^{3/4-\sigma}, E_1(\tau_2, \sigma) < -B\tau_2^{3/4-\sigma}$$

for $1/2 < \sigma < 3/4$ and a suitable constant $B > 0$. In my opinion the analogue of this result, which is an open problem, for $E_2(T, \sigma)$ would be the following assertion:

Every interval $[T, DT]$ for suitable $D > 1$ and $T \geq T_0$ contains points t_1, t_2 such that

$$E_2(t_1, \sigma) > Bt_1^{(3-4\sigma)/2}, E_2(t_2, \sigma) < -Bt_2^{(3-4\sigma)/2}$$

for $1/2 < \sigma < 3/4$ and suitable $B > 0$.

Of course the latter result by continuity trivially implies both $E_2(T, \sigma) = \Omega_{\pm}(T^{(3-4\sigma)/2})$ and the existence of arbitrarily large zeros of $E_2(T, \sigma)$.

Finally, what is the smallest σ such that

$$(42) \quad \int_0^T |\zeta(\tfrac{1}{2} + it)|^4 |\zeta(\sigma + it)|^2 dt \ll T^{1+\varepsilon} ?$$

Trivially (42) holds for $\sigma = 1$, and its truth for $\sigma = 1/2$ (for $\sigma < 1/2$ it is false if ε is small enough) is the hitherto unproved sixth moment of $|\zeta(\frac{1}{2} + it)|$. In fact, at present it seems difficult to find any σ satisfying $\sigma < 1$ such that (42) holds. The bound

$$\int_0^T |\zeta(\tfrac{1}{2} + it)|^4 \left| \sum_{n \leq N} a_n n^{it} \right|^2 dt \ll T^{\varepsilon} (T + T^{1/2} N^2 + T^{3/4} N^{5/4}) \sum_{n \leq N} |a_n|^2,$$

where the a_n 's are arbitrary complex numbers, appears to be a natural tool for attacking this problem. This result is due to J.-M. Deshouillers and H. Iwaniec [3], but the term $T^{1/2} N^2$ is too large to give any $\sigma < 1$ in (42) when we approximate $\zeta(\sigma + it)$ by Dirichlet polynomials of length $\ll t^{1/2}$.

The similar problem of finding σ such that

$$(43) \quad \int_0^T |\zeta(\tfrac{1}{2} + it)|^2 |\zeta(\sigma + it)|^4 dt \ll T^{1+\varepsilon}$$

holds is certainly much less difficult. By using the bound

$$\int_0^T |\zeta(\tfrac{5}{8} + it)|^8 dt \ll T^{1+\varepsilon}$$

(see Ch. 8 of [4]) and the Cauchy-Schwarz inequality it follows that (43) holds for $\sigma = 5/8$. Again I ask whether one can find a value of σ less than $5/8$ for which (43) holds. Similarly as for (42), (43) also cannot hold for $\sigma < 1/2$ if ε is small enough, and its truth for $\sigma = 1/2$ is the sixth moment of $\zeta(s)$ on the critical line.

REFERENCES

- [1] R. Balasubramanian, *On the frequency of Titchmarsh's phenomenon for $\zeta(s)$ IV*, Hardy-Ramanujan **J.9** (1986), 1–10.
- [2] R. Balasubramanian, A. Ivić and K. Ramachandra, *The mean square of the Riemann zeta-function on the line $\sigma = 1$* , L'Enseignement Mathématique **38** (1992), 13–25.
- [3] J.-M. Deshouillers and H. Iwaniec, *Power mean values of the Riemann zeta-function*, Mathematika **29** (1982), 202–212.
- [4] A. Ivić, *The Riemann zeta-function*, John Wiley & Sons, New York, (1985).
- [5] A. Ivić, *The mean values of the Riemann zeta-function*, Tata Institute for Fundamental Research LN's **82**, Bombay 1991 (distr. by Springer Verlag, Berlin etc., 1992).
- [6] A. Ivić, *The moments of the zeta-function on the line $\sigma = 1$* , Nagoya Math. J. **135** (1994), 113–129.
- [7] A. Ivić and A. Perelli, *Mean values of certain zeta-functions on the critical line*, Litovskij Mat. Sbornik **29** (1989), 701–714.
- [8] A. Ivić and Y. Motohashi, *The mean square of the error term for the fourth moment of the zeta-function*, Proc. London Math. Soc. (3) **69** (1994), 309–329.
- [9] D. Joyner, *Distribution theorems for L-functions*, Longman Scientific & Technical, Essex (1986).
- [10] A. Laurinćikas, *The limit theorem for the Riemann zeta-function on the critical line I*, (Russian), Litovskij Mat. Sbornik **27** (1987), 113–132 and II ibid. **27** (1987), 459–500.
- [11] K. Matsumoto, *The mean square of the Riemann zeta-function in the critical strip*, Japan. J. Math. **13** (1989), 1–13.
- [12] K. Matsumoto and T. Meurman, *The mean square of the Riemann zeta-function in the critical strip II*, Acta Arith. **68** (1994), 369–382; III, Acta Arith. **64** (1993), 357–382.
- [13] K. Ramachandra, *Some remarks on the mean value of the Riemann zeta-function and other Dirichlet series IV*, J. Indian Math. Soc. **60** (1994), 107–122.
- [14] K. Ramachandra, *Lectures on the mean-value and omega-theorems for the Riemann zeta-function*, LNs **85**, Tata Institute of Fundamental Research, Bombay 1995 (distr. by Springer Verlag, Berlin etc.).
- [15] P. Shiu, *A Brun-Titchmarsh theorem for multiplicative functions*, J. Reine Angew. Math. **31** (1980), 161–170.

- [16] E.C. Titchmarsh, *The theory of the Riemann zeta-function (2nd ed.)*, Oxford, Clarendon Press, (1986).

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