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# Galois Structure of de Rham Cohomology.

par TED CHINBURG*

## 1. Introduction.

This article has two purposes. The first is to summarize (without proofs ) the results in [C] concerning the Galois module structure of the de Rham cohomology of tame covers of schemes. The second purpose is to prove an alternate interpretation of [C] for irreducible smooth curves over finite fields. The object of [C] is to generalize to schemes the theory of the Galois module structure of tamely ramified rings of integers.

In classical Galois structure theory one considers finite tame Galois extensions $L/K$ of number fields. To generalize this, we recall in §2 Grothendieck and Murre's concept of a tamely ramified $G$-cover $f : X \mapsto Y$ of schemes over a Noetherian ring $A$, where $G$ is a finite group. We then discuss the results of [C] concerning Euler characteristics in Grothendieck groups of $A[G]$-modules of suitable complexes of sheaves of $G$-modules on $X$. In §3 we define via de Rham complexes an invariant $\Psi(X/Y)$ which generalizes the stable isomorphism class of the ring of integers of $L$ in the classical case. We then discuss a conjectural generalization of Martinet's Conjecture when the ground ring $A$ is a finitely generated $\mathbf{Z}[1/m]$-module for some integer $m$ prime to the order of $G$. The main result of [C], which is summarized in §4, is a precise counterpart for smooth projective varieties over a finite field of Fröhlich's conjecture concerning rings of integers. One consequence of this result is that the generalization of Martinet's Conjecture discussed in §3 holds if $A$ is a finite field.

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## 2. Tame covers and Euler characteristics.

We will begin by recalling from [GM] the definition of a tamely ramified cover of schemes.

DEFINITION 2.1. Let $Y$ be a normal scheme which is of finite type over a Noetherian ring $A$. Let $D$ be a Zariski closed subset of $Y$ which is of codimension at least one. A morphism of schemes $f : X \mapsto Y$ is a tamely ramified covering of $Y$ relative to $D$ if the following conditions hold:

(a) $f$ is finite.
(b) $f$ is étale over $Y - D$.
(c) Every irreducible component of $X$ dominates an irreducible component of Y.
(d) $X$ is normal.
(e) Let $y \in D$ have codimension 1 in $Y$ and let $x$ be a point of $X$ over $y$. Then $O_{X,x}/O_{Y,y}$ is a tamely ramified extension of D.V.R.'s, i.e. the associated residue field extension is separable and the ramification degree is prime to the residue characteristic if this characteristic is positive.

DEFINITION 2.2. Let $f : X \mapsto Y$ and $D$ be as in Definition 2.1, and suppose $G$ is a finite group. We will say $f : X \mapsto Y$ is a tame $G$-cover relative to $D$ if $X \times_Y (Y - D) \mapsto Y - D$ is a $G$-torsor when $G$ is regarded as a constant group scheme over $Y - D$ (c.f. [M, §III.4]). This is equivalent to the requirement that $X \times_Y (Y - D) \mapsto Y - D$ is Galois with group $G$ in the sense of [M, p. 43 - 44].

**Example 2.3.** Suppose $L/K$ is a finite Galois extension of global fields which is at most tamely ramified in the usual sense. Let $G = Gal(L/K)$. If $L$ and $K$ are number fields, let $O_L$ and $O_K$ be their rings of integers. The natural morphism $f : X = Spec(O_L) \mapsto Y = Spec(O_K)$ is then a tame $G$-cover relative to the closed subset $D$ of $Y$ consisting of the finitely many closed points over which $f$ ramifies. If $L$ and $K$ are global function fields, then they are the function fields of smooth projective curves $X$ and $Y$. The corresponding morphism $f : X \mapsto Y$ is a tame $G$-cover relative to the closed subset $D$ of $Y$ over which $f$ ramifies.

*Remark 2.4.* We let groups act on rings and modules on the left and on schemes on the right. Suppose $f : X \mapsto Y$ is a tame $G$-cover as in Definition

2.2. Let $R(X)$ be the function ring of $X$ (c.f. [EGA I, 7.1.2]). Since $X$ is normal and Noetherian, the finitely many irreducible components of $X$ are disjoint, and $R(X)$ is the direct sum of the function fields of the generic points of these components. Definition 2.2 implies $R(X)$ is a Galois extension of $R(Y)$ with Galois group $G$ in the sense of [M, p. 43-44]. Because $X$ is the normalization of $Y$ in $R(X)$, the action of $G$ on $R(X)$ gives an action of $G$ on $O_X$. Furthermore, $f$ is affine since $f$ is finite, and [L, Prop. I.9] implies $f$ is surjective.

DEFINITION 2.5. Let $f : X \mapsto Y$ be a tame $G$-cover as in Definition 2.2. A sheaf of $O_Y[G]$-Modules is a sheaf of $O_Y$-modules having a $G$-action which commutes with the action of $O_Y$. A quasi-coherent $O_X$-$G$-Module $T$ is a quasi-coherent sheaf $T$ of $O_X$-modules on $X$ having an action of $G$ which is compatible with the action of $G$ on $O_X$ in the following sense. Suppose $V$ is an open subset of $Y$, $\tau \in G$, $a \in \Gamma(f^{-1}(V), O_X)$ and $m \in \Gamma(f^{-1}(V), T)$. Then $\tau(am) = \tau(a) \cdot \tau(m)$. We will always assume that morphisms between sheaves of $O_Y[G]$-Modules (resp. $O_X$-$G$-Modules) respect both the actions of $G$ and of $O_Y$ (resp. of $G$ and of $O_X$ ).

In Definition 2.5 we have used the terminology $O_X$-$G$-Module rather than $O_X[G]$-Module to signal the fact that the left action of $O_X$ on the underlying sheaf of an $O_X$-$G$-Module is twisted in the indicated way by the action of $G$.

DEFINITION 2.6. A $G$-module $M$ is cohomologically trivial for $G$ if the Tate cohomology group $\hat{H}^i(H, M)$ vanishes for all subgroups $H$ of $G$ and all integers $i$.

The following result underlies the Galois structure invariants we will consider.

THEOREM 2.7. *Suppose $f : X \mapsto Y$ is a tame $G$-cover relative to a divisor $D$ on $Y$ having normal crossings. Let $T$ be a quasi-coherent sheaf of $O_X$-$G$-Modules. Then all of the stalks of the sheaf $f_*T$ of $O_Y[G]$-Modules on $Y$ are cohomologically trivial for $G$.*

The proof of Theorem 2.7 relies on Abhyankhar's Theorem, which states that locally in the étale topology on $Y$, $f : X \mapsto Y$ is induced from a Kummer covering relative to a subgroup of $G$. In [C, Theorem 3.7] a slightly stronger result is proved, in which $f : X \mapsto Y$ is replaced by an arbitrary subquotient cover of $f$.

**Example 2.8.** Suppose $L/K$ is a finite Galois extension of number fields which is at most tamely ramified in the usual sense. Let $I$ be a $Gal(L/K)$-

stable integral ideal of $L$. Theorem 2.7 implies Noether's Theorem that $I$ is cohomologically trivial for $G$. To see this, let $f$ be the natural morphism $X = Spec(O_L) \mapsto Y = Spec(O_K)$ and let $T$ be the sheaf $\tilde{I}$ associated to $I$. The $O_K[G]$-module $I$ is cohomologically trivial for $G$ if and only if all of its localizations at primes of $O_K$ are, and these localizations are the stalks of $f_*\tilde{I}$.

We now consider Euler characteristics of complexes of sheaves of $G$-modules. To do this, we assume for the rest of §1 that $Y$ is proper over $A$.

Let $K^+(Y,G)$ (resp. $K^+(A,G)$) be the category of complexes of quasi-coherent $O_Y[G]$-Modules (resp. $A[G]$-modules) which are bounded below and which have coherent (resp. finitely generated) cohomology. Morphisms in these categories are homotopy classes of morphisms of complexes. A morphism is a quasi-isomorphism if it induces isomorphisms in cohomology.

Let $D^+(Y,G)$ and $D^+(A,G)$ be the localizations of $K^+(Y,G)$ and $K^+(A,G)$, respectively, with respect to the multiplicative systems of quasi-isomorphisms in these categories. Thus $D^+(Y,G)$ has the same objects as $K^+(Y,G)$, and the morphisms of $D^+(Y,G)$ are obtained by formally inverting all quasi-isomorphisms in $K^+(Y,G)$; see [H2] for details.

By [H2, p. 87 - 89], the global section functor $\Gamma$ has a right derived functor $\underline{\underline{R}}\Gamma^+ : D^+(Y,G) \mapsto D^+(A,G)$. Let $F^\bullet$ be a complex in $Ob(K^+(Y,G)) = Ob(D^+(Y,G))$. By [EGA III, Cor. 0.12.4.7] the Cech hypercohomology complex $\mathbf{H}(\mathcal{U}, F^\bullet)$ of $F^\bullet$ with respect to a finite open affine cover $\mathcal{U} = \{U_i\}_{i\in I}$ of $Y$ is isomorphic to $\underline{\underline{R}}\Gamma^+(F^\bullet)$ in $D^+(A,G)$. We now recall the definition of $\mathbf{H}(\mathcal{U}, F^\bullet)$, since the Euler characteristics we will define can be most readily understood in terms of Cech hypercohomology.

For each integer $i \geq 0$ and $i+1$-tuple $(k_0, ..., k_i)$ of elements of the index set $I$ let $U_{k_0,\ldots,k_i} = U_{k_0} \cap \ldots \cap U_{k_i}$. Fix an ordering of the (finite) set $I$. The group of alternating $i$-cochains with coefficients in the sheaf $F^j$ is defined to be

$$C^i(\mathcal{U}, F^j) = \prod_{k_0 < \ldots < k_i} F^j(U_{k_0,\ldots,k_i})$$

where the product is over all $i+1$-tuples of elements of $I$ which are in increasing order. Let $C^\bullet(\mathcal{U}, F^\bullet)$ be the bicomplex whose $(i,j)^{th}$ term is $C^i(\mathcal{U}, F^j)$. The horizontal boundary map $d''$ of $C^\bullet(\mathcal{U}, F^\bullet)$ results from the boundary map of $F^\bullet$, while the vertical boundary map $d'$ is given by the usual Cech coboundary formula ( c.f. [H1, §III.4]). The Cech hypercohomology complex $\mathbf{H}(\mathcal{U}, F^\bullet)$ of $F^\bullet$ is the total complex $\mathrm{Tot}(C^\bullet(\mathcal{U}, F^\bullet))$ of

$C^\bullet(\mathcal{U}, F^\bullet)$. Recall that $\mathrm{Tot}(C^\bullet(\mathcal{U}, F^\bullet))$ has $n^{th}$ term

$$\bigoplus_{i+j=n} C^i(\mathcal{U}, F^j)$$

and the boundary map $d$ of $\mathrm{Tot}(C^\bullet(\mathcal{U}, F^\bullet))$ is defined by $d(x) = d'(x) + (-1)^i d''(x)$ for $x \in C^i(\mathcal{U}, F^j)$. Thus if the only non-zero term of $F^\bullet$ is the sheaf $F^0$ in degree 0, $\mathbf{H}\mathcal{U}, F^\bullet)$ is the usual Cech cohomology complex $\mathbf{H}(\mathcal{U}, F^0)$.

In the following result we do not need to assume $Y$ is normal.

THEOREM 2.9. *Suppose $Y$ is proper over $A$. Let $F^\bullet$ be a bounded complex in $K^+(Y, G)$ such that the stalks of each term of $F^\bullet$ are cohomologically trivial for $G$. Then $\underline{\underline{R}}\Gamma^+(F^\bullet)$ is isomorphic in $D^+(A, G)$ to a bounded complex $M^\bullet$ of finitely generated $A[G]$-modules which are cohomologically trivial for $G$. Let $CT(A[G])$ be the Grothendieck group of all finitely generated $A[G]$-modules which are cohomologically trivial for $G$. The Euler characteristic $\chi(M^\bullet) = \sum(-1)^i(M^i)$ in $CT(A[G])$ depends only on $F^\bullet$, and will be denoted $\chi\underline{\underline{R}}\Gamma^+(F^\bullet)$. If $F^\bullet$ consists of a single non-zero term $F$ in degree 0, then $\chi\underline{\underline{R}}\Gamma^+(F^\bullet)$ will also be denoted by $\chi\underline{\underline{R}}\Gamma^+(F)$.*

A complex $M^\bullet$ with the above properties can be constructed in the following way. Let $\mathcal{U}$ be a finite open affine cover of $Y$. By the inductive procedure of [H1, Lemma III.12.3] (see also [EGA III, Prop. 0.II.9.1]) we can construct a complex $N^\bullet$ of free finitely generated $A[G]$-modules which is bounded above (but not necessarily below) together with a quasi-isomorphism of complexes $N^\bullet \mapsto \mathbf{H}(\mathcal{U}, F^\bullet)$. Suppose $F^j = 0$ for $j < n$, so that the $j^{th}$ term of $\mathbf{H}(\mathcal{U}, F^\bullet)$ is also trivial for $j < n$. Let $M^\bullet$ be the complex which results from $N^\bullet$ by letting $M^j$ equal $N^j$ ( resp. $N^n/\delta(N^{n-1})$, resp. $\{0\}$) if $j > n$ (resp. $j = n$, resp. $j < n$). The resulting morphism $M^\bullet \mapsto \mathbf{H}(\mathcal{U}, F^\bullet)$ is then a quasi-isomorphism. It is shown in the proof of [C, Theorem 2.1] that $M^\bullet$ and $\chi(M^\bullet)$ have all the properties stated in Theorem 2.9.

*Remark 2.10.* Theorem 2.7 provides many examples of complexes of sheaves $F^\bullet$ satisfying the hypotheses of Theorem 2.9. Explicitly, suppose $T^\bullet$ is a bounded complex of quasi-coherent $O_X$-$G$-Modules on $X$ and that the cohomology sheaves of $T^\bullet$ are coherent. Then $F^\bullet = f_*T^\bullet$ is a bounded complex of quasi-coherent $O_Y[G]$-Modules having coherent cohomology sheaves, and the stalks of each term of $F^\bullet$ are cohomologically trivial for $G$. If the underlying ring $A$ is a field, the existence of a complex $M^\bullet$ as in Theorem 2.9 when $F^\bullet = f_*T^\bullet$ was proved by Nakajima in [N1, Theorem 1].

**Example 2.11.** Let $A = \mathbf{Z}$. With the notations of Example 2.8, $\chi\underline{\underline{R}}\Gamma^+(f_*\tilde{I})$ is the stable isomorphism class $(I)$ in $CT(\mathbf{Z}[G])$ of the $Gal(L/K)$-stable $O_L$-ideal $I$. This is because the Cech cohomology complex $\mathbf{H}(\mathcal{U}, f_*\tilde{I})$ of $\tilde{I}$ relative to $\mathcal{U} = \{Spec(O_K)\}$ consists of $I$ in degree 0 and is trivial in all other dimensions. Since $I$ is cohomologically trivial for $G$ by the remarks of Example 2.8, we can let $M^\bullet = \mathbf{H}(\mathcal{U}, f_*\tilde{I})$ in Theorem 2.9.

It is not difficult to use the Cech construction of $M^\bullet$ to verify that the following properties of $\chi\underline{\underline{R}}\Gamma^+$ (c.f. [C, Remarks 2.6 and 2.7]).

PROPOSITION 2.12. *Let $F^\bullet = (F^i)$ be a bounded complex of coherent $O_Y[G]$-Modules whose stalks are cohomologically trivial for $G$. Then*

$$\chi\underline{\underline{R}}\Gamma^+(F^\bullet) = \sum_j (-1)^j \chi\underline{\underline{R}}\Gamma^+(F^j).$$

PROPOSITION 2.13. *Let $H$ be a subgroup of $G$. Suppose $F^\bullet$ (resp. $T^\bullet$) is a bounded complex in $K^+(Y, G)$ (resp. $K^+(Y, H)$), and that the stalks of the terms of $F^\bullet$ (resp. $T^\bullet$) are cohomologically trivial for $G$ (resp. for $H$). Then*

$$\chi\underline{\underline{R}}\Gamma^+(res_{G\mapsto H}(F^\bullet)) = res_{G\mapsto H}(\chi\underline{\underline{R}}\Gamma^+(F^\bullet)) \quad \text{, and}$$

$$\chi\underline{\underline{R}}\Gamma^+(ind_{H\mapsto G}(T^\bullet)) = ind_{H\mapsto G}(\chi\underline{\underline{R}}\Gamma^+(T^\bullet))$$

*where $res_{G\mapsto H}$ (resp. $ind_{H\mapsto G}$) is the map induced by restriction of operators from $G$ to $H$ (resp. by applying the functor $M \mapsto Ind_H^G M = \mathbf{Z}[G] \otimes_{\mathbf{Z}[H]} M$ which induces $H$-modules to $G$). If $H$ is normal in $G$ then*

$$\chi\underline{\underline{R}}\Gamma^+(inv_{G\mapsto G/H}(F^\bullet)) = inv_{G\mapsto G/H}(\chi\underline{\underline{R}}\Gamma^+(F^\bullet))$$

*where $inv_{G\mapsto G/H}$ is the map resulting from the functor $M \mapsto M^H$ from $G$-modules to $G/H$ modules.*

The following result is Proposition 2.4 of [C].

PROPOSITION 2.14. *Suppose $F^\bullet$ is as in Theorem 2.9. Let $H^q(F^\bullet)$ be the $q^{th}$ ( coherent ) cohomology sheaf of $F^\bullet$. Let $\nu : CT(A[G]) \mapsto G_0(A[G])$ be the natural forgetful map to the Grothendieck group $G_0(A[G])$ of all finitely generated $A[G]$-modules. Then*

$$\nu(\chi\underline{\underline{R}}\Gamma^+(F^\bullet)) = \sum_{p,q} (-1)^{p+q} \cdot (H^p(Y, H^q(F^\bullet)))'$$

*where $(M)'$ is the class in $G_0(A[G])$ of the module $M$. If each of the sheaves $F^q$ appearing in $F^\bullet$ are coherent, then*

$$\nu(\chi\underline{\underline{R}}\Gamma^+(F^\bullet)) = \sum_{p,q}(-1)^{p+q}(H^p(Y,F^q))'$$

**Example 2.15.** Suppose that $F^\bullet$ in Proposition 2.14 consists of a single coherent sheaf $F^0$ in dimension 0. Proposition 2.14 shows that $\chi\underline{\underline{R}}\Gamma^+(F^0) = \chi\underline{\underline{R}}\Gamma^+(F^\bullet)$ is a canonical lift to $CT(A[G])$ of the usual coherent Euler characteristic $\sum_p(-1)^p \cdot (H^p(Y,F^0))'$ of $F^0$ in $G_0(A[G])$.

*Remark 2.16.* If the order of $G$ is prime to the residue characteristic of every prime ideal of $A$, then every $A[G]$-module is cohomologically trivial for $G$ and $\nu$ is an isomorphism. Thus Proposition 2.14 gives a way to compute $\chi\underline{\underline{R}}\Gamma^+(F^\bullet)$ for such $A$.

**Example 2.17.** Let us show that if $A$ is a field then

$$\chi\underline{\underline{R}}\Gamma^+(O_Y[G]) = \chi(O_Y)\cdot(A[G]) \tag{2.1}$$

in $CT(A[G])$, where $\chi(O_Y) = \sum_{p=0}^{\infty}(-1)^p \cdot dim_A H^p(Y,O_Y)$ is the Euler characteristic of $O_Y$ over $A$. (If $A$ is a field, (2.1) is contained in Proposition 2.3 of [EL]; see also [N2, Theorem 1].) If $G$ is the trivial group, (2.1) follows from Remark 2.16 and Example 2.15. For arbitrary $G$ one then obtains (2.1) by induction from the trivial subgroup of $G$ using Proposition 2.13.

We end this section by discussing the counterpart of Example 2.11 for global function fields.

THEOREM 2.18. *Let $A = \mathbf{F}_q$ be the finite field with $q$ elements. Suppose $f : X \mapsto Y$ is a tame $G$-cover of smooth projective irreducible curves over $A$ and that $A$ is the field of constants of $Y$. Suppose $E$ is a $G$-stable Weil divisor on $X$, so that $O_X(E)$ is an $O_X$-$G$-Module in the sense of Definition 2.5. By viewing $E$ as a Cartier divisor on $X$ we may regard $f_*O_X(E)$ as a subsheaf of the constant sheaf on $Y$ whose generic fibre is the function field $L$ of $X$. Let $U$ be a non-empty affine open subset of $Y$ and let $S_\infty = Y - U$. We can find an element $\beta \in L$ with the following properties.*

- (a) *$\Gamma(U,O_Y)[G]\cdot\beta$ has finite index in $\Gamma(U,f_*O_X(E))$;*
- (b) *If $y \in S_\infty$, the stalk $(f_*O_X(E))_y$ of $f_*O_X(E)$ at $y$ has finite index in $O_{Y,y}[G]\cdot\beta$.*
- (c) *All of the $G$-modules appearing in (a) and (b) are cohomologically trivial for $G$.*

*For all such $\beta$ we have*
(2.2)
$$\chi\underline{\underline{R}}\Gamma^+(f_*O_X(E)) = (1-g(Y))\cdot(A[G]) + \left(\frac{\Gamma(U, f_*O_X(E))}{\Gamma(U, O_Y)[G]\cdot\beta}\right) - \sum_{y\in S_\infty}\left(\frac{O_{Y,y}[G]\cdot\beta}{(f_*O_X(E))_y}\right)$$

*in $CT(A[G])$, where $g(Y) = dim_A H^1(Y, O_Y)$ is the genus of $Y$.*

**Proof.** Let $K$ be the function field of $Y$. Then $L/K$ is a Galois extension with group $G$, and by the normal basis Theorem we can find an element $\gamma \in L$ such that $K[G]\cdot\gamma = L$. Let $O_Y[G]\cdot\gamma$ be the sheaf of $O_Y[G]$-Modules on $Y$ defined by $\Gamma(V, O_Y[G]\cdot\gamma) = \Gamma(V, O_Y)[G]\cdot\gamma \subset L$ for all open subsets $V$ of $Y$. Over a dense open subset of $Y$ the stalks of $O_Y[G]\cdot\gamma$ and $f_*O_X(E)$ are equal; at all other $y \in Y$, these stalks are rank one $O_{Y,y}[G]$-submodules of $L$ which are taken into one another by multiplication by a sufficiently high power of a uniformizing parameter at $y$. Hence by multiplying $\gamma$ by a non-zero element of $K$ having poles of high order at each point of $S_\infty$ and zeros of large order at enough points of $U = Y - S_\infty$ we arrive at an element $\beta \in L$ which satisfies conditions (a) and (b).

Since $\beta$ is a normal basis generator for $L$ over $K$, $\Gamma(U, O_Y)[G]\cdot\beta$ and $O_{Y,y}[G]\cdot\beta$ are cohomologically trivial for $G$. By Theorem 2.7, the stalks of $f_*O_X(E)$ are cohomologically trivial for $G$. The localizations of the $\Gamma(U, O_Y)[G]$-module $\Gamma(U, f_*O_X(E))$ at prime ideals of $\Gamma(U, O_Y)$ are the stalks of $f_*O_X(E)$ at the points of $U$. It follows that $\Gamma(U, f_*O_X(E))$ is also cohomologically trivial for $G$, proving (c).

We now prove (2.2). We can find an open affine subset $V$ of $Y$ which contains $S_\infty$ such that $f_*O_X(E)$ and $O_Y[G]\cdot\beta$ have equal stalks over $V - S_\infty$. Since $U \cap V \subset V - S_\infty$ this implies

$$\Gamma(U\cap V, f_*O_X(E)) = \Gamma(U\cap V, O_Y)[G]\cdot\beta. \tag{2.3}$$

We may also conclude using property (b) of Theorem 2.18 that $\Gamma(V, f_*O_X(E)) \subseteq \Gamma(V, O_Y[G]\cdot\beta)$ and

$$\frac{\Gamma(V, O_Y[G]\cdot\beta)}{\Gamma(V, f_*O_X(E))} = \bigoplus_{y\in S_\infty}\left(\frac{O_{Y,y}[G]\cdot\beta}{(f_*O_X(E))_y}\right). \tag{2.4}$$

In view of Theorem 2.18(a) we now have a diagram

$$
\begin{array}{ccc}
\Gamma(U, O_Y[G]\cdot\beta) \oplus \Gamma(V, O_Y[G]\cdot\beta) & \longrightarrow & \Gamma(U\cap V, O_Y[G]\cdot\beta) \\
\downarrow & & \| \\
\Gamma(U, f_*O_X(E)) \oplus \Gamma(V, O_Y[G]\cdot\beta) & \longrightarrow & \Gamma(U\cap V, f_*O_X(E)) \\
\uparrow & & \| \\
\Gamma(U, f_*O_X(E)) \oplus \Gamma(V, f_*O_X(E)) & \longrightarrow & \Gamma(U\cap V, f_*O_X(E))
\end{array}
\tag{2.5}
$$

in which the first and third rows are the Cech cohomology complexes $\mathbf{H}(\mathcal{U}, O_Y[G]\cdot\beta)$ and $\mathbf{H}(\mathcal{U}, f_*O_X(E))$ with respect to $\mathcal{U} = \{U, V\}$. The vertical arrows, either up or down, in the first column of this diagram are injective.

We now compute $\chi\underline{\underline{R}}\Gamma^+(O_Y[G]\cdot\beta)$ and $\chi\underline{\underline{R}}\Gamma^+(f_*O_X(E))$ using $\mathbf{H}(\mathcal{U}, O_Y[G]\cdot\beta)$ and $\mathbf{H}(\mathcal{U}, f_*O_X(E))$ in the way described after just after Theorem 2.9. By comparing the Euler characteristic in $CT(A[G])$ (in the sense of Theorem 2.9) of the middle row of (2.5) with the Euler characteristics of the top and bottom rows we find

$$
\begin{aligned}
&\chi\underline{\underline{R}}\Gamma^+(O_Y[G]\cdot\beta) + \left(\frac{\Gamma(U, f_*O_X(E))}{\Gamma(U, O_Y[G]\cdot\beta)}\right) = \\
&\chi\underline{\underline{R}}\Gamma^+(f_*O_X(E)) + \left(\frac{\Gamma(V, O_Y[G]\beta)}{\Gamma(V, f_*O_X(E))}\right)
\end{aligned}
\tag{2.6}
$$

Since $O_Y[G]\cdot\beta$ is isomorphic to $O_Y[G]$ we have

$$
\chi\underline{\underline{R}}\Gamma^+(O_Y[G]\cdot\beta) = (1-g(Y))\cdot(A[G]). \tag{2.7}
$$

from Example 2.17. Combining (2.6), (2.4) and (2.7) proves (2.2).

Specializing Theorem 2.18 and rewriting the last term on the right side of (2.2) gives the following result.

COROLLARY 2.19. *With the notation of Theorem 2.18, suppose $E$ is the zero divisor, so $O_X(E) = O_X$. Let $O_L = \Gamma(U, f_*O_X)$ and $O_K = \Gamma(U, O_Y)$. Then $O_L$ (resp. $O_K$ ) is the ring of elements of the function field $L$ of $X$ (resp. $K$ of $Y$) which are regular off of $S_\infty$. Let $O_{L,\infty}$ (resp. $O_{K,\infty}$) be ring of elements of $L$ (resp. $K$) which are regular above $S_\infty$. Then for $\beta \in L$ as in Theorem 2.18,*

$$
\chi\underline{\underline{R}}\Gamma^+(f_*O_X) = (1-g(Y))\cdot(A[G]) + \left(\frac{O_L}{O_K[G]\beta}\right) - \left(\frac{O_{K,\infty}[G]\beta}{O_{L,\infty}}\right) \tag{2.8}
$$

*in* $CT(A[G])$.

### 3. de Rham Galois structure invariants.

For the rest of this paper we assume $Y$ be a normal scheme which is proper and of finite type over a Noetherian ring $A$. The connected components of $Y$ are then irreducible; we assume that all of these components have a fixed dimension $d$. We suppose $f : X \mapsto Y$ is a tame $G$-cover relative to a divisor $D$ on $Y$ in the sense of Definitions 2.2 and 2.1. For simplicity, we assume $D$ has strictly normal crossings; a weaker hypothesis is used in §4 of [C].

Since $f$ is finite, $X$ is normal, proper and of finite type over $A$, and all of the connected components of $X$ are irreducible and of dimension $d$. We will write the sheaf $\Omega_{X/Spec(A)}$ of differentials on $X$ as $\Omega_X$. Let $\Omega^i_X = \wedge^i \Omega_X$ be the $i^{th}$ exterior power of $\Omega_X$ for $i \geq 0$. The action of $G$ on $O_X$ described in Remark 2.4 gives rise to an action of $G$ on $\Omega^i_X$ which makes $\Omega^i_X$ into a coherent sheaf of $O_X$-$G$-Modules in the sense of Definition 2.5.

From Remark 2.10 we have the following result.

PROPOSITION 3.1. *The direct image* $f_*\Omega^i_X$ *is a coherent sheaf of* $O_Y[G]$-*Modules whose stalks are cohomologically trivial for* $G$.

Hence by Theorem 2.9 we can make

DEFINITION 3.2. Let

$$\Psi(X/Y) = \sum_{i=0}^{d} (-1)^i \cdot (d-i) \cdot \chi \underline{\underline{R}}\Gamma^+(f_*\Omega^i_X)$$

in $CT(A[G])$, where $d = dim(X)$.

**Example 3.3.** If $X$ has dimension $d = 1$, then $\Psi(X/Y) = \chi \underline{\underline{R}}\Gamma^+(f_*O_X)$. In particular, if we let $A = \mathbf{Z}$ in Example 2.8 then $\Psi(X/Y)$ is the stable isomorphism class in $CT(\mathbf{Z}[G])$ of the ring of integers of $L$.

The $\Omega^i_X$ are coherent, and $H^j(Y, f_*\Omega^i_X) = H^j(X, \Omega^i_X)$ because $f$ is finite. Hence Proposition 2.14 gives

PROPOSITION 3.4. *Let* $\nu : CT(A[G]) \mapsto G_0(A[G])$ *be the natural forgetful homomorphism to the Grothendieck group of all finitely generated* $A[G]$-*modules. Then*

$$\nu(\Psi(X/Y)) = \sum_{\substack{0 \leq i \leq d \\ 0 \leq j}} (-1)^{i+j} \cdot (d-i) \cdot (H^j(X, \Omega^i_X))'$$

*where $(M)'$ is the class in $G_0(A[G])$ of the $A[G]$-module $M$.*

We now develop a generalization to schemes of the statement of Martinet's Conjecture about tame rings of integers.

DEFINITION 3.5. Let $CT(A[G])^{red} = CT(A[G])/$ { free $A[G]$-modules } Suppose $B \mapsto A$ is a homomorphism or Noetherian rings such that $A$ is a finitely generated module over the image of $B$. Restriction of operators from $A[G]$ to $B[G]$ then induces a homomorphism $res^{stab}_{A \mapsto B} : CT(A[G]) \mapsto CT(B[G])^{red}$.

*Remark 3.6.* Suppose $B = \mathbf{Z}[1/m]$ for some integer $m$ prime to the order of $G$. The natural map from locally free finitely generated $B[G]$-modules to $B[G]$-modules which are cohomologically trivial for $G$ identifies the locally free classgroup $Cl(B[G])$ of $B[G]$ with $CT(B[G])^{red}$. Suppose $\mathcal{M}$ is any maximal $B$-order in $\mathbf{Q}[G]$ containing $B[G]$. The kernel subgroup $D(B[G])$ of $Cl(B[G]) = CT(B[G])^{red}$ may be defined to be the kernel of the homomorphism $Cl(B[G]) \mapsto Cl(\mathcal{M})$ induced by tensoring modules with $\mathcal{M}$ over $B[G]$.

CONJECTURE 3.7. (Kernel Conjecture) Suppose $B = \mathbf{Z}[1/m]$ for some integer $m$ prime to the order of $G$, and that there is a ring homomorphism $B \mapsto A$ making $A$ a finitely generated $B$-module. Suppose $X$ and $Y$ are projective over $A$ and that $X$ and $Spec(A)$ are regular. Then $res^{stab}_{A \mapsto B}(\Psi(X/Y)) \in CT(B[G])^{red} = Cl(B[G])$ lies in $D(B[G])$.

It is a natural question to what extent this Conjecture is true under weaker hypotheses.

**Example 3.8.** Suppose $X$, $Y$ and $A = B = \mathbf{Z}$ are as in Example 3.3, so that $res^{stab}_{A \mapsto \mathbf{Z}}(\Psi(X/Y)) = (O_L)$ in $Cl(\mathbf{Z}[G])$. The assertion that $(O_L)$ lies in $D(\mathbf{Z}[G])$ is Martinet's Conjecture and was proved by Fröhlich (see [F, §1]).

The following Theorem is deduced in [C, Theorem 4.11] from a sharper result, which is recalled in Theorem 4.6 below.

THEOREM 3.9. *The Kernel Conjecture is true if $A$ is a finite field.*

**Remark 3.10.** If $A$ is a finite field then $res_{A \mapsto \mathbf{Z}}(\Psi(X/Y))$ does not change when $A$ is replaced by a subfield of $A$. In particular, to prove Theorem 3.9, one can reduce to the case $A = \mathbf{F}_p$ and $B = \mathbf{Z}$.

In Example 4.13 of [C] it is shown that the Kernel Conjecture for tame covers of integral models of modular curves concerns the Galois module structure of weight two cusp forms. We end this section with a result about $res^{stab}_{A\mapsto \mathbf{Z}}(\Psi(X/Y))$ when $f : X \mapsto Y$ is a tame G-cover of smooth projective irreducible curves over a finite field $A$ as in Theorem 2.18 and Corollary 2.19.

PROPOSITION 3.11. *With the notations of Corollary 2.19,*

$$res^{stab}_{A\mapsto \mathbf{Z}}(\Psi(X/Y)) = \left(\frac{O_L}{O_K[G]\beta}\right) - \left(\frac{O_{K,\infty}[G]\beta}{O_{L,\infty}}\right) \tag{3.1}$$

*in $Cl(\mathbf{Z}[G]) = CT(\mathbf{Z}[G])^{red}$. There is an integer $n$ depending only on $G$ with the following property. Suppose that the degree over $\mathbf{F}_p$ of the residue field of each point of $S_\infty$ is divisible by $n$. Then*

$$res^{stab}_{A\mapsto \mathbf{Z}}(\Psi(X/Y)) = \left(\frac{O_L}{O_K[G]\beta}\right). \tag{3.2}$$

To prove Proposition 3.11, note that if $A$ is a finite field of characteristic $p$, then $res^{stab}_{A\mapsto \mathbf{Z}}(A[G]) = 0$. Hence (3.1) follows immediately from Corollary 2.19. One now deduces (3.2) from (3.1) by applying the following Lemma with $R = O_{K,\infty}$ to the $R[G]$-module $M = \left(\frac{O_{K,\infty}[G]\beta}{O_{L,\infty}}\right)$.

LEMMA 3.12. *Suppose $R$ is a ring of characteristic $p > 0$ which is either a finite field or an excellent Dedekind ring having finite residue fields. Suppose $A$ is a finite subfield of $R$, and let $G$ be a finite group. There is an integer $n$ depending only on $G$ with the following property. Suppose $M$ is a finite $R[G]$-module which is cohomologically trivial for $G$. Then*

(a) *$M$ is a projective $A[G]$-module, and*
(b) *If the degree of each residue field of $R$ over $\mathbf{F}_p$ is divisible by $n$ then $res^{stab}_{A\mapsto \mathbf{Z}}(M) = 0$ in $Cl(\mathbf{Z}[G])$.*

**Proof.** To prove part (a), it will suffice to show $Ext^1_{A[G]}(M, M') = 0$ for all $A[G]$-modules $M'$. The spectral sequence $H^i(G, Ext^j_A(M, M')) \Longrightarrow Ext^{i+j}_{A[G]}(M, M')$ degenerates to give $H^1(G, Hom_A(M, M')) = Ext^1_{A[G]}(M, M')$. Since $p \cdot Hom_A(M, M') = 0$, it will now suffice by [S1, Chap. IX, Cor. to Thm. 4] to show $H^1(G_p, Hom_A(M, M')) = 0$ if $G_p$ is a $p$-Sylow subgroup of $G$. The argument of [S1, Thm. IX.5] shows $M$ is a free $A[G_p]$-module. It follows that $Hom_A(M, M')$ is also a free $A[G_p]$-module, so $H^1(G_p, Hom_A(M, M')) = 0$ and (a) is proved.

To prove (b), note that since $M$ is finite, $M$ is supported on a finite set of prime ideals of $R$. Hence we can reduce to the case in which $R$ is a discrete valuation ring. A power of the maximal ideal of $R$ annihilates $M$. Hence $M$ is a module for the completion of $R$, and we can reduce to the case in which $R$ is complete. Since $R$ has characteristic $p$ and finite residue field, the Teichmuller lift of the multiplicative group of the residue field of $R$ gives rise to an embedding of the residue field of $R$ into $R$. Thus we can reduce to the case in which $R$ is a finite field. In view of (a), it will suffice to show there is an integer $n$ depending only on $G$ such that if $n$ divides the relative degree $[R : \mathbf{F}_p]$ then $res^{stab}_{R \mapsto \mathbf{Z}}(M) = 0$ for all finite projective $R[G]$-modules $M$.

Let $\overline{R}$ be an algebraic closure of $R$. By [S2, Chap. 14], there are up to isomorphism only finitely many projective indecomposable $\overline{R}[G]$-modules, and each finitely generated projective $\overline{R}[G]$-module is a direct sum of projective indecomposables. Furthermore, two projective $R[G]$-modules are isomorphic if and only if they become isomorphic on tensoring with $\overline{R}$ over $R$. Finally, each projective indecomposable $\overline{R}[G]$-module is defined over the subfield $T$ of $\overline{R}$ obtained by adjoining to $\mathbf{F}_p$ all roots of unity of order dividing that of $G$. There is an $n$ depending only on $G$ such that if $n$ divides $[R : \mathbf{F}_p]$ then $R$ contains $T$ and $[R : T]$ is divisible by the ( finite) order of $Cl(\mathbf{Z}[G])$. Hence each projective $R[G]$-module $M$ has the form $M = R \otimes_T M'$ for some projective $T[G]$-module $M'$. Then $res^{stab}_{R \mapsto \mathbf{Z}}(M) = [R : T] \cdot res^{stab}_{T \mapsto \mathbf{Z}}(M') = 0$ in $Cl(\mathbf{Z}[G])$, which completes the proof.

## 4. Root numbers and Galois structure over finite fields.

In this section we assume $X$ and $Y$ are projective schemes over $A = \mathbf{F}_p$ and that $X$ is regular. Since $A$ is perfect, this implies $X$ is smooth over $A$. As in §3 we assume $f : X \mapsto Y$ is a tame $G$ cover over divisor $D$ on $Y$ which has strictly normal crossings.

By Remark 2.4, $f : X \mapsto Y$ is finite and surjective. Hence for each element $y$ of the set $Y^0$ of closed points of $Y$ there is an $x(y) \in X^0$ lying over $y$. Define $G_{x(y)}$ ( resp. $I_{x(y)}$) to be the decomposition group (resp. the inertia group) of $x(y)$ in $G$. The Frobenius $Frob(x(y))$ of $x(y)$ over $y$ is the unique element of $G_{x(y)}/I_{x(y)}$ which induces the automorphism $\alpha \mapsto \alpha^{p^{deg(y)}}$ of the residue field $k(x(y))$ of $x(y)$, where $deg(y)$ is the degree of the residue field $k(y)$ over $\mathbf{F}_p$.

Let $V$ be a finite dimensional complex representation of $G$, and let $Y'$ be a Zariski open or closed subset of $Y$. The Artin $L$-function of $V$ with respect to $Y'$ is

$$(4.1) \qquad L(Y',V,t) = \prod_{y \in Y' \cap Y^0} det(1 - Frob(x(y))\ t^{deg(y)} \mid V^{I_{x(y)}})^{-1}.$$

Since $Y'$ is open or closed in $Y$, it follows from Grothendieck's Theorem on the rationality of L-series [M, Thm. VI.13.3] that $L(Y',V,t)$ is a finite product

$$(4.2) \qquad L(Y',V,t) = \prod_{i,k} (1 - a_{i,k}\ t)^{(-1)^{i+1}}$$

for some $a_{i,k} \in \overline{\mathbf{Q}}$. Define

$$(4.3) \qquad \epsilon(Y',V) = \prod_{i,k} (-a_{i,k})^{(-1)^{i+1}}.$$

Using work of Milne and Illusie, it is shown in [C, Theorem 6.2] that the class $\Psi(X/Y)$ in $CT(\mathbf{F}_p[G])$ both determines and is determined by the $p$-adic absolute values of the $\epsilon(Y,V)$ as $V$ varies over all of the complex irreducible representations of $G$. We will not state this result precisely here, but we will recall its consequences to $res^{stab}_{A\mapsto \mathbf{Z}}(\Psi(X/Y)) \in Cl(\mathbf{Z}[G])$.

For $F$ a perfect field let $\overline{F}$ be an algebraic closure of $F$ and let $\Omega_F = Gal(\overline{F}/F)$. Let $J(E)$ denote the group of ideles of a number field $E$. Define $J(\overline{\mathbf{Q}})$ to be the direct limit over all number fields $E$ of $J(E)$. Let $R_G$ be the additive group of virtual characters over $\overline{\mathbf{Q}}$ of the finite group $G$. Let $U(\mathbf{Z}[G])$ be the unit ideles of the idele group $J(\mathbf{Q}[G])$ of $\mathbf{Q}[G]$. As in [F] we have a determinant map $Det : J(\mathbf{Q}[G]) \mapsto Hom_{\Omega_{\mathbf{Q}}}(R_G, J(\overline{\mathbf{Q}}))$. Fröhlich's Hom-description of the classgroup $Cl(\mathbf{Z}[G])$ states that there is an isomorphism

$$(4.4) \qquad \frac{Hom_{\Omega_{\mathbf{Q}}}(R_G, J(\overline{\mathbf{Q}}))}{Hom_{\Omega_{\mathbf{Q}}}(R_G, \overline{\mathbf{Q}}^*) \cdot Det(U(\mathbf{Z}[G]))} \xrightarrow{\mu} Cl(\mathbf{Z}[G])$$

normalized in the following way. Suppose $\alpha = (\alpha_v) \in J(\mathbf{Q}[G])$ is an idele of $\mathbf{Q}[G]$ such that $\alpha_\infty = 1$ if $\infty$ is the infinite place of $\mathbf{Q}$. Then $M_\alpha = \bigcap_{v \text{ finite}} \mathbf{Z}_v[G]\alpha_v$ is a locally free rank one $\mathbf{Z}[G]$-module. The homomorphism $\mu$ is the unique one sending the class of $Det(\alpha) \in Hom_{\Omega_{\mathbf{Q}}}(R_G, J(\overline{\mathbf{Q}}))$ to the class of $M_\alpha$ for all $\alpha$ as above.

DEFINITION 4.1. For each place $v$ of $\mathbf{Q}$ let $i_v : (\overline{\mathbf{Q}})^*_v = (\overline{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathbf{Q}_v)^* \hookrightarrow J(\overline{\mathbf{Q}})$ be the natural inclusion. Let $|\ \ |_v$ be the unique extension to $\overline{\mathbf{Q}}_v$ of the usual v-adic absolute value on $\mathbf{Q}_v$. We will also use $|\ |_v$ to denote the unique function $|\ |_v : (\overline{\mathbf{Q}})_v \mapsto (\overline{\mathbf{Q}})_v$ such that $|\ |_v \circ t = t \circ |\ |_v$ for all embeddings $t : \overline{\mathbf{Q}}_v \hookrightarrow (\overline{\mathbf{Q}})_v$ over $\mathbf{Q}_v$. We can write each element $\alpha$ of $J(\overline{\mathbf{Q}})$ as $\alpha = (\alpha_v)_v$ with $\alpha_v \in (\overline{\mathbf{Q}})_v^*$. Let $|\ |_{tot} : J(\overline{\mathbf{Q}}) \mapsto J(\overline{\mathbf{Q}})$ be the homomorphism for which $|(\alpha_v)_v|_{tot} = (|\alpha_v|_v)_v$.

DEFINITION 4.2. For each complex representation $V$ of $G$ let $\overline{V}$ be the dual of $V$ and let $\chi_V$ be the character of $V$. Suppose $Y'$ is a Zariski open or closed subset of $Y$ and that $v$ is a place of $\mathbf{Q}$. Define $\epsilon(Y')$ (resp. $\epsilon_v(Y')$, resp. $|\epsilon_v(Y')|_{tot}$) to be the function in $Hom(R_G, \overline{\mathbf{Q}}^*)$ (resp. $Hom(R_G, J(\overline{\mathbf{Q}}))$, resp. $Hom(R_G, J(\overline{\mathbf{Q}}))$) which sends the character $\chi_V \in R_G$ to $\epsilon(Y', \overline{V})$ (resp. $i_v\epsilon(Y', \overline{V})$, resp. $|i_v\epsilon(Y', \overline{V})|_{tot}$ ). Let $\infty$ be the infinite place of $\mathbf{Q}$. The finite place of $\mathbf{Q}$ determined by a rational prime $l$ will also be denoted by $l$.

From the Euler product (4.1) one sees that $L(Y', V^\lambda, t) = L(Y', V, t)^\lambda$ for $\lambda \in Aut(\mathbf{C}/\mathbf{Q})$. This together with (4.2) and (4.3) show

LEMMA 4.3. *$\epsilon(Y')$ and $\epsilon_v(Y')$ are $\Omega_{\overline{\mathbf{Q}}}$-equivariant for all places $v$ of $\mathbf{Q}$.*

Recall that a representation $V$ of $G$ is symplectic if there is a non-degenerate alternative bilinear form on $V$ which is $G$-invariant. We now define a counterpart of the root number class defined by Cassou-Noguès for tame finite Galois extensions of number fields.

LEMMA 4.4. *If $V$ is symplectic then $\chi_V$ is real valued and $\epsilon(Y, V) = \epsilon(Y, \overline{V})$ is totally real. There is a function $h_\infty \in Hom_{\Omega_{\mathbf{Q}}}(R_G, J(\overline{\mathbf{Q}}))$ defined by*

$$h_\infty(\chi_V) = \left\{ \begin{array}{ll} \dfrac{\epsilon_\infty(Y)(\chi_V)}{|\epsilon_\infty(Y)|_{tot}(\chi_V)} & \text{if } V \text{ is symplectic} \\ \\ 1 & \text{otherwise} \end{array} \right\}$$

*Hence the root number class $W_{X/Y} = \mu(h_\infty)$ is a well defined element of $Cl(\mathbf{Z}[G])$.*

**Proof.** This follows directly from Lemma 4.3. Note that for symplectic $V$, $h_\infty(\chi_V)$ is the idele with trivial finite components and component in $\overline{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathbf{R}$ given by the signs ($= \pm 1$) of $\epsilon(Y, \overline{V})$ at infinity.

DEFINITION 4.5. (c.f. [M, Remark I.3.7]) The different $\delta_{X/Y}$ of $X$ over $Y$ is the annihilator of $\Omega^1_{X/Y}$ in $O_X$. The closed subscheme $B_{X/Y}$ of $X$ defined by $\delta_{X/Y}$ is the branch locus of $f : X \mapsto Y$ in $X$. The set $b = b_{X/Y} = f(B_{X/Y})$ is a closed subset of $Y$ because $f$ is finite, and will be called the branch locus of $f$ in $Y$. Let $U = U_{X/Y}$ be the open subset $Y - b_{X/Y}$ of $Y$.

We can now state the main result of [C].

THEOREM 4.6. *Let $f : X \mapsto Y$ be a tame $G$-cover of projective schemes over $A = \mathbf{F}_p$ relative to a divisor $D$ on $Y$ which has strictly normal crossings. Suppose $X$ is regular. Then*

$$res^{stab}_{\mathbf{F}_p \mapsto \mathbf{Z}}(\Psi(X/Y)) = W_{X/Y} + R_{X/Y}$$

*where the root number class $W_{X/Y}$ is defined in Lemma 4.4, and the ramification class $R_{X/Y}$ is defined as follows. Let $b = b_{X/Y}$ be the branch locus of $f : X \mapsto Y$ in $Y$. Then $|\epsilon_p(b)|^{-1}_{tot} \cdot \epsilon_\infty(b)$ lies in $Hom_{\Omega_\mathbf{Q}}(R_G, J(\overline{\mathbf{Q}}))$, so we may define*

$$R_{X/Y} = \mu(|\epsilon_p(b)|^{-1}_{tot} \cdot \epsilon_\infty(b))$$

*in $Cl(\mathbf{Z}[G])$.*

The following result is Theorem 6.9 of [C].

THEOREM 4.7. *The classes $W_{X/Y}$ and $R_{X/Y}$ lie in the Kernel group $D(\mathbf{Z}[G])$. The order of $W_{X/Y}$ is one or two, and $W_{X/Y}$ is trivial if $G$ has no symplectic representations. The class $R_{X/Y}$ is trivial if $X \mapsto Y$ is étale or if $dim(X) = 1$.*

In view of Remark 3.10, Theorems 4.6 and 4.7 establish Theorem 3.9 (the Kernel Conjecture over finite fields).

**Example 4.8** Suppose $f : X \mapsto Y$ is a tame $G$-cover of smooth projective irreducible curves over $A = \mathbf{F}_p$. Then $b \subset Y$ consists of the finitely many closed points of $Y$ over which $f$ is ramified. Let $V$ be a complex representation of $G$. By equations (4.3), (4.2) and (4.1),

$$(4.5)\qquad \begin{aligned} \epsilon(b, \overline{V}) &= \prod_{y \in b} det(-Frob(x(y)) \mid \overline{V}^{I_{x(y)}})^{-1} \\ &= \prod_{y \in b} det(-Frob(x(y)) \mid V^{I_{x(y)}}) \end{aligned}$$

where $x(y)$ is a closed point of $X$ over $y$ and $I_{x(y)}$ is the inertia group of $x(y)$. Thus $\epsilon(b)(\chi_V) = \epsilon(b, \overline{V})$ is the product over $y \in b$ of the local non-ramified characteristic of $V$ at $y$ defined by Fröhlich in [F, eq. (1.1), p. 149].

In particular, since $\epsilon(b, \overline{V})$ is a root of unity, $|\epsilon_p(b)|_{tot}$ is trivial. The results of [F] on non-ramified characteristics imply $\epsilon_\infty(b)$ lies in $Det(\mathbf{Z}_\infty[G]^*) \subset Det(U(\mathbf{Z}[G]))$. Thus (4.4) shows

$$R_{X/Y} = \mu(|\epsilon_p(b)|_{tot}^{-1} \cdot \epsilon_\infty(b)) = 0$$

as stated in Theorem 4.7.

*Remark 4.9* In dimensions greater than 1, $R_{X/Y}$ can be non-trivial. For example, in [C, Example 6.13] it is shown that if $n > 2$ is prime, there is a tame Kummer $G$-cover $X \mapsto Y$ of projective spaces over a finite field of characteristic $p$ such that $res^{stab}_{\mathbf{F}_p \mapsto \mathbf{Z}}(\Psi(X/Y)) = R_{X/Y}$ and $R_{X/Y}$ has exact order $n$.

We conclude by restating Theorems 4.6 and 4.7 for irreducible $X$ of dimension 1 in a way that parallels Taylor's Theorem concerning Fröhlich's conjecture for tame rings of integers.

COROLLARY 4.10. *Let $f : X \mapsto Y$ be a tame $G$-cover of smooth projective irreducible curves over $A = \mathbf{F}_p$. Let $S_\infty$ be a finite non-empty set of closed points of $Y$. Let $O_L$ ( resp. $O_K$ ) be the ring of elements of the function field $L$ of $X$ ( resp. $K$ of $Y$ ) which are regular off of $S_\infty$. Let $O_{L,\infty}$ ( resp. $O_{K,\infty}$ ) be ring of elements of $L$ ( resp. $K$ ) which are regular above $S_\infty$. There is a normal basis generator $\beta$ of $L$ over $K$ such that $O_L$ contains $O_K[G] \cdot \beta$ with finite index and $O_{K,\infty}[G] \cdot \beta$ contains $O_{L,\infty}$ with finite index, where all of these $G$-modules are cohomologically trivial for $G$. For all such $\beta$ one has*

$$\left(\frac{O_L}{O_K[G]\beta}\right) - \left(\frac{O_{K,\infty}[G]\beta}{O_{L,\infty}}\right) = W_{X/Y} \tag{4.6}$$

*in $Cl(\mathbf{Z}[G])$, where $W_{X/Y}$ is as in Lemma 4.4. There is an integer $n$ depending only on $G$ with the following property. Suppose that the degree over $\mathbf{F}_p$ of the residue field of each point of $S_\infty$ is divisible by $n$. Then (4.6) can be simplified to*

$$(\frac{O_L}{O_K[G]\beta}) = W_{X/Y} \tag{4.7}$$

*in $Cl(\mathbf{Z}[G])$.*

**Proof.** Combine Proposition 3.11 with Theorems 4.6 and 4.7.

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