

HARM VOSKUIL

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Non archimedean Hopf surfaces.

par HARM VOSKUIL

0. Introduction

We study the non-archimedean Hopf surfaces. A Hopf surface is a surface defined over a complete field K , which has $K^2 - \{(0,0)\}$ as its universal covering. So it can be described as $K^2 - \{(0,0)\}/\Gamma$, where Γ is a discrete group acting discontinuously on $K^2 - \{(0,0)\}$.

The complex Hopf surfaces are very well-known. They have been studied in detail by Kodaira (See [Ko.1] and [Ko.2]).

The p -adic Hopf surfaces are less known, although they are treated as examples in some articles (See [GG], [Mus.1], [Mus.2] and [U]). All those articles mention only the diagonal Hopf surfaces $K^2 - \{(0,0)\}/\Gamma$ with Γ generated by a single element γ such that $\gamma(z_1, z_2) = (\alpha z_1, \beta z_2)$ with $|\alpha|, |\beta| < 1$. The most detailed study is given by Mustafin (See [Mus.1] and [Mus.2]). So there will be some overlap with his work.

This article is divided into three parts. In the first paragraph we will describe the group Γ . We will prove that $\Gamma \simeq \mathbb{Z}/l\mathbb{Z} \times \mathbb{Z}$ for some $l \in \mathbb{Z}_{>0}$. So these results are the same as in the complex case.

In the second paragraph we will give some pure affinoid coverings of a Hopf surface X , such that the reduction consists of non-singular components. Here we will use the theory of toroidal embeddings (see [KKMS], [O,1] and [O.2]).

In the third paragraph we will determine the cohomology of the line bundles on a Hopf surface. We will show that there is a Serre duality for the line bundles. This is also stated in [U] when $\text{char}(K) = 0$.

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1. The structure of the group Γ

We will first recall some basic definitions.

DEFINITIONS. Let K be a complete non-archimedean valued field.

An *affinoid algebra* A over K is a K -algebra which is a finite extension of $K \langle z_1, \dots, z_n \rangle$ for some n .

An *affinoid space* $Sp(A)$ is the set of all maximal ideals of the affinoid algebra A .

On A we define a (semi-) norm : the *spectral (semi-) norm* $\|f\| = \sup_{x \in Sp(A)} |f(x)|$. The spectral semi-norm is a norm if there are no nilpotent elements $\neq 0$ in A .

Example : The set $Y = \{(z_1, z_2) \in K^2 \mid |z_1| \leq 1, |z_2| \leq 1\}$ is an affinoid space. The affinoid algebra belonging to Y is $K \langle z_1, z_2 \rangle$.

DEFINITIONS. A surface Y is called *separated* if Y has an admissible affinoid covering $\{Y_i \mid i \in I\}$ such that if $Y_i \cap Y_j \neq \emptyset$ then $Y_i \cap Y_j$ is affinoid and the canonical homomorphism $\mathcal{O}(Y_i) \hat{\otimes} \mathcal{O}(Y_j) \rightarrow \mathcal{O}(Y_i \cap Y_j)$ is surjective.

We write $U \in Sp(A)$ and say U is *relatively compact* in $Sp(A)$ if there exists an affinoid generating system $\{f_1, \dots, f_r\}$ of A over K such that :

$$U \subset \{x \in Sp(A) \mid |f_1(x)| < 1, \dots, |f_r(x)| < 1\}.$$

A surface Y is called *proper* over K if Y is separated and has two finite affinoid coverings $\{X_i^{(1)} \mid i = 1..n\}$ and $\{X_i^{(2)} \mid i = 1..n\}$ such that $X_i^{(1)} \in X_i^{(2)}$ for all $i = 1..n$.

A *Hopf surface* is a proper rigid analytic surface that has $K^2 - \{(0,0)\}$ as its universal analytic covering.

Remark. In [U] a surface that we call proper is called compact.

In order to show that our definitions of a Hopf surface is meaningful we have to show that $K^2 - \{(0,0)\}$ is simply connected. We will do this in the following lemma.

DEFINITION. A connected analytic space X is called *simply connected* if the only connected analytic covering of X is equal to $id : X \rightarrow X$.

LEMMA 1.1. *The analytic space $K^2 - \{(0,0)\}$ is simply connected.*

Proof. Let us write $U = K^2 - \{(0,0)\} = U_1 \cup U_2$, where $U_1 = K^* \times K$ and $U_2 = K \times K^*$. Since K and K^* are simply connected, the same is true of $K \times K^*$ and $K^* \times K^*$ (This is theorem 1 in [vdP]). Now $U_1 \cup U_2$ is also simply connected, since U_1, U_2 and $U_1 \cap U_2$ are simply connected. Indeed let S be a locally constant sheaf on U_1 , then $S|_{U_i}$ is constant since U_i is simply connected (See [vdP]). This shows that S is constant on U , since $S(U_1)|_{U_1 \cap U_2} = S(U_2)|_{U_1 \cap U_2}$.

DEFINITION. A group Γ acts *discontinuously* on an analytic space X if for every affinoid subspace $A \subset X$ the set $\{\gamma \in \Gamma | A \cap \gamma(A) \neq \emptyset\}$ is finite.

LEMMA 1.2. A Hopf surface is a proper rigid analytic surface of the form $K^2 - \{(0,0)\}/\Gamma$. Here Γ is a group of automorphisms of $K^2 - \{(0,0)\}$ that acts discontinuously and without fixed points.

Proof. The universal covering space of a Hopf surface X is $U = K^2 - \{(0,0)\}$. Let π be the analytic map $\pi : U \rightarrow X$. Let Γ be the group of covering transformations of U , so $\Gamma = \{\gamma : U \rightarrow U | \pi \circ \gamma = \pi\}$.

We have to show that $U/\Gamma \simeq X$. Clearly Γ is discrete and $U/\Gamma \rightarrow X$ is a covering of X . So we only have to prove that $U/\Gamma \rightarrow X$ is bijective. Let us look at $U \times_X U = \{(u_1, u_2) | \pi(u_1) = \pi(u_2)\}$. Now the projection on the first factor $p_1 : U \times_X U \rightarrow U$ is again an analytical covering. Since U is simply connected, we must have $p_1 : Z \xrightarrow{\sim} U$ for every connected component Z of $U \times_X U$. The same is true for $p_2 : U \times_X U \rightarrow U$, the projection on the second factor.

Let $(a, b) \in U \times_X U$ and let Z be the connected component of $U \times_X U$ containing (a, b) . Now we have the following commutative diagram :

$$\begin{array}{ccc}
 & Z & \\
 p_1 \swarrow & & \searrow p_2 \\
 U & \xrightarrow{\gamma} & U
 \end{array}$$

Here $\gamma \in \Gamma$ and $\gamma(a) = b$. This shows that $U/\Gamma \simeq X$.

Since X is proper, there exists a finite covering $\{X_i | i \in I\}$ of X . We may assume that $\pi^{-1}(X_i)$ is a disjoint union of copies of X_i . Now the covering $\mathcal{C} = \{Y \subseteq K^2 - \{(0,0)\} \mid Y \in \pi^{-1}(X_i) \text{ for some } i \in I\}$ of $K^2 - \{(0,0)\}$ shows that Γ acts discontinuously and without fixed points.

LEMMA 1.3. *An analytic automorphism g of $K^2 - \{(0,0)\}$ can be extended uniquely to an analytic automorphism of K^2 .*

Proof. Let g be defined by $g(z_1, z_2) = (g_1(z_1, z_2), g_2(z_1, z_2))$. Let $U = K^2 - \{(0,0)\}$ be $U = U_1 \cup U_2$ and $U_1 = K^* \times K$ and $U_2 = K \times K^*$. We can expand g_1 and g_2 into a convergent power series on U_1 and U_2 :

$$g_1|_{U_1} = \sum_{m \geq 0} a_{n,m} z_1^n z_2^m$$

$$g_1|_{U_2} = \sum_{n \geq 0} b_{n,m} z_1^n z_2^m$$

These two power series have to be equal on $X_1 \cap X_2$ so we have

$$g_1|_U = \sum_{n,m \geq 0} a_{n,m} z_1^n z_2^m.$$

This power series of g_1 is also holomorphic in $(0,0)$. So g_1 is an analytic function on K^2 . The same is true of g_2 . Therefore we have a unique extension of g to an analytic automorphism of K^2 . It is clear that $g(0,0) = (0,0)$.

DEFINITIONS. Let $|K^*| := \{|a| \mid a \in K^*\}$ be the norm group of K . Let $R \in |K^*|$ and $R > 1$. We now define :

$$B_R = \{(z_1, z_2) \in K^2 \mid \max(|z_1|, |z_2|) \leq R\}$$

$$\partial B_R = \{(z_1, z_2) \in K^2 \mid \max(|z_1|, |z_2|) = R\}$$

A *contraction* $\gamma \in \Gamma$ is an automorphism of K^2 such that :

$$\gamma(\partial B_R) \subset B_R - \partial B_R$$

LEMMA 1.4. *Let $F \subset K^2 - \partial B_R$ be a connected affinoid subspace. Then either $F \subset K^2 - B_R$ or $F \subset B_R - \partial B_R$.*

Proof. Let $R' \in |K^*|$, $R' > R$. We consider the following affinoid subspaces of K^2 :

$$\begin{aligned} I_1 &= \{(z_1, z_2) \in K^2 \mid |z_1| \leq |z_2| \leq R\} \\ I_2 &= \{(z_1, z_2) \in K^2 \mid |z_2| \leq |z_1| \leq R\} \\ I_3 &= \{(z_1, z_2) \in K^2 \mid |z_1| \leq |z_2|, R \leq |z_2| \leq R'\} \\ I_4 &= \{(z_1, z_2) \in K^2 \mid |z_2| \leq |z_1|, R \leq |z_1| \leq R'\} \end{aligned}$$

For $R' > R$ sufficient large we have :

$$F = F_1 \cup F_2 \cup F_3 \cup F_4 \quad \text{where} \quad F_i := F \cap I_i, \quad i = 1 \dots 4.$$

Because $(I_1 \cup I_2) \cap (I_3 \cup I_4) \subseteq \partial B_R$ we have $(F_1 \cup F_2) \cap (F_3 \cup F_4) = \emptyset$. Since F is connected, either $F_1 \cup F_2 = \emptyset$ or $F_3 \cup F_4 = \emptyset$. This proves the lemma.

LEMMA 1.5. *The group Γ contains a contraction γ .*

Proof. The subspace $\partial B_R \subset K^2$ is the union of the two affinoid subspaces $\{(z_1, z_2) \in K^2 \mid |z_1| = R, |z_2| \leq R\}$ and $\{(z_1, z_2) \in K^2 \mid |z_2| = R, |z_1| \leq R\}$. The intersection of these two subspaces is connected and non-empty, so ∂B_R is connected.

Furthermore since the Hopf surface $X = K^2 - \{(0,0)\}/\Gamma$ is proper, we know that Γ is not finite. Indeed, suppose Γ is finite. Now $\mathcal{O}(X) = \mathcal{O}(K^2 - \{(0,0)\})^\Gamma$ is not finite dimensional over K . Since $\mathcal{O}(K^2 - \{(0,0)\})$ is not finite dimensional. This shows that X cannot be proper (See [BGR] or [Ki.1]).

Since Γ is not finite, there exists a $\gamma \in \Gamma$ such that $\gamma(\partial B_R) \cap \partial B_R = \emptyset$. Now applying the previous lemma, we have one of the following :

- 1) $\gamma(\partial B_R) \subset B_R - \partial B_R$
- 2) $\gamma(\partial B_R) \subset K^2 - B_R$

In the first case γ is already a contraction, so then the lemma is true. In the second case we have : $B_R \cap \gamma(\partial B_R) = \emptyset$. We now apply lemma 1.4 with $F = \gamma^{-1}(B_R)$. So we have : $\gamma^{-1}(B_R) \subset B_R - \partial B_R$, since $\gamma^{-1}((0,0)) = (0,0)$. This proves that $\gamma^{-1} \in \Gamma$ is a contraction.

PROPOSITION 1.1. *The group Γ contains a contraction γ such that*

$$\Gamma_0 = \langle \gamma \rangle$$

is in the centre of Γ and $[\Gamma : \Gamma_0] < \infty$.

Proof. Let $\gamma \in \Gamma$ be a contraction defined by

$$\gamma(z_1, z_2) = (a(z_1, z_2), b(z_1, z_2)),$$

where $a(z_1, z_2) = \sum_{n+m \geq 1} a_{n,m} z_1^n z_2^m$ and $b(z_1, z_2) = \sum_{n+m \geq 1} b_{n,m} z_1^n z_2^m$.

Since $\gamma(\partial B_R) \subset B_R - \partial B_R$ we have :

$$R > \max_{z \in \partial B_R} |a(z_1, z_2)| = \max_{z \in \partial B_R} \left| \sum_{n+m \geq 1} a_{n,m} z_1^n z_2^m \right| = \max |a_{n,m}| R^{n+m}.$$

A similar result is true for $b(z_1, z_2)$, so we may conclude :

$$\exists r \in |K^*|, \quad r > R, \quad \gamma(B_R) \subset B_r.$$

The linear part of γ has a matrix $\begin{pmatrix} a_{10} & a_{01} \\ b_{10} & b_{01} \end{pmatrix}$. All coefficients have an absolute value < 1 . In particular the order of γ is not finite. It is clear that if $R' \leq R$ then $\gamma(B_{R'}) \subset B_S$, $S := \frac{R'}{R}$, because :

$$\max_{n+m \geq 1} |a_{n,m}| (R')^{n+m} = \max_{n+m \geq 1} |a_{n,m}| R^{n+m} \left(\frac{R'}{R} \right)^{n+m} \leq r \cdot \frac{R'}{R}.$$

If $R' > R$ we look at the subspace

$$Y = \{(z_1, z_2) \in K^2 \mid R \leq \max(|z_1|, |z_2|) \leq R'\} \subset B_{R'}.$$

The space Y is the union of two affinoid subspaces Y_1 and Y_2 , where

$$Y_1 = \{(z_1, z_2) \in K^2 \mid R \leq |z_1| \leq R', |z_2| \leq |z_1|\}$$

and

$$Y_2 = \{(z_1, z_2) \in K^2 \mid R \leq |z_2| \leq R', |z_1| \leq |z_2|\}.$$

Since γ is not of finite order and Γ acts discontinuously on $K^2 - \{(0,0)\}$, we have :

$$\exists n > 0, \gamma^n(Y) \cap Y = \emptyset$$

In particular we have : $\gamma^n(Y) \cap \partial B_R = \emptyset$.

Now Y is connected and we may apply lemma 1.4, therefore we have :

$$\gamma^n(Y) \subset K^2 - B_R \text{ or } \gamma^n(Y) \subset B_R.$$

Since $\gamma^n(B_R) \subset B_R$ and $B_R \cup Y$ is also connected, we must have : $\gamma^n(B_R) \subset B_R$.

Now we have proved that every point $p \in K^2 - \{(0,0)\}$ has a Γ_0 -image in the subspace

$$Z = \{(z_1, z_2) \in K^2 - \{(0,0)\} | \rho \leq \max(|z_1|, |z_2|) \leq R\} \subseteq K^2 - \{(0,0)\},$$

where $\rho < R$ is taken such that $B_\rho \subset \gamma(B_R)$. This subspace Z is the union of two affinoid subspace Z_1 and Z_2 , where

$$Z_1 = \{(z_1, z_2) \in K^2 | \rho \leq |z_1| \leq R, |z_2| \leq |z_1|\}$$

and

$$Z_2 = \{(z_1, z_2) \in K^2 | \rho \leq |z_2| \leq R, |z_1| \leq |z_2|\}.$$

If $[\Gamma : \Gamma_0]$ were not finite there would be an infinite number of elements $\alpha \in \Gamma$ such that :

$$\alpha(Z_1) \cap Z_1 \neq \emptyset, \alpha(Z_2) \cap Z_2 \neq \emptyset.$$

Since Γ acts discontinuously on $K^2 - \{(0,0)\}$, we must have $[\Gamma : \Gamma_0]$ is finite.

Now we may suppose that $\Gamma_0 \subset \Gamma$ is a normal subgroup, since we can replace Γ_0 by the intersection of all subgroups conjugated with Γ_0 . So for an elements $a \in \Gamma$ we have $a\gamma a^{-1} = \gamma^n$ for some $n \in \mathbb{Z}$. The linear part of γ has eigenvalues with absolute value < 1 . This shows that only $a\gamma a^{-1} = \gamma$ can occur. This proves that Γ_0 is in the centre of Γ .

THEOREM 1.1. *There exist global parameters t_1, t_2 of K^2 such that a contraction γ has the following form :*

$$\gamma(t_1, t_2) = (\alpha_1 t_1 + \lambda t_2^m, \alpha_2 t_2).$$

Here $0 < |\alpha_1| \leq |\alpha_2| < 1$ and $\lambda = 0$ if $\alpha_1 \neq \alpha_2^m$ otherwise $\lambda \in K$.

Proof. This will be proved in the following three lemmas.

LEMMA 1.6. *There exist formal parameters $t_1, t_2 \in K[[z_1, z_2]]$ such that $\gamma(t_1, t_2)$ has the form described in theorem 1.1 above.*

Proof. It is clear that γ also acts on the formal local ring $K[[z_1, z_2]]$. We denote this action with $\tilde{\gamma}$. Let $\underline{m}, \underline{m}^2, \underline{m}^3, \dots$ be the powers of the maximal ideal $\underline{m}$ of $K[[z_1, z_2]]$.

Now, after possibly an extension of degree two of K , $\tilde{\gamma}$ has two eigenvalues α_1, α_2 on $\underline{m}/\underline{m}^2$. Let us take $|\alpha_1| \leq |\alpha_2|$. We can make a linear transformation of the variables z_1 and z_2 such that the linear part of γ , which is the matrix of $\tilde{\gamma}$ on $\underline{m}/\underline{m}^2$, has the form $\begin{pmatrix} \alpha_1 & * \\ 0 & \alpha_2 \end{pmatrix}$ and $* \neq 0$ only if $\alpha_1 = \alpha_2$.

With respect to this basis $z_1, z_2, z_1^2, z_1 z_2, z_2^2, \dots, z_2^n$ of $\underline{m}/\underline{m}^{n+1}$, $n \geq 1$ we find $\tilde{\gamma}$ has a triangular matrix :

$$\begin{pmatrix} \alpha_1 & & & & \\ & \alpha_2 & & & * \\ & & \alpha_1^2 & & \\ & 0 & & \ddots & \\ & & & & \alpha_2^n \end{pmatrix}$$

If $\alpha_1 \neq \alpha_2$ then the eigenvalue α_2 only occurs once. So modulo $\underline{m}^{n+1}$ there is an unique one-dimensional eigenspace belonging to α_2 . Taking the limit $n \rightarrow \infty$ we get an unique formal power series $t_2 = z_2 + \dots$ such that $\tilde{\gamma}(t_2) = \alpha_2 t_2$.

If $\alpha_1 = \alpha_2$ then the value eigenvalue α_2 occurs twice. So modulo $\underline{m}^{n+1}$ there is an unique two-dimensional eigenspace belonging to α_2 . Taking the limit $n \rightarrow \infty$ we can find in this eigenspace a formal power series $t_2 = z_2 + \dots$ such that $\tilde{\gamma}(t_2) = \alpha_2 t_2$.

If $\alpha_1 \neq \alpha_2^m \forall m \geq 1$ then also the eigenvalue α_1 occurs only once. Again we find an unique one-dimensional eigenspace belonging to α_1 and a power series $t_1 = z_1 + \dots \in K[[z_1, z_2]]$ such that $\tilde{\gamma}(t_1) = \alpha_1 t_1$.

If $\alpha_1 = \alpha_2^m$ for a certain $m \geq 1$ then the eigenvalue α_1 always occurs twice for $n \geq m \geq 1$. It has also an eigenvector t_2^m . Now we can find a power series $t_1 \in K[[z_1, z_2]]$ such that $\tilde{\gamma}(t_1) = \alpha_1 t_1 + \lambda t_2^m$. This t_1 is not unique, we could also have taken $t_1 + \mu t_2^m, \mu \in K$. This proves the lemma for some formal parameters t_1, t_2 in $(0, 0)$.

LEMMA 1.7. *The parameters t_1 and t_2 constructed in lemma 1.6 are holomorphic functions on K^2 .*

Proof. Let us choose an $R \in |K^*|, R \gg 0$. Let

$$V = \{f \in \mathcal{O}(B_R) | f(0,0) = 0\}$$

be the Banach space of functions that are holomorphic on B_R . On V we have the sup-norm.

The contraction γ induces an action $\tilde{\gamma} : V \rightarrow V$ on V . In the proof of proposition 1.1 we have shown that $\exists r \in |K^*| r < R$ $\gamma(B_R) \subset B_r$. Since $\gamma(B_R) \subset B_r \subset B_R$, the operator $\tilde{\gamma}$ acting on V is compact. The p -adic theory of compact operators (See [G]) tells us that for every $\lambda \in K^*$ we have :

- 1) $K_\lambda = \bigcup_{n \geq 1} \ker((\tilde{\gamma} - \lambda)^n : V \rightarrow V)$ is finite dimensional
- 2) K_λ has a $\tilde{\gamma}$ -invariant closed complement W_λ in V and $(\tilde{\gamma} - \lambda) : W_\lambda \xrightarrow{\sim} W_\lambda$

So we can suppose $V = K_\lambda \oplus W_\lambda$ for some $\lambda \in K^*$. Furthermore we have $V/(z_1, z_2)^n V \simeq \underline{m}/\underline{m}^{n+1}$, where $\underline{m}$ is the maximal ideal of $K[[z_1, z_2]]$. As in the previous lemma this shows that the eigenspace K_λ for $\lambda = \alpha_1$ or $\lambda = \alpha_2$ has dimension 1 or 2. Specially the parameters t_1 and t_2 of lemma 1.6 are in fact holomorphic functions, since they are holomorphic on any $B_R, R \gg 0, R \in |K^*|$.

LEMMA 1.8. *The map $t : K^2 \rightarrow K^2$ defined by $t(z_1, z_2) = (t_1, t_2)$ is invertible, so t_1, t_2 are global parameters of K^2 and $t : K^2 \rightarrow K^2$ is an isomorphism.*

Proof. Let $\rho \in |K^*|$ be sufficiently small. Then the map

$$t : B_\rho = \{(z_1, z_2) \in K^2 | \max(|z_1|, |z_2|) \leq \rho\} \rightarrow B_\rho$$

is an isomorphism. This can be seen by considering the linear part of t . Let s_0 be the inverse of t . Let δ be the transformation on the second B_ρ defined by $\delta(a_1, a_2) = (\alpha_1 a_1 + \lambda a_2^n, \alpha_2 a_2)$. It is clear that $t \circ \gamma = \delta \circ t$.

For every $R \in |K^*|, R > \rho$ there exists an $n \geq 1$ such that $\delta^n(B_R) \subset B_\rho$. Let s be $s : B_R \xrightarrow{\delta^n} B_\rho \xrightarrow{s_0} B_\rho \xrightarrow{\gamma^{-n}} K^2$, so $s = \gamma^{-n} \circ s_0 \circ \delta^n$.

Now we have :

$$\begin{aligned} t \circ s &= t\gamma^{-n}s_0\delta^n = \delta^{-1}t\gamma\gamma^{-n}s_0\delta^n = \delta^{-n}ts_0\delta^n = \delta^{-n}\delta^n = 1 \text{ and} \\ s \circ t &= \gamma^{-n}s_0\delta^n t = \gamma^{-n}s_0\delta^{n-1}t\gamma = \gamma^{-n}s_0t\gamma^n = \gamma^{-n}\gamma^n = 1. \end{aligned}$$

So the maps s do not depend on the choice of n . We can glue them together into a map $s : K^2 \rightarrow K^2$. Of course $s \circ t = t \circ s = id$, since the germ of $s \circ t$ and $t \circ s$ in $(0,0)$ is the identity map.

Remark. Another way to prove the previous lemma would be the following. The map $t : K^2 - \{(0,0)\} \rightarrow K^2 - \{(0,0)\}$ is already invertable on a small polydisc $B_\rho - \{(0,0)\}$ around $(0,0)$. This gives an isomorphism φ :

$$\begin{array}{ccc} K^2 - \{(0,0)\} / \langle \gamma \rangle & \xrightleftharpoons[\varphi^{-1}]{\varphi} & K^2 - \{(0,0)\} / \langle \delta \rangle \\ \uparrow & & \uparrow \\ K^2 - \{(0,0)\} & \xrightleftharpoons[L]{t} & K^2 - \{(0,0)\} \end{array}$$

Since $K^2 - \{(0,0)\}$ is simply connected (lemma 1.1), there exists a lifting L of φ^{-1} . We can choose the lifting L such that $L \circ t = t \circ L = 1$.

THEOREM 1.2. *The group Γ is abelian and $\Gamma \cong \mathbb{Z} \times \mathbb{Z}/l\mathbb{Z}$.*

Proof. Let $\gamma \in \Gamma$ be a contraction lying in the centre of Γ .

First we look at the case where $\gamma(z_1, z_2) = (\alpha z_1, \alpha z_2)$, $0 < |\alpha| < 1$.

Now we have :

$$\delta \in \Gamma \implies \delta\gamma = \gamma\delta \implies \delta(\alpha z_1, \alpha z_2) = \alpha.\delta(z_1, z_2).$$

So δ is linear, $\delta(z_1, z_2) = (\beta_1 z_1 + \lambda z_2, \beta_2 z_2)$ for a suitable choice of coordinates.

If $\exists \delta \in \Gamma$ with $\beta_1 = 1$ then $\delta(z, 0) = (z, 0)$. Since Γ acts without fixed points, we have $\delta = 1$. Therefore the map $\varphi : \Gamma \rightarrow K$ defined by $\varphi(\delta) = \beta_1$ -coordinate is injective. Now we can conclude that Γ is abelian. Since Γ acts discontinuously we must have $\Gamma \cong \mathbb{Z} \times \Gamma_{torsion}$ and $\mathbb{Z} \subset \Gamma$ is generated by a contraction. The injectivity of φ shows that $\Gamma_{torsion} \cong \mathbb{Z}/l\mathbb{Z}$ for some $l \in \mathbb{Z}_{\geq 1}$. Now we look at the case where

$$\gamma(z_1, z_2) = (\alpha_1 z_1, \alpha_2 z_2), \quad 0 < |\alpha_1| \leq |\alpha_2| < 1, \alpha_1 \neq \alpha_2.$$

The eigenspace belonging to α_1 is :

- a) 1-dimensional if $\alpha_1 \neq \alpha_2^m, \forall m \geq 1$ or
- b) 2-dimensional if $\exists m \in \mathbb{Z}_{>1}, \alpha_1 = \alpha_2^m$.

In case *a* we have :

$$\delta \in \Gamma \implies \delta\gamma = \gamma\delta \implies \delta(z_1, z_2) = (\beta_1 z_1, \beta_2 z_2)$$

This shows that Γ has to be abelian. Since Γ acts discontinuously, we must have $\Gamma \cong \mathbb{Z} \times \Gamma_{\text{torsion}}$ and $\mathbb{Z} \subset \Gamma$ is generated by a contraction. Now since the element $\delta : (z_1, z_2) \rightarrow (\beta_1 z_1, \beta_2 z_2)$ is fixed point free we must have $\Gamma_{\text{torsion}} \simeq \mathbb{Z}/l\mathbb{Z}$. Clearly Γ_{torsion} is generated by $\tilde{\omega} : (z_1, z_2) \rightarrow (\omega^l z_1, \omega z_2), \omega^l = 1$ and $\text{g.c.d.}(l, 1) = 1$.

In case *b* we have :

$$\delta \in \Gamma \implies \delta\gamma = \gamma\delta \implies \delta(z_1, z_2) = (\beta_1 z_1 + \mu z_2^m, \beta_2 z_2).$$

Again the map $\varphi : \Gamma \rightarrow K$ defined by $\varphi(\delta) = \beta_1$ - coordinate is injective. Therefore Γ is abelian and we have $\Gamma \simeq \mathbb{Z} \times \mathbb{Z}/l\mathbb{Z}$.

Now we consider the case where $\gamma(z_1, z_2) = (\alpha_1 z_1 + \lambda z_2^m, \alpha_2 z_2), \alpha_2^m = \alpha_1$.

Let $\delta \in \Gamma$, then we have :

$$\delta\gamma = \gamma\delta \implies \delta(z_1, z_2) = (\beta_1 z_1 + \mu z_2^m, \beta_2 z_2), \beta_1 = \beta_2^m.$$

Again Γ is abelian and therefore : $\Gamma \simeq \mathbb{Z} \times \mathbb{Z}/l\mathbb{Z}$.

Remark. We can also describe the generator of the torsion subgroup explicitly when the group $\mathbb{Z} \subset \Gamma$ is generated by a contraction γ of the form :

$$\gamma(z_1, z_2) = (\alpha_1 z_1 + \lambda z_2^m, \alpha_2 z_2), \alpha_2^m = \alpha_1.$$

Let $\tilde{\omega}$ be a generator of $\mathbb{Z}/l\mathbb{Z}$. Then $\tilde{\omega}$ has the following form :

$$\tilde{\omega} : (z_1, z_2) \rightarrow (\omega(z_1 + \mu z_2^m), \omega^j z_2), \omega^l = 1, (l, j) = 1.$$

Now we have :

$$\begin{aligned} \tilde{\omega}^l : (z_1, z_2) &= (\omega^l z_1 + (\omega^l + \omega^{l-1} \omega^{jm} + \omega^{l-2} \omega^{2jm} + \dots \\ &\quad + \omega \omega^{(l-1)jm}) \mu z_2^m, \omega^{jl} z_2). \end{aligned}$$

Since $\tilde{\omega}^l = 1$ we must have :

$$\begin{aligned} 0 &= \mu \sum_{k=1}^l \omega^k \omega^{(l-k)jm} = \mu \omega^{jml} \sum_{k=1}^l \omega^{k(1-jm)} = \mu \sum_{k=1}^l \omega^{k(1-jm)} \\ &\Leftrightarrow \mu = 0 \vee \sum_{k=1}^l \omega^{(1-jm)k} = 0 \end{aligned}$$

Now ω is a primitive l -th root of unity, so we have :

$$\sum_{k=1}^l \omega^{k(1-jm)} = \begin{cases} l & \text{if } jm \equiv 1 \pmod{l} \\ 0 & \text{otherwise} \end{cases}$$

So there are no restrictions on $\tilde{\omega}$ when $mj \not\equiv 1 \pmod{l}$. When $j m \equiv 1 \pmod{l}$ then of course $(l, m) = 1$ and $\mu = 0$ because $l \equiv 0$ cannot occur (when $\text{char}(K) = p > 0$ there are no p -th roots of unity $\neq 1$).

Since $\tilde{\omega}$ has to commute with γ , we have :

$$\tilde{\omega}\delta = \delta\tilde{\omega} \Leftrightarrow \lambda = 0 \vee jm \equiv 1 \pmod{l}.$$

This gives us all the possibilities for $\tilde{\omega}$:

$$\lambda \neq 0 \implies jm \equiv 1 \pmod{l}, (l, m) = 1, \mu = 0$$

$$\lambda = 0 \implies jm \not\equiv 1 \pmod{l}, \mu \in K \text{ or } jm \equiv 1 \pmod{l}, (l, m) = 1, \mu = 0.$$

THEOREM 1.3. *Let Γ be generated by a contraction γ and let X be the Hopf surface $K^2 - \{(0, 0)\}/\Gamma$. Then the field $\mathcal{M}(X)$ of meromorphic functions on X is :*

$$1) K \left(\frac{z_1^a}{z_2^b} \right), \text{ if } \gamma(z_1, z_2) = (\alpha_1 z_1, \alpha_2 z_2), \alpha_1^a = \alpha_2^b, g.c.d.(a, b) = 1, \\ a, b \in \mathbb{Z}_{>0}.$$

$$2) K \left(\frac{z_1^p - \lambda^{p-1} z_1 z_2^{mp-m}}{z_2^m} \right), \text{ if } \gamma(z_1, z_2) = (\alpha_1 z_1 + \lambda z_2^m, \alpha_2 z_2), \\ \alpha_2^m = \alpha_1, \lambda \neq 0, \text{char}(K) = p > 0.$$

$$3) K \text{ in all other cases.}$$

Proof. We have the following identities :

$$\begin{aligned} \mathcal{M}(X) &= \{f \mid f \text{ is meromorphic on } K^2 - \{(0, 0)\} \text{ and } \gamma - \text{invariant}\} \\ &= \{f \mid f \text{ is meromorphic on } K^2 \text{ and } \gamma - \text{invariant}\}. \end{aligned}$$

Since K^2 is a quasi-Stein space, we can now write : $f = \frac{t}{s}$, $t \in \mathcal{O}(K^2)$. (The proof of this fact is the same as the one given in [FP] theorem VI.3.5 for $(K^*)^n$). We can choose t, s in such a way that they are minimal, i.e. have only a finite number of zeroes in common. Let

$$t = \sum_{n,m \geq 0} a_{n,m} z_1^n z_2^m \in \mathcal{O}(K^2) \text{ and } s = \sum_{n,m \geq 0} b_{n,m} z_1^n z_2^m \in \mathcal{O}(K^2).$$

Let us first consider the case where γ has the form : $\gamma(z_1, z_2) = (\alpha_1 z_1, \alpha_2 z_2)$, $0 < |\alpha_1| \leq |\alpha_2| < 1$. Then clearly we have : $\gamma(t) = ct \implies c = \alpha_1^k \alpha_2^l$ for some $k, l \in \mathbb{Z}_{\geq 0}$.

Now suppose that $\alpha_1^a \neq \alpha_2^b \forall a, b \in \mathbb{Z}$ and $(a, b) \neq (0, 0)$. Then it is clear that :

$$\begin{aligned} \gamma(f) = f \implies \gamma(t) = ct \wedge \gamma(s) = cs \implies t &= \lambda z_1^k z_2^l \wedge s = \mu z_1^k z_2^l \\ \implies f &= \frac{t}{s} \in K. \end{aligned}$$

So we have : $\mathcal{M}(X) = K$.

Next we suppose that $\alpha_1^a = \alpha_2^b$ for some $a, b \in \mathbb{Z}_{\geq 0}, (a, b) \neq (0, 0)$. We can choose a, b minimal, such that $\text{g.d.c.}(a, b) = 1$. Then we have $\alpha_1^d = \alpha_2^c \implies (d, c) = n(a, b)$ for some $n \in \mathbb{Z}$. Now a monomial $z_1^k z_2^l$ with $\gamma(z_1^k, z_2^l) = cz_1^k z_2^l$ for a fixed $c = \alpha_1^{k_0} \alpha_2^{l_0}$ is of the form $z_1^k z_2^l$ with $(k, l) = (k_0, l_0) + n(a, -b)$ for some $n \in \mathbb{Z}$. This shows that :

$$\gamma(f) = f \implies \gamma(t) = ct \wedge \gamma(s) = cs \implies \frac{t}{s} \in K \left(\frac{z_1^a}{z_2^b} \right) \implies \mathcal{M}(X) = K \left(\frac{z_1^a}{z_2^b} \right).$$

Let us now consider the case where γ has the form :

$$\gamma(z_1, z_2) = (\alpha_1 z_1 + \lambda z_2^m, \alpha_2 z_2), \alpha_2^m = \alpha_1, \lambda \neq 0.$$

We can replace z_1 by $\lambda^{-1} z_1$, then γ has the form :

$$\gamma(z_1, z_2) = (\alpha_1 z_1 + z_2^m, \alpha_2 z_2), \alpha_2^m = \alpha_1.$$

Every monomial $z_1^k z_2^l$ can be written as $\left(\frac{z_1}{z_2^m} \right)^k z_2^{l+km}$. Let us take $x := \frac{z_1}{z_2^m}$ and z_2 as new variables. Then we have :

$$\gamma(x) = x + 1, \gamma(z_2) = \alpha_2 z_2.$$

Let g be polynomial in the variables x and z_2 with $\gamma(g) = c \cdot g$. Then we clearly have $c = \alpha_2^K$ for some $K \in \mathbb{Z}_{\geq 0}$. This shows that $g = z_2^K h$, where h is a polynomial in x with $\gamma(h) = h$. Let us take $h = \sum_{i=0}^s a_i x^i$ and let s be the highest power of x such that $a_s \neq 0$. Then we have:

$$\begin{aligned} \gamma(h) = h &\implies \sum_{i=0}^s a_i x^i = \sum_{i=0}^s a_i (x+1)^i \\ &\implies sa_s + a_{s-1} = a_{s-1} \\ &\implies s = 0 \vee a_s = 0. \end{aligned}$$

Since $a_s \neq 0$, we must have $s = 0$. When $\text{char}(K) = 0$ then $s = 0$ and $h \in K$. So in this case $\mathcal{M}(X) = K$.

But when $\text{char}(K) = p > 0$, then we see $p|s$. Now we look at the polynomial $x^p - x$. We have: $\gamma(x^p - x) = (x+1)^p - (x+1) = x^p - x$.

So any polynomial of the form $\sum a_i (x^p - x)^i$ is γ -invariant. The proof given above also shows that polynomials $(x^p - x)^i$ form a basis of the γ -invariant polynomials. This shows that:

$$\mathcal{M}(X) = K(x^p - x) = K\left(\frac{z_1^p}{z_2^{mp}} - \frac{z_1}{z_2^m}\right) = K\left(\frac{z_1^p - z_1 z_2^{m(p-m)}}{z_2^{mp}}\right).$$

2. Affinoid coverings and reductions

We first construct a fundamental domain for the action of the group Γ , where Γ is generated by a contraction. Then we will study some special affinoid subspaces of K^2 and their reduction. We will use this to construct a pure covering of $K^2 - \{(0,0)\}$, which is Γ -invariant. This will give us a pure affinoid covering of the Hopf surface $X = K^2 - \{(0,0)\}/\Gamma$.

DEFINITION. We call a subspace $F \subset K^2 - \{(0,0)\}$ a *fundamental domain* for the action of the group Γ , if F has the following properties:

- 1) $K^2 - \{(0,0)\} = \bigcup_{\gamma \in \Gamma} \gamma(F)$.
- 2) There exists a finite affinoid covering $\{Sp(A_i)\}_{i=1}^n$ of F .
- 3) The only action of Γ on F itself is the identification of a finite number of affinoid subspaces $B_k \subset F$, where $B_k \subset Sp(A_i)$ is defined by a finite number s of equations:

$$|f_j| = c_j, \quad j = 1 \dots s, \quad f_j \in A_i, \quad c_j \in |K^*|.$$

So the subspaces B_k are of the form

$$B_k = Sp(A_i < \frac{f_i}{c_j}, \frac{c_j}{f_j} \mid j = 1 \dots s >).$$

PROPOSITION 2.1. *Let Γ be generated by a contraction*

$$\gamma : (z_1, z_2) \rightarrow (\alpha_1(z_1 + \lambda z_2^m), \alpha_2 z_2),$$

where $0 < |\alpha_1| \leq |\alpha_2| < 1$ and $\lambda = 0$ if $\alpha_1 \neq \alpha_2^m$, otherwise $\lambda \in K$. If we choose $|\lambda| < 1$ then Γ has a fundamental domain F defined by :

$$F := \{(z_1, z_2) \in K^2 - \{(0, 0)\} \mid |z_1| \leq 1, |z_2| \leq 1, (|z_1| \geq |\alpha_1| \vee |z_2| \geq |\alpha_2|)\}.$$

Proof. Let us first show that we may choose $|\lambda| < 1$. We can replace the coordinate z_1 by εz_1 , $\varepsilon \in K^*$. Then γ is defined by :

$$\gamma(\varepsilon z_1, z_2) = (\alpha_1(\varepsilon z_1 + \varepsilon \lambda z_2^m), \alpha_2 z_2), \lambda \neq 0$$

Now take $\varepsilon = \mu \lambda^{-1}$, $|\mu| < 1$. This gives us the desired form of γ :

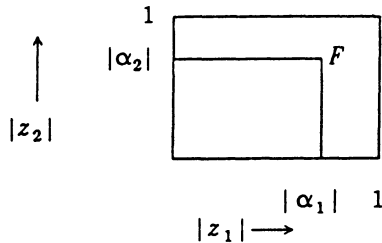
$$\gamma(z_1, z_2) = (\alpha_1(z_1 + \mu z_2^m), \alpha_2 z_2), |\mu| < 1.$$

A straightforward calculation now shows that :

$$\bigcup_{i \in \mathbb{Z}} \gamma^i(F) = K^2 - \{(0, 0)\},$$

$$\gamma^i(F) \cap \gamma^{i+1}(F) = \left\{ (z_1, z_2) \in K^2 - \{(0, 0)\} \mid |z_2| = |\alpha_2|^i \wedge |z_1| \leq |\alpha_1|^i \right. \\ \left. \text{or } |z_1| = |\alpha_1|^i \wedge |z_2| \leq |\alpha_2|^i \right\},$$

$$\gamma^i(F) \cap \gamma^j(F) = \emptyset \text{ if } |i - j| \neq 0,$$



This shows that the subspace F satisfies the first property of our definition of a fundamental domain.

The only action of Γ on F is the identification of $\gamma^{-1}(F) \cap F$ and $F \cap \gamma(F)$. This gives the following identifications of affinoid subspaces of F :

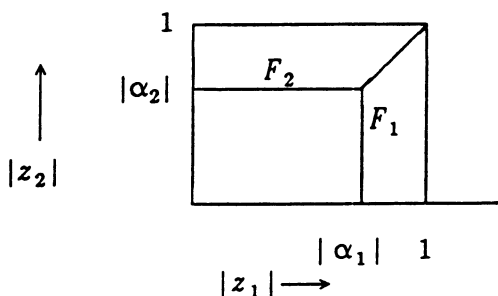
$$\gamma(C_1) = C_2, \quad \gamma(C_3) = C_4.$$

Here C_i , $i = 1 \dots 4$ are defined by :

$$\begin{aligned} C_1 &= \{(z_1, z_2) \mid |z_1| = 1, |z_2| \leq 1\} \\ C_2 &= \{(z_1, z_2) \mid |z_1| = |\alpha_1|, |z_2| \leq |\alpha_2|\} \\ C_3 &= \{(z_1, z_2) \mid |z_2| = 1, |z_1| \leq 1\} \\ C_4 &= \{(z_1, z_2) \mid |z_2| = |\alpha_2|, |z_1| \leq |\alpha_1|\}. \end{aligned}$$

We will now show that F can be covered by a finite number of affinoid subspaces, such that $C_i, i = 1 \dots 4$ satisfy property 3 of our definition.

If $|\alpha_1|^k = |\alpha_2|^l$, $\lambda = 0$, $k, l \in \mathbb{Z}_{>0}$ then Γ also preserves the area given by $\{(z_1, z_2) \in K^2 - \{(0,0)\} \mid \left| \frac{z_1^k}{z_2^l} \right| = 1\}$. This gives a γ -invariant partition of the domain F into two affinoid subspaces F_1 and F_2 .



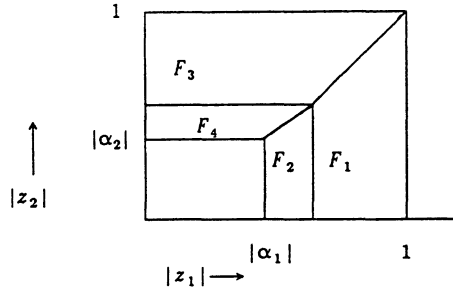
We have :

$$\begin{aligned} F_1 &= \{(z_1, z_2) \in K^2 - \{(0,0)\} \mid |\alpha_1| \leq |z_1| \leq 1, |z_2^l| \leq |z_1^k|\} \\ F_2 &= \{(z_1, z_2) \in K^2 - \{(0,0)\} \mid |\alpha_2| \leq |z_2| \leq 1, |z_2^l| \geq |z_1^k|\}. \end{aligned}$$

The affinoid subspaces $C_1, C_2 \subset F_1$ are defined by $|z_1| = 1$ and $|z_1| = |\alpha_2|$ respectively. The subspaces $C_3, C_4 \subset F_2$ are defined by $|z_2| = 1$ and $|z_2| = |\alpha_1|$ respectively. This shows that F is a fundamental domain.

If γ has the form $\gamma(z_1, z_2) = (\alpha_1(z_1 + \lambda z_2^m), \alpha_2 z_2)$, $|\lambda| < 1$, then again $F = F_1 \cup F_2$ as above with $k = 1$, $l = m$. Since $|\lambda| < 1$ the area $\{(z_1, z_2) \in K^2 - \{(0, 0)\} \mid \left| \frac{z_1}{z_2^m} \right| = 1\}$ is Γ -invariant. Again F is a fundamental domain.

If $|\alpha_1^k| \neq |\alpha_2^l|$, $\forall k, l \in \mathbb{Z}_{>0}$ then we can find an $s \in \mathbb{Z}_{>0}$ such that $|\alpha_2^s| < |\alpha_1|$ since $0 < |\alpha_1| < |\alpha_2| < 1$. Now the areas defined by $\left| \frac{z_1}{z_2^s} \right| = 1$ and by $\left| \frac{z_1}{z_2} \right| = \left| \frac{\alpha_1}{\alpha_2} \right|$ have a non-empty intersection P in F . Here P is defined by $|z_2^{s-1}| = \left| \frac{\alpha_1}{\alpha_2} \right|$, $|z_1^{s-1}| = \left| \frac{\alpha_1}{\alpha_2} \right|$. This gives us a finite affinoid covering of F by F_1, F_2, F_3 and F_4 (See figure below). The subspaces $C_i \subset F_i$, $i = 1..4$ have property 2 of our definition. So F is a fundamental domain.



DEFINITIONS. Let A be an affinoid algebra and $Sp(A)$ its affinoid space. A subspace $X \subset Sp(A)$ is called a *rational domain* if there exists a set $\{f_0, f_1, \dots, f_n\}$ generating the unit ideal of A such that X is defined by :

$$\begin{aligned} X &= \{x \in Sp(A) \mid |f_i(x)| \leq |f_0(x)|, i = 1..n\} \\ &= \left\{ x \in Sp(A) \mid \left| \frac{f_i(x)}{f_0(x)} \right| \leq 1, i = 1..n \right\}. \end{aligned}$$

The rational domain X is an affinoid subspace of $Sp(A)$ and has as its affinoid algebra $A < \frac{f_i}{f_0} \mid i = 1..n > \cong A < x_1..x_n > / < f_0 x_i - f_i \mid i = 1..n >$ (See [BGR] or [FP].).

We will only use rational subspaces of

$$Y = Sp(K < z_1, z_2 >) \cong \{(z_1, z_2) \in K^2 \mid |z_1| \leq 1, |z_2| \leq 1\}.$$

In particular we will restrict ourselves to those rational subspaces $X \subset Y$ where the $f_i, i = 0..n$, are monomials $cz_1^k z_2^l$, $k, l \in \mathbb{Z}_{\geq 0}$, $c \in K^*$. Such a subspace $X \subset Y$ will be called a *monomial rational subspace* (of Y with respect to the affinoid generating set $\{z_1, z_2\}$).

Example. The affinoid covering of the fundamental domain F constructed in the proof of proposition 2.1 consists of a finite number of monomial rational subspaces of Y .

We will only show this for the affinoid space, F_1 when $|\alpha_1^k| = |\alpha_2^l|$ for some $k, l \in \mathbb{Z}_{>0}$. All the other cases are similar. Let F_1 be as in proposition 2.1, so we have :

$$\begin{aligned} F_1 &= \{(z_1, z_2) \in K^2 \mid |\alpha_1| \leq |z_1| \leq 1, |z_2^l| \leq |z_1^k|\} \\ &= \{(z_1, z_2) \in Y \mid \left| \frac{\alpha_1}{z_1} \right| \leq 1, \left| \frac{z_2^l}{z_1^k} \right| \leq 1\} \\ &= \{(z_1, z_2) \in Y \mid \left| \frac{z_2^l}{z_1^k} \right| \leq 1, \left| \frac{\alpha_1^s z_1^{k-s}}{z_1^k} \right| \leq 1, s = 1..k\}. \end{aligned}$$

It is clear that the set $\{z_1^k, \alpha_1 z_1^{k-1}, \dots, \alpha_1^k, z_2^l\}$ generates the unit ideal of $K \langle z_1, z_2 \rangle$, so F_1 is a monomial rational subspace of Y .

Remark. Let $v : K^2 \rightarrow (\mathbb{R} \cup \{-\infty\})^2$ be the map defined by :

$$(z_1, z_2) \rightarrow (\log |z_1|, \log |z_2|).$$

The image $v(Y)$ of Y is given by :

$$v(Y) = \{(x_1, x_2) \in (\mathbb{R} \cup \{-\infty\})^2 \cap v(K^2) \mid x_1 \leq 0, x_2 \leq 0\}.$$

The image of a monomial rational subspace of Y is a convex domain in $v(Y)$ defined by a finite number s of rational inequalities

$$n_i x_1 + m_i x_2 \leq \log |c_{n_i, m_i}|, \quad i = 1..s$$

coming from the monomial inequalities

$$\left| \frac{f_i}{f_0} \right| = \left| \frac{z_1^{n_i} z_2^{m_i}}{c_{n_i, m_i}} \right| \leq 1, \quad i = 1..s, \quad n_i, m_i \in \mathbb{Z}, \quad c_{n_i, m_i} \in K^*.$$

PROPOSITION 2.2. *A convex domain $C \subseteq v(Y)$, is the image $v(X)$ of a monomial rational subspace $X \subseteq Y$ if and only if C satisfies the following two conditions a and b :*

a) *C is defined by a finite number s of rational inequalities :*

$$n_i x_i + m_i x_2 \leq \log |c_{n_i, m_i}|, \quad n_i, m_i \in \mathbb{Z}, \quad c_{n_i, m_i} \in K^*, \quad i = 1..s.$$

(When K is contained in the algebraic closure of a local field then we can normalize the valuation on K such that $\log |K^| \subseteq \mathbb{Q}$. Then all the coefficients of these inequalities are really rational.)*

b) *C has one of the following properties :*

- 1) $\{(x_1, x_2) \in (\mathbb{R} \cup \{-\infty\})^2 \cap v(Y) \mid x_1, x_2 \leq a\} \subseteq C$ for some $a \in (\log |K^*|) \cap \mathbb{R}_{<0}$
- 2) $C \subseteq \{(x_1, x_2) \in (\mathbb{R} \cup \{-\infty\})^2 \cap v(Y) \mid c \leq x_1 \leq 0\}$ for some $c \in (\log |K^*|) \cap \mathbb{R}_{<0}$
- 3) $C \subseteq \{(x_1, x_2) \in (\mathbb{R} \cup \{-\infty\})^2 \cap v(Y) \mid c \leq x_2 \leq 0\}$ for some $c \in (\log |K^*|) \cap \mathbb{R}_{<0}$
- 4) $C \subseteq \{(x_1, x_2) \in (\mathbb{R} \cup \{-\infty\})^2 \cap v(Y) \mid c \leq x_1, x_2 \leq 0\}$ for some $c \in (\log |K^*|) \cap \mathbb{R}_{<0}$.

Proof. Let $X = \{z \in Y \mid \left| \frac{f_i(z)}{f_0(z)} \right| \leq 1, \quad i = 1..s\} \subseteq Y$ be a monomial rational subspace. In the last remark above we have already shown that the image $C = v(X) \subseteq v(Y)$ is given by a finite number of rational inequalities. So we only have to prove that $C = v(X)$ satisfies condition b.

Now $f_0(z)$ is one of the following monomials :

- 1) $f_0 = c$, $c \in K^*$
- 2) $f_0 = cz_1^k$, $c \in K^*$, $k \in \mathbb{Z}_{>0}$
- 3) $f_0 = cz_2^l$, $c \in K^*$, $l \in \mathbb{Z}_{>0}$
- 4) $f_0 = cz_1^k z_2^l$, $c \in K^*$, $k, l \in \mathbb{Z}_{>0}$.

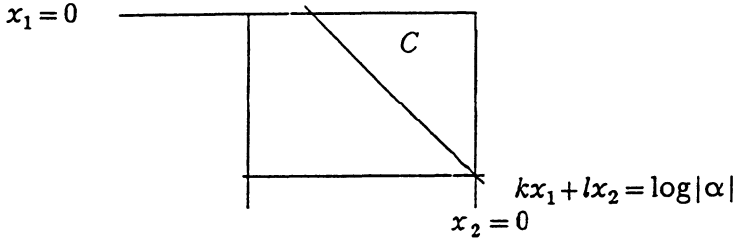
In case 1 we have $f_0 = c \in K^*$, so all the monomials $\frac{f_i(z)}{f_0(z)}$ are monomials $c_i z_1^{n_i} z_2^{m_i}$ with $n_i, m_i \in \mathbb{Z}_{\geq 0}$, $c_i \in K^*$. This shows that $C = v(X)$ has property b1.

In the other cases we see that some $f_i = \alpha \in K^*$, since $f_0, \dots, f_s$ generate the unit ideal in $K \langle z_1, z_2 \rangle$. This shows that in the cases 2,3 and 4 the convex domain $C = v(X)$ satisfies conditions b2, b3 and b4 respectively.

We will only prove this explicitly in case 4. Now $z \in X$ satisfies $\left| \frac{\alpha}{z_1^k z_2^l} \right| \leq 1$ for some $k, l \in \mathbb{Z}_{>0}$, $\alpha \in K^*$. Therefore $v(X)$ satisfies the inequality : $-kx_1 - lx_2 \leq -\log |\alpha|$. Since $x_1, x_2 \leq 0$ we have :

$$x_1 \geq \frac{\log |\alpha|}{k}, \quad x_2 \geq \frac{\log |\alpha|}{l}$$

$$\Rightarrow x_1, x_2 \geq \min \left(\frac{\log |\alpha|}{k}, \frac{\log |\alpha|}{l} \right) \geq c \text{ for some } c \in \log |K^*| \cap \mathbb{R}_{<0}.$$



This shows that $C = v(X)$ satisfies condition b4.

Now we will show that a convex domain C that satisfies conditions a and b is the image $v(X)$ of a monomial rational subspace $X \subseteq Y$. Let C be defined by the rational inequalities :

$$n_i x_1 + m_i x_2 \leq \log |c_{n_i, m_i}|, \quad n_i, m_i \in \mathbb{Z}, \quad c_{n_i, m_i} \in K^*, \quad i = 1..s.$$

Now $z \in v^{-1}(C)$ satisfies the inequalities :

$$\left| \frac{z_1^{n_i} z_2^{m_i}}{c_{n_i, m_i}} \right| \leq 1, \quad i = 1..s.$$

Let n, m be defined by

$$n = \min(\{0, n_1, n_2, \dots, n_s\}) \text{ and } m = \min(\{0, m_1, m_2, \dots, m_s\}).$$

Now we take $f_0 = z_1^n z_2^m$ and define $f_i \in K \langle z_1, z_2 \rangle$ by

$$\frac{f_i(z)}{f_0(z)} = \frac{z_1^{n_i} z_2^{m_i}}{c_{n_i, m_i}}, \quad i = 1 \dots s.$$

So we have $f_i(z) = \frac{z_1^{n_i+n} z_2^{m_i+m}}{c_{n_i, m_i}}, \quad i = 1 \dots s.$

If $f_0(z) = 1$ then $X = \{z \in Y \mid \left| \frac{f_i(z)}{f_0(z)} \right| \leq 1, \quad i = 1 \dots s\}$ is a monomial rational subspace of Y and $v^{-1}(C) = X$ and C satisfies condition a and $b1$. If some $f_i(z) \in K^*$ for $s \geq i \geq 1$ then $v^{-1}(C)$ is again a monomial rational subspace of Y , since the f_i generate the unit ideal in $K \langle z_1, z_2 \rangle$.

Now suppose $f_0 = z_1^n z_2^m, n, m \in \mathbb{Z}_{>0}$ and $f_i \notin K, i = 1 \dots s$.

If C satisfies condition $b4$ we can find an element $c \in K^*$ such that $\left| \frac{c}{z_1^n z_2^m} \right| \leq 1$ for all $z \in v^{-1}(C)$. So taking $f_{s+1} = c$ we find a monomial rational subspace $X = v^{-1}(C)$ of Y defined by $\left| \frac{f_i(z)}{f_0(z)} \right| \leq 1, \quad i = 1 \dots s+1$.

If C satisfies condition $b2$ we can find a $c \in K^*$ such that $\left| \frac{c}{z_1} \right| \leq 1$ for all $z \in v^{-1}(C)$. Furthermore by the definition of m there exists an $f_i(z)$ such that $\frac{f_i(z)}{f_0(z)} = \frac{z_1^{n_i}}{c_{n_i, m} z_2^m}, \quad m > 0$.

From this we see :

$$\begin{aligned} \left| \frac{f_i(z)}{f_0(z)} \right| &= \left| \frac{z_1^{n_i}}{c_{n_i, m} z_2^m} \right| \leq 1 \\ \Rightarrow \begin{cases} \left| \frac{c}{z_1} \right|^{-n_i} \left| \frac{1}{z_2^m} \right| \leq |c_{n_i, m} c^{-n_i}| & \text{if } n_i \leq 1 \text{ and } \left| \frac{c}{z_i} \right| \leq 1 \\ |z_1|^{n_i} \left| \frac{1}{z_2^m} \right| \leq |c_{n_i, m}| & \text{if } n_i \geq 0 \text{ and } |z_1| \leq 1 \end{cases} \\ \Rightarrow \left| \frac{1}{z_2^m} \right| &\leq |\alpha| \quad \text{for some } \alpha \in K^*. \end{aligned}$$

So C satisfies condition $b4$, therefore we know $v^{-1}(C) \subseteq Y$ is a monomial rational subspace. If C satisfies condition $b3$ we again find that C must satisfy condition $b4$ if $f_0 = z_1^n z_2^m, n, m \in \mathbb{Z}_{>0}$. If $f_0 = z_1^n z_2^m, n, m > 0$ then C cannot satisfy condition $b1$.

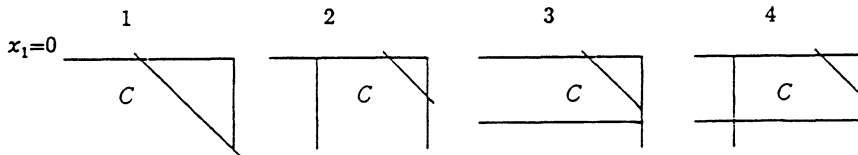
The situations with $f_0 = z_1^n$ or $f_0 = z_2^m$ are similar.

Remark. Using proposition 2.2 above we can now describe explicitly the convex doamins $C \subseteq v(Y)$ such that $v^{-1}(C) = X$ is a monomial rational subspace $X \subseteq Y$.

In the next table we give description of C and X in the case $\text{int}(C) = \emptyset$.

C	X
The empty set $\emptyset$	$ cz_1 \geq 1$ and $ c < 1$
a point $P = (\frac{1}{n} \log c_1 , \frac{1}{m} \log c_2)$	$ z_1^n = c_1 , z_2^m = c_2 \leq 1, c_1, c_2 \in K^*$
a halfline $x_1 = \frac{1}{n} \log c_1 $	$ z_1^n = c_1 \leq 1, c_1 \in K^*$
or $x_2 = \frac{1}{m} \log c_2 $	$ z_2^m = c_2 \leq 1, c_2 \in K^*$
a line segment:	
$nx_1 + mx_2 = \log c_1 $	$ z_1^n z_2^m = c_1 \leq 1, c_1 \in K^*$
and $\frac{1}{k} \log c_2 \leq x_1 \leq \frac{1}{l} \log c_3 $	$ c_2 \leq z_1 ^k, z_1^l \leq c_3 \leq 1, c_2, c_3 \in K^*$
or $\frac{1}{k} \log c_4 \leq x_2 \leq \frac{1}{l} \log c_5 $	$ c_4 \leq z_2 ^k, z_2^l \leq c_5 \leq 1, c_4, c_5 \in K^*$

If $\text{int}(C) \neq \emptyset$ then C can have one of the following forms. The numbering corresponds with the one of property b in proposition 2.2.



DEFINITION. We define $\sqrt{|K^*|}$ as being the set

$$\sqrt{|K^*|} = \{x \in \mathbb{R} \mid x^n \in |K^*| \text{ for some } n \in \mathbb{Z}_{>0}\}.$$

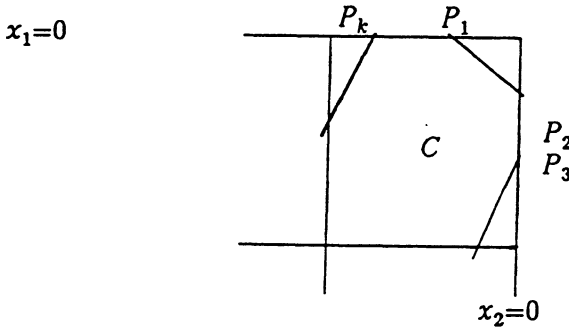
Let $C \subseteq v(Y)$ be a convex doamin. A point $P \in C \neq \emptyset$ is called an *extremal point of C* if there exists no line segment $[P_1, P_2] \subseteq C$ with $P \neq P_1, P_2$ such that $P \in [P_1, P_2]$.

LEMMA 2.1. Let $|K^*| = \sqrt{|K^*|}$ and let $X \subseteq Y$ be a monomial rational domain such that $v(X) = C \neq \emptyset$ is a convex domain in $v(Y)$. Let A be the affinoid algebra of X . For a polynomial $f \in K[z_1, z_2, z_1^{-1}, z_2^{-1}] \cap A$ we have $\|f\| = \|\sum a_{n,m} z_1^n z_2^m\| = \max |a_{n,m}| \|z_1^n z_2^m\|$ where $\|\cdot\|$ denotes the spectral norm on the affinoid algebra A .

Let $P_1 \dots P_k$ be the extremal points of the convex domain C . Then we have :

$$\|z_1^n z_2^m\| = \max \{|a^n b^m| \mid (a, b) \in \bigcup_{i=1}^k v^{-1}(P_i)\}.$$

Proof. Since $|K^*| = \sqrt{|K^*|}$ and K is non-archimedean we have $\|f\| = \max |a_{n,m}| \|z_1^n z_2^m\|$.



A monomial $z_1^n z_2^m$ has norm $|c|$ on the line $nx_1 + mx_2 = \log |c|$ in $v(Y)$. The maximal value $|c| \in |K^*|$ such that the line $nx_1 + mx_2 = \log |c|$ has at least one point in common with the convex domain C is equal to $\|z_1^n z_2^m\|$. It is clear that this rational line can contain at most two points P_i . This only occurs when the monomial belongs to a rational line on the boundary of C . This shows that we have indeed :

$$\|z_1^n z_2^m\| = \max\{|a^n b^m| \mid (a, b) \in \bigcup_{i=1}^k v^{-1}(P_i)\}.$$

So for a polynomial $f = \sum a_{n,m} z_1^n z_2^m \in K[z_1, z_2, z_1^{-1}, z_2^{-1}] \cap A$ we have :

$$\|f\| = \max_{n,m} |a_{n,m}| \max_{1 \leq i \leq k} |a_i^n b_i^m|$$

Here $(a_i, b_i) \in v^{-1}(P_i)$ are chosen in $v^{-1}(P_i)$.

DEFINITIONS. Let A be an affinoid algebra, with spectral norm $\|\cdot\|$ and let $X = Sp(A)$. Let K^0 be the ring of integers of K , i.e.

$$K^0 := \{x \in K \mid |x| \leq 1\}.$$

We define the K^0 -module A^0 by $A^0 := \{f \in A \mid \|f\| \leq 1\}$. Now we define the K^0 -submodule $A^{00} \subset A^0$ by $A^{00} := \{f \in A \mid \|f\| < 1\}$. We call $\overline{A} = A^0/A^{00}$ the *reduction* of A and $\overline{X} = \text{spec}(\overline{A})$ the *reduction* of X .

We have a map $R : X \rightarrow \overline{X}$. The image $R(m)$ of a maximal ideal m of A is a maximal ideal of $\overline{A}$ defined by :

$$R(m) = \text{the image of } m \cap A^0 \text{ in } \overline{A} = A^0/A^{00}.$$

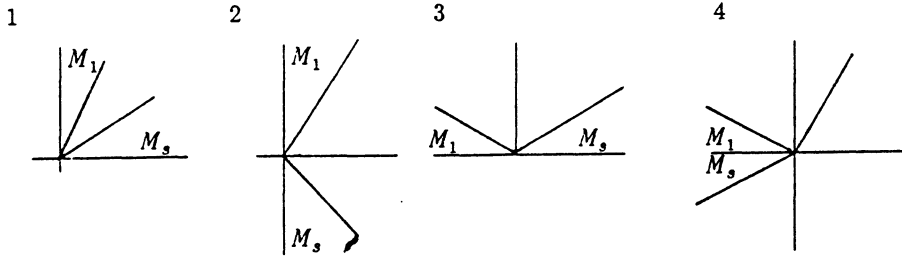
The map R is surjective onto the set of closed points of $\overline{X}$ (see [BGR] p.270).

Remark. Let us take a monomial rational domain $X \subseteq Y$, $X \neq \emptyset$ such that $C = v(X)$ is a convex domain $\neq \emptyset$ in $v(Y)$. We can now associate to an extremal point P_i of C the monomials $z_1^n z_2^m$, $n, m \in \mathbb{Z}$ that are in the affinoid algebra A of X and attain their maximal value $\|z_1^n z_2^m\|$ in $v^{-1}(P_i)$. This gives a partition of the monomials in A .

Let f map the monomials $z_1^n z_2^m$ into $\mathbb{Z}^2$ and be defined by :

$$f(z_1^n z_2^m) = (n, m)$$

Let M_i be the set $M_i := \{f(z_1^n z_2^m) \mid z_1^n z_2^m \text{ is a monomial in } A \text{ and attains its maximal value } \|z_1^n z_2^m\| \text{ in } v^{-1}(P_i)\}$.



In the pictures above we have drawn the partitions. The figures 1,2,3 and 4 correspond to monomial rational domains $X \subseteq Y$ that have property b1, b2, b3 and b4 respectively (see proposition 2.2).

The line between the areas M_i and M_{i+1} belongs to both, since it corresponds to the monomials that have their maximum value in both $v^{-1}(P_i)$ and $v^{-1}(P_{i+1})$.

LEMMA 2.2. *Let $X \subseteq Y$ be a monomial rational domain with affinoid algebra A . Let $|K^*| = \sqrt{|K^*|}$. Then there is a 1 - 1 correspondance between the minimal prime ideals p_i of $\bar{A}$ and the extremal points P_i of $C = v(X)$. In fact we have :*

$$p_i = \{\bar{f} \in \bar{A} \mid |f(a_i, b_i)| < 1, (a_i, b_i) \in v^{-1}(P_i)\}$$

Proof. Since $|K^*| = \sqrt{|K^*|}$, we can choose for every monomial $z_1^n z_2^m \in A$, a $c_{n,m} \in K^*$ such that $\|z_1^n z_2^m\| = c_{n,m}$. Now the K° -module A° is generated by the elements $x_{n,m} := \frac{z_1^n z_2^m}{c_{n,m}}$. So the $\bar{K}$ -module $\bar{A}$ is generated by the images $\bar{x}_{n,m}$ of $x_{n,m}$ in $\bar{A}$. A straightforward calculation shows that :

$$\bar{x}_{n,m} \cdot \bar{x}_{k,l} = \delta \cdot \bar{x}_{n+k, m+l} \text{ for some } \delta \in \bar{K}.$$

Furthermore $\delta \in \bar{K}^*$ if and only if $\bar{x}_{n,m}$ and $\bar{x}_{k,l}$ are the images of monomials belonging to the same area M_i .

This shows that we have indeed for every extremal point P_i of $C = v(X)$ a minimal prime ideal $p_i = \{\bar{f} \in \bar{A} \mid |f(a_i, b_i)| < 1, (a_i, b_i) \in v^{-1}(P_i)\}$. The ideal p_i is generated by the elements $\bar{x}_{n,m}$ with $(n, m) \notin M_i$, so $z_1^n z_2^m$ does not reach its maximum $\|z_1^n z_2^m\|$ in $v^{-1}(P_i)$.

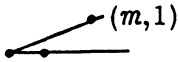
Remark. Let us look at $B_i = \overline{A}/p_i$. We see that this ring is generated over $\overline{K}$ by the elements $\overline{x}_{n,m}, (n,m) \in M_i \cap \mathbb{Z}^2$. We can choose the constants $c_{n,m}$ such that the multiplication in B_i is given by :

$$\overline{x}_{n,m} \cdot \overline{x}_{k,l} = \overline{x}_{n+k,m+l}, (n,m), (k,l) \in M_i \cap \mathbb{Z}^2.$$

Since the areas $M_i \subseteq \mathbb{Z}^2$ are rational, the semigroup of points in $M_i \cap \mathbb{Z}^2$ is generated by a finite number of elements. This shows that B_i is a finitely generated $\overline{K}$ -algebra.

It is clear that the quotient field of B_i is $\overline{K}(z_1, z_2)$.

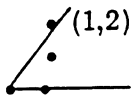
Examples. We identify the monomials $\overline{x}_{n,m} \in B_i = \overline{A}/P_i$ with the points $(n,m) \in M_i \cap \mathbb{Z}^2$. For convenience we choose B such that one of the borderlines goes through the point $(1,0)$. This can always be done by using a transformation by an element of $GL(2, \mathbb{Z})$.



Now the generators of B over $\overline{K}$ are $\overline{x}_{1,0}$ and $\overline{x}_{m,1}$.

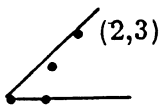
There are no relations between the generators, so we have :

$$B = \overline{K}[\overline{x}_{1,0}, \overline{x}_{m,1}].$$



The generators of B are $\overline{x}_{1,2}, \overline{x}_{1,1}$ and $\overline{x}_{1,0}$. We have the relations : $\overline{x}_{1,1}^2 = \overline{x}_{1,2} \cdot \overline{x}_{1,0}$. So we have :

$$B = \overline{K}[\overline{x}_{1,2}, \overline{x}_{1,1}, \overline{x}_{1,0}] / (\overline{x}_{1,1}^2 - \overline{x}_{1,2} \cdot \overline{x}_{1,0}).$$



The generators of B are $\overline{x}_{1,0}, \overline{x}_{1,1}$ and $\overline{x}_{2,3}$. We have the relations : $\overline{x}_{2,3} \cdot \overline{x}_{1,0} = \overline{x}_{1,1}^3$. So we have :

$$B = \overline{K}[\overline{x}_{1,0}, \overline{x}_{1,1}, \overline{x}_{2,3}] / (\overline{x}_{1,1}^3 - \overline{x}_{2,3} \cdot \overline{x}_{1,0}).$$

Remark. Our description of the algebra B is in complete accordance with the theory of toroidal embeddings as described in [KKMS], [0.1] and [0.2]. We will now state and use some results and definitions from it.

DEFINITIONS. Let M be the set of monomials $z_1^n z_2^m$, $n, m \in \mathbb{Z}$.

Now $M \simeq \mathbb{Z}^2$ where the isomorphism is given by the map

$$f : z_1^n z_2^m \rightarrow (n, m).$$

Let $M_{\mathbb{R}}$ be $M_{\mathbb{R}} := M \otimes \mathbb{R} \simeq \mathbb{R}^2$.

We call a semi-group $S \subseteq M$ *saturated* if S satisfies :

$$nr \in S \Rightarrow r \in S, \text{ where } r \in M \text{ and } n \in \mathbb{Z}_{>0}$$

We call a convex domain in $M_{\mathbb{R}}$ bounded by two rational halflines starting in the origin a (*convex rational polyhedral*) *cone*. We will always assume that the cone does not contain a linear subspace. For a semi-group $S \subseteq M$ we define the space $X_S := \text{spec } \overline{K}[f^{-1}(S)]$. For a cone $\sigma \subset M_{\mathbb{R}}$, the set $\sigma \cap M$ is a saturated finitely generated semi-group. Moreover any finitely generated semi-group S , not containing a line, has the form $\sigma \cap M$ for some cone σ .

THEOREM 2.1. *Let $S \subset M$ be a semi-group that generates M as a group. Let $\sigma \subset M_{\mathbb{R}}$ be a cone such that $\text{int}(\sigma) \neq \emptyset$, so*

$$\sigma = \{\lambda(ae_1 + be_2) + \mu(ce_1 + de_2) | \lambda, \mu \in \mathbb{R}_{\geq 0}\}$$

for some $\mathbb{Z}$ -basis $\{e_1, e_2\}$ of M and $a, b, c, d \in \mathbb{Z}$ with $\text{g.c.d.}(a, b) = \text{g.c.d.}(c, d) = 1$ and $n = \left| \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right| \neq 0$. Now we have :

- a) *The space X_S is normal if and only if S is saturated.*
- b) *The space X_{σ} is non-singular if and only if the semi-group $\sigma \cap M$ is generated by a $\mathbb{Z}$ -basis of M .*
- c) *If $\overline{K}$ contains the n -th roots of unity then $X_{\sigma} \cong \mathbb{A}_{\overline{K}}^2 / \mu$, where μ is a cyclic group of order n acting diagonally on $\mathbb{A}_{\overline{K}}^2$.*

Proof. All this is proved in [KKMS] Ch.I §1. We shall recall the proof of part c of the theorem, because this will give us a nice and explicit description of X_{σ} .

Let σ be as in the theorem. We can choose a $\mathbb{Z}$ -basis $\{f_1, f_2\}$ of M such that $\sigma = \{\lambda f_1 + \mu(kf_1 + lf_2) | \lambda, \mu \in \mathbb{R}_{\geq 0}\} \subset M_{\mathbb{R}}$, where $k, l \in \mathbb{Z}$ with $\text{g.c.d.}(k, l) = 1$. We may assume $l > 0$, since we always can replace f_2 by $-f_2$. So we have $n = |\det(kf_1 + lf_2, f_1)| = l$, since $\det(f_1, f_2) = \pm 1$.

If $n = l = 1$ then $\{f_1, kf_1 + lf_2\}$ is a $\mathbb{Z}$ -basis of M . These two elements also generate $\sigma \cap M$. This makes it clear that :

$$X_{\sigma} = \text{spec } \overline{K}[f^{-1}(f_1), f^{-1}(kf_1 + lf_2)] \cong \mathbb{A}_{\overline{K}}^2$$

Let $l \neq 1$ and $l > 0$. Now the semi-group $\sigma \cap M$ is not generated by f_1 and $kf_1 + lf_2$. Let $M^* = \mathbb{Z} \cdot \frac{1}{l}f_1 \oplus \mathbb{Z} \cdot f_2$. Now $\{\frac{1}{l}f_1, \frac{k}{l}f_1 + f_2\}$ is a $\mathbb{Z}$ -basis of M^* and the two elements also generate the semi-group $\sigma \cap M^*$.

Let g map the monomials $x^u y^v$, $u, v \in \mathbb{Z}$ into M^* and be defined by

$$g : x^u y^v \rightarrow u \cdot \frac{1}{l}f_1 + v \cdot f_2$$

It is clear that $\text{spec} \overline{K}[g^{-1}(\sigma \cap M^*)] \cong \mathbb{A}_{\overline{K}}^2$ and $\sigma \cap M \subseteq \sigma \cap M^*$. Moreover we have $\text{spec} \overline{K}[g^{-1}(\sigma \cap M)] \cong \text{spec} \overline{K}[f^{-1}(\sigma \cap M)]$. This can be seen by using the map : $z_1 \rightarrow x_1^l, z_2 \rightarrow y$.

If $\overline{K}$ contains a primitive l -th root of unity ζ we can describe $X_\sigma \cong \text{spec} \overline{K}[g^{-1}(\sigma \cap M)]$ as in the statement of the theorem. We can define an action $\tilde{\zeta}$ on $\mathbb{A}_{\overline{K}}^2 \cong \text{spec} \overline{K}[g^{-1}(\sigma \cap M^*)]$ by :

$$\tilde{\zeta}(x) = \zeta \cdot x, \quad \tilde{\zeta}(y) = y$$

The invariants of the group $\mu := \langle \tilde{\zeta} \rangle$ are generated by the monomials $x^r y^s$, $r, s \in \mathbb{Z}$ that are in $g^{-1}(\sigma \cap M^*)$. So we have :

$$\overline{K}[f^{-1}(\sigma \cap M)] = \overline{K}[g^{-1}(\sigma \cap M)] = K[g^{-1}(\sigma \cap M^*)]^\mu.$$

This shows that : $X_\sigma = \mathbb{A}_{\overline{K}}^2 / \mu$.

LEMMA 2.3. Let $X \subseteq Y$ be a monomial rational domain such that $C = v(X)$ is a convex domain in $v(Y)$ with $\text{int}(C) \neq \emptyset$. Let $P_1, \dots, P_S$ be the extremal points of C . Let M_i be the cone associated to the extremal point P_i (See the remark just before lemma 2.2). Let $n_i = \left| \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right|$, where $ax + by = 0$ and $cx + dy = 0$ are the bordelines of M_i . Let $\overline{K}$ contain all the n_i -th roots of unity for $i = 1..s$. Now the reduction $\overline{X}$ of the monomial rational domain X is the following :

- a) Every extremal point P_i corresponds to exactly one affine surface $\mathbb{A}_{\overline{K}}^2 / \mu_i$
- b) If the line-segment $[P_i P_j]$ is part of the boundary of C , then the surfaces belonging to P_i and P_j have exactly one affine line $\mathbb{A}_{\overline{K}}^1$ in common.
- c) If the line-segment $[P_i P_j]$ is not contained in the boundary of C , then the surfaces belonging to P_i and P_j have exactly one point in common.

Proof. Let A be the affinoid algebra of X .

In lemma 2.2 we proved that there is a 1-1 correspondance between the points P_i and the minimal prime ideals p_i of $\overline{A}$. Now theorem 2.1 shows that $\text{spec}(B_i) := \text{spec}(\overline{A}/p_i)$ is the surface $\mathbb{A}_{\overline{K}}^2/\mu_i$ defined by the cone M_i .

Suppose the line-segment $[P_i P_j]$ is part of the boundary of C . The monomials in $\overline{A}$ that have an image $\neq 0$ in both B_i and B_j are the monomials $\left(\frac{z_1^n z_2^m}{c_{n,m}}\right)^l$, $l \in \mathbb{Z}_{\geq 0}$ which correspond to the rational line $P_i P_j$. This shows that the surfaces belonging to P_i and P_j have exactly one affine line $\mathbb{A}_{\overline{K}}^1$ in common.

Now suppose the line-segment $[P_i P_j]$ is not contained in the boundary of C . In this case there are no monomials in $\overline{A}$ which have a non-zero image in both B_i and B_j . So the surfaces belonging to P_i and P_j can have at most one point in common. Of course they have the point defined by $\frac{z_1^n z_2^m}{c_{n,m}} = 0$ for all $\frac{z_1^n z_2^m}{c_{n,m}}$ in $\overline{A}$ in common, since $\text{int}(C) \neq \emptyset$.

Remark : We are looking for admissible affinoid coverings of $K^2 - \{(0,0)\}$ that are invariant under the action of the group $\Gamma = \langle \gamma \rangle$. To find such coverings we use the fundamental domain of Γ given in proposition 2.1.

First we need the notion of a pure covering, since we want the reductions of the affinoid space to glue together nicely.

DEFINITION. Let Z be a rigid analytic space.

A *pure covering* $\mathcal{U} = (U_i)$ of Z is an admissible covering by affinoid subspaces U_i satisfying the following conditions :

- 1) For each i , U_i intersects a finite number of U_j
- 2) If $U_i \cap U_j \neq \emptyset$ then there exists a Zariski-open affine set $V_{ij} \subset \overline{U}_i$ such that $U_i \cap U_j = R_i^{-1}(V_{ij})$, where $R_i : U_i \rightarrow \overline{U}_i$ is the reduction, and $U_i \cap U_j$ is an affinoid space having reduction $R_{ij} : U_i \cap U_j \rightarrow V_{ij}$.

The word admissible in the definition means admissible with respect to a certain Grothendieck topology on Z .

Remark : There is a 1-1 correspondence between pure coverings $\mathcal{U}$ of a rigid analytic space Z and formal schemes $\mathfrak{X}$ over K^0 such that the generic fibre of the map $\mathfrak{X} \rightarrow \text{Spf } K^0$ is the space Z . In this case the closed fibre of the map $\mathfrak{X} \rightarrow \text{Spf } K^0$ is the reduction of Z with respect to the pure covering $\mathcal{U}$.

Indeed for an affinoid subspace $U_i \subset Z$, $U_i \in \mathcal{U}$ with affinoid algebra $A_i = K \langle x_1 \dots x_n \rangle / I$ we have :

$$A_i^\circ = \varprojlim A_i^\circ / m^s A_i^\circ = \varprojlim (K^\circ[x_1 \dots x_n] / I) / m^s (K^\circ[x_1 \dots x_n] / I).$$

Here is $m = K^{\circ\circ}$ if the valuation is discrete, otherwise we take $m = (\pi)$ for some $0 \neq \pi \in K^{\circ\circ}$.

Now $\text{Spf } A_i^\circ \subset \text{spec } K^\circ[x_1 \dots x_n] / I$ is the subspace defined by the ideal m . This shows that the map $\text{Spf } A_i^\circ \rightarrow \text{Spf } K^\circ$ has $\text{Sp}(A_i) = U_i$ as its generic fibre and $\overline{\text{Sp}(A_i)} = \overline{U_i}$ as its closed fibre. Now the properties 1 and 2 of the definition of a pure covering show that all maps $\text{Spf } A_i^\circ \rightarrow \text{Spf } K^\circ$ glue together nicely. So we get a formal scheme $\mathfrak{X} \rightarrow \text{Spf } K^\circ$ with Z as its generic fibre and the reduction of Z with respect to the pure covering $\mathcal{U}$ as its closed fibre.

LEMMA 2.4. *Let (X_i) be a covering of $K^2 - \{(0,0)\}$, such that every X_i is a monomial rational domain and $\text{int}(C_i) \neq \emptyset$, where $C_i = v(X_i)$. Now (C_i) is a covering of $v(K^2 - \{(0,0)\})$ by convex rational domains.*

The covering (X_i) of $K^2 - \{(0,0)\}$ is pure if and only if :

- 1) *For each i , $C_i \cap C_j \neq \emptyset$ for at most a finite number of C_j .*
- 2) *$\forall i, j$, $C_i \cap C_j \neq \emptyset \Leftrightarrow C_i \cap C_j$ is a point P or*

*$C_i \cap C_j = C_i \cap L = C_j \cap L$, where L is a
rational line.*

Proof. Let us first show that a covering as described in the statement of the lemma is pure. If $C_i \cap C_j = \emptyset$ then also $X_i \cap X_j = \emptyset$. Since C_i intersects only a finite number of C_j , our covering satisfies condition 1 of the definition.

Let us assume $C_i \cap C_j \neq \emptyset$, so $C_i \cap C_j$ is a point P or a rational line L such that $C_i \cap L = C_j \cap L$. Now clearly $v^{-1}(C_i \cap C_j)$ is a affinoid subspace of $X_i = v^{-1}(C_i)$, since it is given by an equation $|x_{n,m}| = \left| \frac{z_1^n z_2^m}{c_{n,m}} \right| = 1$ in X_i if $C_i \cap C_j = L$, where L is the rational line $nx_1 + mx_2 = \log |c_{n,m}|$. When $C_i \cap C_j$ is a point P then $v^{-1}(C_i \cap C_j)$ is given by two such equations, coming from the two rational lines on the boundary of C_i intersecting each other in the extremal point P . The situation in C_j is identical.

In lemma 2.3 we proved that $\overline{X_i}$ is affine, so $v^{-1}(\overline{C_i \cap C_j})$ is also affine in $\overline{X_i}$. The set $v^{-1}(\overline{C_i \cap C_j})$ is defined in $\overline{X_i}$ by one or two equations of the form $\overline{x}_{n,m} \neq 0$, so $v^{-1}(\overline{C_i \cap C_j}) \subset \overline{X_i}$ is open affine subset.

Since $v^{-1}(C_i \cap C_j)$ describes the same affinoid subset of $K^2 - \{(0,0)\}$ in both X_i and X_j , we can identify $v^{-1}(\overline{C_i \cap C_j})$ in both $\overline{X_i}$ and $\overline{X_j}$. This shows that our covering satisfies condition 2 of the definition, since it is clear that $v^{-1}(C_i \cap C_j) = R_i^{-1}(v^{-1}(\overline{C_i \cap C_j}))$.

Let us now show that a covering (X_i) of $K^2 - \{(0,0)\}$ such that the covering (C_i) of $v(K^2 - \{(0,0)\})$ does not satisfy condition 1 or 2 in the statement of the lemma is not pure. There are now three possibilities :

- 1) There exists an i such that $C_i \cap C_j \neq \emptyset$ for an infinite number of C_j
- 2) There exists i, j such that $C_i \cap C_j = C_i \cap C_j \cap L$, but $C_i \cap L \neq C_j \cap L$, L is a rational line.
- 3) $\text{Int}(C_i \cap C_j) \neq \emptyset$ for some i, j .

It is easy to see that in all three cases the covering is not pure, since it does not satisfy some of the conditions in the definition above.

In case 1 it is clear that the covering (X_i) does not satisfy condition 1 of the definition, since X_i has a non-empty intersection with an infinite number of C_j .

In cases 2 and 3 the covering does not satisfy condition 2 of the definition above. In case 2 we have : $X_i \cap X_j \neq R_i^{-1}(V_{ij})$ or $X_i \cap X_j \neq R_j^{-1}(V_{ij})$. In case 3 $v^{-1}(\overline{C_i \cap C_j})$ is not open in $\overline{X_i}$ and $\overline{X_j}$.

Example : Let the group Γ be generated by a contraction γ . In proposition 2.1 we constructed a fundamental domain F for the action of Γ . The finite affinoid covering (F_i) of F we gave, where F_i is a monomial rational domain, can be used to give a pure covering of $K^2 - \{(0,0)\}$ that is Γ -invariant. Indeed the covering $(\gamma^j(F_i))_{j \in \mathbb{Z}}$ is Γ -invariant and pure by the lemma above if γ has the form with $\lambda = 0$. If $\lambda \neq 0$ then we can choose a small enough value of $|\lambda|$ such that $\gamma^i(F_k) \cap \gamma^{i+1}(F_l)$ and $\gamma^i(F_k) \cap \gamma^i(F_l)$, $k, l = 1, 2$ are pure by the lemma above. Since $\gamma^j(F_k) \cap \gamma^j(F_l) = \emptyset$ if $|i - j| \geq 2$ the covering $(\gamma^j(F_i))_{j \in \mathbb{Z}}$ is again pure.

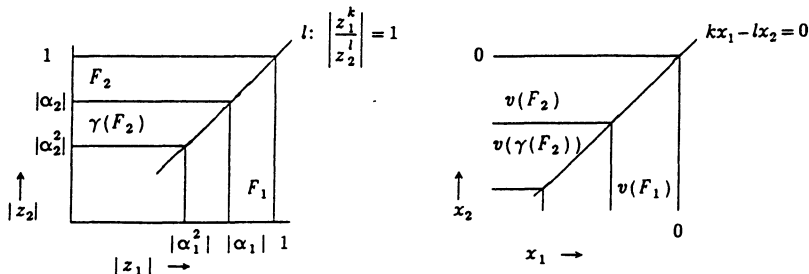
We will now study in some detail the case where γ is defined by :

$$\gamma(z_1, z_2) = (\alpha_1(z_1 + \lambda z_2^m), \alpha_2 z_2), \quad 0 < |\alpha_1| \leq |\alpha_2| < 1, |\lambda| < 1$$

and $\lambda = 0$ if $\alpha_1 \neq \alpha_2^m$ and satisfies the extra condition : $|\alpha_1^k| = |\alpha_2^l|$ for some $k, l \in \mathbb{Z}_{>0}$.

In proposition 2.1. we showed that the fundamental domain F can be covered by the following two affinoid subspaces :

$$F_1 = Sp\ K < z_1, \frac{\alpha_1}{z_1}, \frac{z_2^l}{z_1^k} > \text{ and } F_2 = Sp\ K < z_2, \frac{\alpha_2}{z_2}, \frac{z_1^k}{z_2^l} >$$



In the next lemma we will study the special case $|\alpha_1| = |\alpha_2|$. Then we will study the case $|\alpha_1^k| = |\alpha_2^l|$.

LEMMA 2.5. *Let Γ be generated by a contraction γ such that $|\alpha_1| = |\alpha_2|$. Let $\{\gamma^i(F_1), \gamma^j(F_2) | i, j \in \mathbb{Z}\}$ be the pure Γ -invariant covering C of $K^2 - \{(0, 0)\}$ given above. The reduction of $K^2 - \{(0, 0)\}$ with respect to this covering C has for every extremal point P of the convex domains $v(\gamma^i(F_j)), j = 1, 2, i \in \mathbb{Z}$, a surface $\tilde{\mathbb{P}}_K^2$, i.e. a $\mathbb{P}_K^2$ blown up in one point. The surface $\tilde{\mathbb{P}}_K^2$ corresponding to the extremal points P and $\gamma(P)$ have one $\mathbb{P}_K^1$ in common, this line is exceptional in the $\tilde{\mathbb{P}}_K^2$ belonging to P and ordinary in the other.*

Proof. Let $\gamma^i(A_j), j = 1, 2, i \in \mathbb{Z}$ be the affinoid algebra belonging to $\gamma^i(F_j)$. Let $[\gamma^i(A_j)]_P$ be the component of the reduction of $\gamma^i(A_j)$ that corresponds to the extremal point P of the convex domain

$$v(\gamma^i(A_j)) \subset v(K^2 - \{(0, 0)\}).$$

Since the covering C is Γ -invariant and Γ acts transitively on the sets of extremal points, it is sufficient to look at one extremal point P . We choose $P = (\log |\alpha_1|, \log |\alpha_2|)$. The point P is an extremal point of the following four convex domains in $v(K^2 - \{(0, 0)\})$: $v(F_1)$, $v(F_2)$, $v(\gamma(F_1))$ and $v(\gamma(F_2))$. So we have to consider the following affinoid algebras and their

reduction in P :

$$\begin{aligned} A_1 &= K < z_1, \frac{\alpha}{z_1}, \frac{z_2}{z_1} > \quad [\overline{A_1}]_P = \overline{K} \left[\frac{\overline{\alpha}}{z_1}, \frac{\overline{z_2}}{z_1} \right] \\ \gamma(A_1) &= K < \frac{z_1}{\alpha}, \frac{\alpha^2}{z_1}, \frac{z_2}{z_1} > \quad [\gamma(\overline{A_1})]_P = \overline{K} \left[\frac{\overline{z_1}}{\alpha}, \frac{\overline{z_2}}{z_1} \right] \\ A_2 &= K < z_2, \frac{\alpha}{z_2}, \frac{z_1}{z_2} > \quad [\overline{A_2}]_P = \overline{K} \left[\frac{\overline{\alpha}}{z_2}, \frac{\overline{z_1}}{z_2} \right] \\ \gamma(A_2) &= K < \frac{z_2}{\alpha}, \frac{\alpha^2}{z_2}, \frac{z_1}{z_2} > \quad [\gamma(\overline{A_2})]_P = \overline{K} \left[\frac{\overline{z_2}}{\alpha}, \frac{\overline{z_1}}{z_2} \right] \end{aligned}$$

Here $\alpha \in K$ is chosen such that $|\alpha| = |\alpha_1| = |\alpha_2|$. We will now glue the reductions together.

The glueing of $[\overline{A_1}]_P$ and $[\gamma(\overline{A_1})]_P$ along their intersection defined by $\frac{\overline{\alpha}}{z_1} \neq 0, \frac{\overline{z_1}}{\alpha} \neq 0$ and identifying $\left(\frac{\overline{\alpha}}{z_1}\right)^{-1}$ with $\frac{\overline{z_1}}{\alpha}$ gives us a $\mathbb{P}_K^1 \times \mathbb{A}_K^1$ and has coordinate ring $\overline{K}[x_1, x_3] \times \overline{K}\left[\frac{y_0}{y_1}\right]$ where $\frac{x_1}{x_3} = \frac{\overline{z_1}}{\alpha}, \frac{x_3}{x_1} = \frac{\overline{\alpha}}{z_1}$ and $\frac{y_0}{y_1} = \frac{\overline{z_2}}{z_1}$.

The glueing of $[\overline{A_2}]_P$ and $[\gamma(\overline{A_2})]_P$ gives us again a $\mathbb{P}_K^1 \times \mathbb{A}_K^1$ defined by $\overline{K}[x_0, x_2] \times \overline{K}\left[\frac{y_1}{y_0}\right]$, where $\frac{x_2}{x_0} = \frac{\overline{\alpha}}{z_2}, \frac{x_0}{x_2} = \frac{\overline{z_2}}{\alpha}$ and $\frac{y_1}{y_0} = \frac{\overline{z_1}}{z_2}$.

Now we have to glue these two surfaces $\mathbb{P}_K^1 \times \mathbb{A}_K^1$ along their intersection given by $\frac{y_0}{y_1} \neq 0, \frac{y_1}{y_0} \neq 0$. So we identify $\left(\frac{y_0}{y_1}\right)^{-1}$ with $\frac{y_1}{y_0}$ and use the relations :

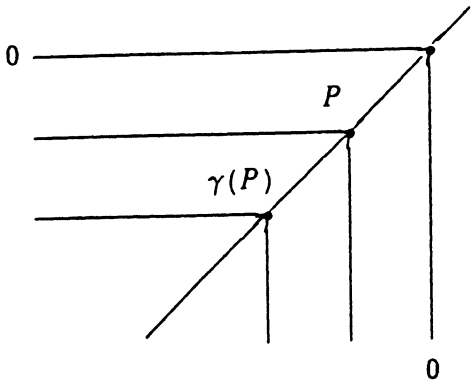
$$\frac{x_1}{x_3} \cdot \frac{x_2}{x_0} = \frac{\overline{z_1}}{\alpha} \cdot \frac{\overline{\alpha}}{z_2} = \frac{\overline{z_1}}{z_2} = \frac{y_0}{y_1} \quad \text{and} \quad \frac{x_3}{x_1} \cdot \frac{x_0}{x_2} = \frac{y_1}{y_0}.$$

So we have to identify x_0 with x_1 , or x_3 with x_2 . We choose $x_2 = x_3$. Now the glueing gives us a homogeneous coordinate ring

$$\overline{K}[x_0, x_1, x_2] \times \overline{K}[y_0, y_1] / (x_0 y_0 - x_1 y_1).$$

This is the coordinate ring of a surface $\tilde{\mathbb{P}}_K^2$ (see [H]). We have

$$\frac{x_0}{x_2} = \frac{\overline{z_2}}{\alpha}, \frac{x_1}{x_2} = \frac{\overline{z_1}}{\alpha}, \frac{y_0}{y_1} = \frac{\overline{z_2}}{z_1}, \text{ etc...}$$



In the point $\gamma(P)$ we also find a surface $\tilde{\mathbb{P}}^2_K$ given by the homogeneous coordinate ring $\overline{K} \left[\frac{x_0}{\alpha}, \frac{x_1}{\alpha}, x_2 \right] \times \overline{K} \left[\frac{y_0}{\alpha}, \frac{y_1}{\alpha} \right] / \left(\frac{x_0}{\alpha} \frac{y_0}{\alpha} - \frac{x_1 y_1}{\alpha \alpha} \right)$. We will now describe the intersection of these two surfaces $\tilde{\mathbb{P}}^2_K$. In the $\tilde{\mathbb{P}}^2_K$ belonging to P this intersection is determined by $\left| \frac{z_1}{\alpha} \right|, \left| \frac{z_2}{\alpha} \right| < 1$. In the coordinate ring this space is determined by $x_0 = x_1 = 0$. So we find the exceptional line given by $\overline{K}[0, 0, 1] \times \overline{K}[y_0, y_1]$ in the $\tilde{\mathbb{P}}^2_K$ belonging to P .

In the $\tilde{\mathbb{P}}^2_K$ belonging to $\gamma(P)$ the intersection is defined by $\left| \frac{\alpha^2}{z_1} \right|, \left| \frac{\alpha^2}{z_2} \right| < 1$. So in the coordinate ring this space is given by

$$x_2 / \left(\frac{x_0}{\alpha} \right) = x_2 / \left(\frac{x_1}{\alpha} \right) = 0,$$

therefore $x_2 = 0$. We find the ordinary line given by

$$\overline{K} \left[\frac{x_0}{\alpha}, \frac{x_1}{\alpha}, 0 \right] \times \overline{K} \left[\frac{y_0}{\alpha}, \frac{y_1}{\alpha} \right] / \left(\frac{x_0 y_0}{\alpha \alpha} - \frac{x_1 y_1}{\alpha \alpha} \right)$$

in the $\tilde{\mathbb{P}}^2_K$ belonging to $\gamma(P)$.

So the reduction of $K^2 - \{(0, 0)\}$ with respect to the covering $\mathcal{C}$ is given by a string of surfaces $\tilde{\mathbb{P}}^2_K$, glued together as in the figure by identifying an exceptional line e in one $\tilde{\mathbb{P}}^2_K$ with an ordinary line o in the next surface $\tilde{\mathbb{P}}^2_K$.

$$\overset{o}{\text{---}} \tilde{\mathbb{P}}^2_K \overset{e}{\text{---}} \tilde{\mathbb{P}}^2_K \overset{e}{\text{---}} \tilde{\mathbb{P}}^2_K \overset{e}{\text{---}} \tilde{\mathbb{P}}^2_K \overset{e}{\text{---}} \tilde{\mathbb{P}}^2_K \overset{e}{\text{---}}$$

Remark : Since the group Γ above acts transitively on the set of extremal points, the reduction of the Hopf surface $K^2 - \{(0,0)\}/\Gamma$ is given by a surface $\tilde{\mathbb{P}}_{\overline{K}}^2/\sim$, where $\sim$ is the relation that identifies the exceptional line e with the ordinary line o , when we use the covering $\mathcal{C}$ of $K^2 - \{(0,0)\}$ given above.

THEOREM 2.2. *Let $\Gamma = \langle \gamma \rangle$ be generated by a contraction γ with $|\alpha_1^k| = |\alpha_2^l|$ for some $k, l \in \mathbb{Z}_{>0}$ with g.c.d. $(k, l) = 1$ and suppose K contains a primitive $k \cdot l$ -th root of unity. Let F be the fundamental domain of Γ constructed in proposition 2.1. Let $\mathcal{C}$ be the pure Γ -invariant covering $\{\gamma^i(F_1), \gamma^j(F_2) \mid i, j \in \mathbb{Z}\}$ of $K^2 - \{(0,0)\}$.*

Every extremal point P of a convex domain $v(\gamma^i(F_j)), j = 1, 2, i \in \mathbb{Z}$ corresponds to a surface $\tilde{\mathbb{P}}_{\overline{K}}^2/\mu_{k,l}$ in the reduction. The group $\mu_{k,l}$ is a finite cyclic group of order kl acting diagonally on $\tilde{\mathbb{P}}_{\overline{K}}^2$. The surfaces $\tilde{\mathbb{P}}_{\overline{K}}^2/\mu_{k,l}$ belonging to the extremal points P and $\gamma(P)$ have a line in common. This line is the image of the exceptional line in the $\tilde{\mathbb{P}}_{\overline{K}}^2/\mu_{k,l}$ belonging to P and it is the image of an ordinary line in the surface belonging to the extremal point $\gamma(P)$.

Proof. We only have to prove the theorem for one extremal point P , since Γ acts transitively on the set of extremal points. We choose $P = (\log |\alpha_1|, \log |\alpha_2|)$. So P is an extremal point of the following four convex domains in $v(K^2 - \{(0,0)\})$:

$$v(F_1), v(F_2), v(\gamma(F_1)) \text{ and } v(\gamma(F_2)).$$

The associated affinoid algebras and their reductions in P are :

$$\begin{aligned} A_1 &= K \left\langle z_1, \frac{\alpha_1}{z_1}, \frac{z_2^l}{z_1^k} \right\rangle & [\overline{A_1}]_P &= \overline{K} \left[\frac{\overline{\alpha_1}}{z_1}, \dots, \frac{\overline{z_2^l}}{z_1^k} \right] \\ \gamma(A_1) &= K \left\langle \frac{z_1}{\alpha_1}, \frac{\alpha_1^2}{z_1}, \frac{z_2^l}{z_1^k} \right\rangle & [\overline{\gamma(A_1)}]_P &= \overline{K} \left[\frac{\overline{z_1}}{\alpha_1}, \dots, \frac{\overline{z_2^l}}{z_1^k} \right] \\ A_2 &= K \left\langle z_2, \frac{\alpha_2}{z_2}, \frac{z_1^k}{z_2^l} \right\rangle & [\overline{A_2}]_P &= \overline{K} \left[\frac{\overline{\alpha_2}}{z_2}, \dots, \frac{\overline{z_1^k}}{z_2^l} \right] \\ \gamma(A_2) &= K \left\langle \frac{z_2}{\alpha_2}, \frac{\alpha_2^2}{z_2}, \frac{z_1^k}{z_2^l} \right\rangle & [\overline{\gamma(A_2)}]_P &= \overline{K} \left[\frac{\overline{z_2}}{\alpha_2}, \dots, \frac{\overline{z_1^k}}{z_2^l} \right] \end{aligned}$$

Since $\text{char}(K) \nmid k, l$, we can use theorem 2.1.c to get a more convenient description of $[\overline{A_i}]_P, i = 1, 2$ and $[\gamma(\overline{A_i})]_P, i = 1, 2$. Let us take new variables u, v such that $u^l = \frac{z_1}{\alpha_1}$ and $v^l = \frac{z_2}{\alpha_2}$. Now we have :

$$\begin{aligned} [\overline{A_1}]_P &= \overline{K} \left[u^{-1}, \frac{\overline{z_2}}{\alpha_2} / u^k \right]^\mu, \quad \mu = \langle \tilde{\zeta} \rangle \\ [\gamma(\overline{A_1})]_P &= \overline{K} \left[u, \frac{\overline{z_2}}{\alpha_2} / u^k \right]^\mu \\ [\overline{A_2}]_P &= \overline{K} \left[v^{-1}, \frac{\overline{z_1}}{\alpha_1} / v^l \right]^\mu \\ [\gamma(\overline{A_2})]_P &= \overline{K} \left[v, \frac{\overline{z_1}}{\alpha_1} / v^l \right]^\mu \end{aligned}$$

Here $\tilde{\zeta}$ is defined by $\tilde{\zeta}(u) = \zeta^k \cdot u, \tilde{\zeta}(v) = \zeta^l \cdot v, \tilde{\zeta}\left(\frac{\overline{z_2}}{\alpha_2}\right) = \frac{\overline{z_2}}{\alpha_2}, \tilde{\zeta}\left(\frac{\overline{z_1}}{\alpha_1}\right) = \frac{\overline{z_1}}{\alpha_1}$, where ζ is a primitive $k \cdot l$ -th root of unity.

Furthermore we have :

$$\begin{aligned} (u^{-1})^r \left(\frac{v}{u} \right)^s &\in \overline{K}[u^{-1}, v/u]^\mu \\ \Leftrightarrow kl - kr + (l - k)s & \\ \Leftrightarrow k|s \wedge l|r + s, \text{ since } g.c.d. (k, l) = 1 & \\ \Leftrightarrow (u^{-1})^r \left(\frac{v}{u} \right)^s &\in \overline{K} \left[u^{-1}, \frac{v^k}{u^k} \right]^\mu = \overline{K} \left[u^{-1}, \frac{\overline{z_2}}{\alpha_2} / u^k \right]^\mu = [\overline{A_1}]_P. \end{aligned}$$

This shows that we have in fact :

$$\begin{aligned} [\overline{A_1}]_P &= \overline{K} \left[u^{-1}, \frac{\overline{z_2}}{\alpha_2} / u^k \right]^\mu = \overline{K} \left[u^{-1}, \frac{v^k}{u^k} \right]^\mu = \overline{K} \left[u^{-1}, \frac{v}{u} \right]^\mu. \\ [\gamma(\overline{A_1})]_P &= \overline{K} \left[u, \frac{\overline{z_2}}{\alpha_2} / u^k \right]^\mu = \overline{K} \left[u, \frac{v^k}{u^k} \right]^\mu = \overline{K} \left[u, \frac{v}{u} \right]^\mu \\ [\overline{A_2}]_P &= \overline{K} \left[v^{-1}, \frac{\overline{z_1}}{\alpha_1} / v^l \right]^\mu = \overline{K} \left[v^{-1}, \frac{u^l}{v^l} \right]^\mu = \overline{K} \left[v^{-1}, \frac{u}{v} \right]^\mu \\ [\gamma(\overline{A_2})]_P &= \overline{K} \left[v, \frac{\overline{z_1}}{\alpha_1} / v^l \right]^\mu = \overline{K} \left[v, \frac{u^l}{v^l} \right]^\mu = \overline{K} \left[v, \frac{u}{v} \right]^\mu. \end{aligned}$$

The group $\mu = \mu_{k,l}$ works identical on the intersections of the affine spaces. So we can interchange the group action and the glueing.

In lemma 2.5 we already proved that the glueing gives us a surface $\tilde{\mathbb{P}}_{\bar{K}}^2$. The group μ acts diagonally on this surface $\tilde{\mathbb{P}}_{\bar{K}}^2$.

Let $K[x_0, x_1, x_2] \times K[y_0, y_1]/(x_0 y_0 - x_1 y_1)$ be the homogeneous coordinate ring of $\tilde{\mathbb{P}}_{\bar{K}}^2$. Here $\frac{x_0}{x_2} = u$, $\frac{x_1}{x_2} = v$, $\frac{y_0}{y_1} = \frac{u}{v}$ etc. This shows that the action of μ is defined by :

$$\begin{aligned}\tilde{\zeta}(x_0) &= \zeta^k x_0 \\ \tilde{\zeta}(x_1) &= \zeta^l x_1 \\ \tilde{\zeta}(x_2) &= x_2 \\ \tilde{\zeta}(y_0) &= \zeta^l y_0 \\ \tilde{\zeta}(y_1) &= \zeta^k y_1\end{aligned}\quad \zeta^{kl} = 1, \mu = \langle \tilde{\zeta} \rangle.$$

Since the finite group μ acts diagonally, the invariants of μ are generated by monomials. If we forget about degree the generators are :

$$x_2, x_0^i x_1^{l-i}, i = 0 \dots l, x_1^i y_0^{k-i}, i = 0 \dots k.$$

The homogeneous algebra of invariants is generated by the monomials of degree kl in the x_i and/or y_i that are μ -invariant. So the generators are :

$$\begin{aligned}x_2^{kl-i-j} x_0^i x_1^j y_0^{r k-j} y_1^{s l-i} & \quad r l + s k - i - j = kl \\ y_0^{kl}, y_1^{kl} \\ x_2^{kl-ik-jl} x_0^{jl} x_1^{ik} & \quad ik + jl \leq kl.\end{aligned}$$

Of course there are relations between these generators. One directly sees that there are relations of the form $s_1 s_2 = s_3 s_4$, where the s_i are some generators such that $s_1 s_2$ is the same monomial as $s_3 s_4$.

The relation $x_0 y_0 - x_1 y_1 = 0$ gives rise to the following set of relations :

$$x_2^{kl-i-j} x_0^i x_1^j y_0^{al-j} y_1^{bk-i} - x_2^{kl-r-s} x_1^r x_0^s y_0^{al-r} y_1^{bk-s} = 0,$$

where $i + j = r + s$ and $al + bk - i - j = kl$.

This shows that we can reduce the number of generators. We will not go into this any further.

Since the surfaces $\tilde{\mathbb{P}}_{\bar{K}}^2/\mu_{k,l}$, belonging to the extremal points P and $\gamma(P)$, are the images of surfaces $\tilde{\mathbb{P}}_{\bar{K}}^2$, belonging to P and $\gamma(P)$, which intersect each other as in lemma 2.5, the last part of the theorem is clear.

Remark : Let $\mathcal{C} = (X_i)$ be a pure affinoid covering of $K^2 - \{(0,0)\}$ such that all X_i are monomial rational domains. Every extremal point P of a convex rational domain $C_i = v(X_i)$ gives a surface in the reduction of $K^2 - \{(0,0)\}$ with respect to $\mathcal{C}$. The surface belonging to P is the surface given by a conic decomposition of $\mathbb{R}^2$. Every cone σ gives a surface $X_{\check{\sigma}}$, where $\check{\sigma}$ is the dual cone of σ . So $\check{\sigma}$ is given by the points $(n, m) \in \mathbb{R}^2$ such that the lines $nx + my$ have only values ≥ 0 on σ . The surface $X_{\check{\sigma}}$ is the component of the reduction of X_i in P and σ is the cone given by the two half-lines through P bounding $C_i = v(X_i)$. We have $X_{\check{\sigma}} = \text{spec } K[f^{-1}(\check{\sigma} \cap \mathbb{Z}^2)]$, where f is the map $f : z_1^n z_2^m \rightarrow (n, m)$. In fact we have $\check{\sigma} = f(\overline{B_i})$, where B_i is the set of the monomials in the affinoid algebra A_i of X_i that obtain their maximum value in P . The glueing of the surfaces $X_{\check{\sigma}}$ for the cones σ defined by P gives us the surface in the reduction belonging to P .

We will now study coverings $\mathcal{C}$ such that to every extremal point P belongs a non-singular surface.

DEFINITIONS. A conic decomposition is called *regular* if the surface $X_{\check{\sigma}}$ is non-singular for every cone σ in the decomposition.

A regular conic decomposition is called *minimal* if there are no cones σ_i and σ_j in the decomposition such that the decomposition obtained by replacing σ_i and σ_j by their union $\sigma_i \cup \sigma_j$ is again a regular conic decomposition.

Remark : In theorem 2.1.b we showed that the surface $X_{\check{\sigma}}$ is non-singular if and only if the semigroup $\check{\sigma} \cap \mathbb{Z}^2$ is generated by a $\mathbb{Z}$ -basis of $\mathbb{Z}^2$.

This shows that if $\sigma = \{\lambda(ae_1 + be_2) + \mu(ce_1 + de_2) \mid \lambda, \mu \in \mathbb{R}_{\geq 0}\}$ where $\{e_1, e_2\}$ is a basis of $\mathbb{Z}^2$ and $a, b, c, d \in \mathbb{Z}$ with $\text{g.c.d.}(a, b) = \text{g.c.d.}(c, d) = 1$ the surface $X_{\check{\sigma}}$ is non-singular if and only if $\det\left(\begin{pmatrix} a & c \\ b & d \end{pmatrix}\right) = \pm 1$.

DEFINITION. We denote the rational ruled surfaces $\mathbb{P}(\mathcal{O}(m) \oplus \mathcal{O})$ by Σ_m . (See [H]). Sometimes these surfaces are called the *Hirzebruch surfaces* in the literature.

THEOREM 2.3.

- a) A minimal regular conic decomposition corresponds with one of the following surfaces :

$$\Sigma_0, \mathbb{P}_K^2, \Sigma_m, m \geq 2$$

- b) *Every non-minimal regular conic decomposition gives a non-singular surface which can be obtained from one of the surfaces above by a finite succession of blow ups.*

Proof. These facts are proved in [O.1] and [O.2]. We will not recall the proof here. In the next remark we will describe the minimal regular conic decompositions.

Remark : In the pictures below we give the minimal regular conic decompositions and the surfaces defined by them.

$$\begin{array}{c}
 \sigma_1 \quad | \quad \sigma_2 \\
 \hline
 \sigma_4 \quad | \quad \sigma_3 \\
 x=0
 \end{array}
 \quad y=0
 \quad \Sigma_0 = \mathbb{P}_K^1 \times \mathbb{P}_K^1$$

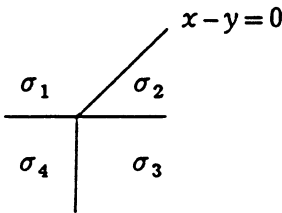
$$\begin{array}{c}
 \sigma_1 \quad \diagup \quad x-y=0 \\
 \hline
 \sigma_3 \quad | \quad \sigma_2
 \end{array}
 \quad \mathbb{P}_K^2$$

$$\begin{array}{c}
 \sigma_1 \quad \diagup \quad x-my=0 \\
 \hline
 \sigma_4 \quad | \quad \sigma_3
 \end{array}
 \quad \Sigma_m, \quad m \geq 2$$

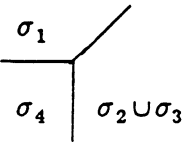
So a minimal regular conic decomposition consists of 3 or 4 cones as above.

If a regular conic decomposition R_1 is not minimal then there are two cones σ_i and σ_j in R_1 such that replacing σ_i and σ_j by their union $\sigma_i \cup \sigma_j$ gives another regular conic decomposition R_2 . The surface defined by R_2 is a blowing down of the surface defined by R_1 . By induction this process stops if we reach a minimal regular conic decomposition as above. Below

we show an example.

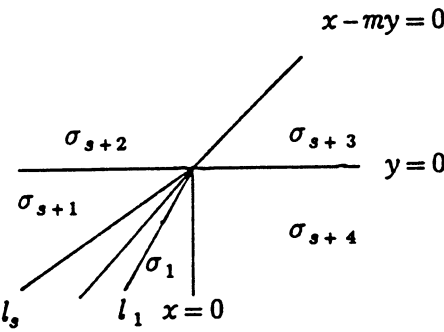


R_1 defines a $\widetilde{\mathbb{P}}^2_K = \Sigma_1$



R_2 defines a $\mathbb{P}^2_K$

For later use we give the following decomposition :

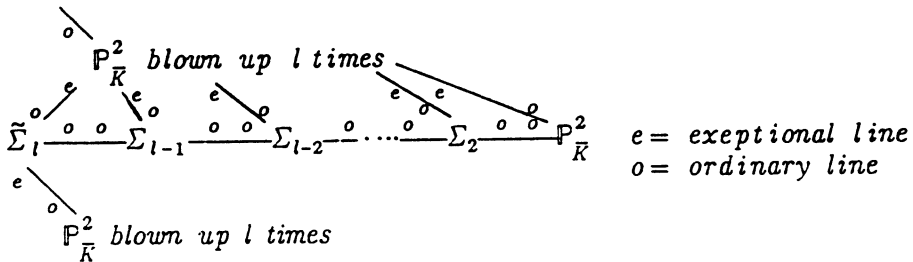


Here l_i is given by $x - iy = 0$, $i = 1 \cdots s$. The regular conic decomposition R_{s-i+1} is given by replacing $\sigma_{s+1}, \sigma_s, \dots, \sigma_i$ by $\sigma_{s+1} \cup \sigma_s \cup \dots \cup \sigma_i$. Since R_s defines a surface Σ_m , R_0 defines a surface Σ_m blown up s times.

THEOREM 2.4. *Let Γ be generated by a contraction γ . There exists a pure Γ -invariant affinoid covering $\mathcal{C} = (X_i)$ of $K^2 - \{(0, 0)\}$ where the X_i are monomial rational domains such that every extremal point P of $C_i = v(X_i)$ gives a non-singular surface in the reduction.*

Proof. If $|\alpha_1| = |\alpha_2|$ we have proved this in lemma 2.5. All other cases are proved in the next proposition.

PROPOSITION 2.3. *Let Γ be generated by a contraction γ with $0 < |\alpha_1| < |\alpha_2| < 1$. Now there exists for every $l \in \mathbb{Z}_{>1}$ such that $|\alpha_2^l| < |\alpha_1|$ a pure affinoid covering as stated in theorem 2.4 above. The reduction is as shown in the figure below.*



Proof. Let F be the fundamental domain of Γ constructed in proposition 2.1. Let v denote as before the map $v : (z_1, z_2) \rightarrow (\log |z_1|, \log |z_2|)$.

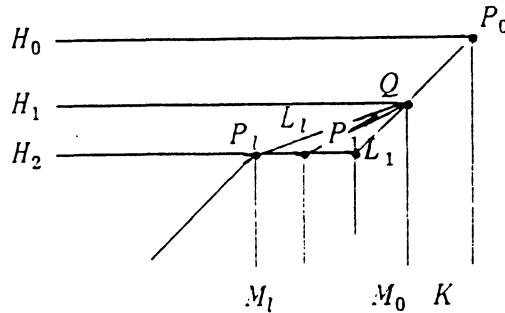
We now look at $v(F)$. Let the point $Q = (q_1, q_2) \in v(F)$ be defined by :

$$\begin{cases} x_1 - x_2 = 0 \\ x_1 - lx_2 = \log |\alpha_1| - l \cdot \log |\alpha_2| \end{cases}$$

This point Q is in $v(F)$ since $0 < |\alpha_2^l| < |\alpha_1| < |\alpha_2| < 1$ and therefore we have

$$0 > q_1 = q_2 = \frac{\log |\alpha_1| - l \cdot \log |\alpha_2|}{1 - l} > \log |\alpha_2| > \log |\alpha_1|.$$

We now cover the area $v(F)$ by a finite number of convex domains as shown in the figure.



The line-segments drawn in the figure above are the following :

L_1	$x_1 - x_2 = 0$	$x_2 \in [\log \alpha_2 , 0]$	
L_i	$x_1 - ix_2 = q_1 - iq_2$	$x_2 \in [\log \alpha_2 , q_2]$	$i = 2 \dots l$
M_0	$x_1 = q_1$	$x_2 \in [-\infty, q_2]$	
M_i	$x_1 = q_1 - iq_2 + i \log \alpha_2 $	$x_2 \in [-\infty, \log \alpha_2]$	$i = 1 \dots l$
K	$x_1 = 0$	$x_2 \in [-\infty, 0]$	
H_0	$x_2 = 0$	$x_1 \in [-\infty, 0]$	
H_1	$x_2 = q_2$	$x_1 \in [-\infty, q_1]$	
H_2	$x_2 = \log \alpha_2 $	$x_1 \in [-\infty, \log \alpha_2]$	

The extremal points of the convex domains are Q , $P_i, i = 0 \dots l$. The points P_i are given by $P_0 = (0, 0)$ and $P_i = (q_1 - iq_2 + i \log |\alpha_2|, \log |\alpha_2|)$, $i = 1 \dots l$.

Now proposition 2.2 tells us that the covering of $v(F)$ with convex domains as above corresponds with a covering of F with monomial rational domains. Furthermore lemma 2.4 shows us that the covering of $K^2 - \{(0, 0)\}$ arising from this covering of the fundamental domain F by the action of Γ is a pure affinoid covering of $K^2 - \{(0, 0)\}$.

The extremal points $\mathcal{Q}$, $P_i, i = 0..l$ give surfaces in the reduction of $K^2 - \{(0,0)\}$. The remark before theorem 2.4 shows that the surfaces are the following :

to $\mathcal{Q}$ belongs a $\mathbb{P}_{\bar{K}}^2$ blown up l times.

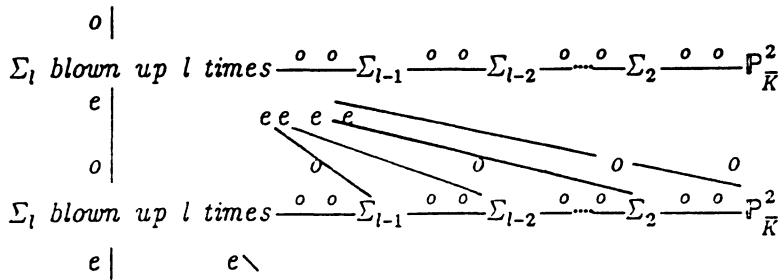
to P_1 belongs a $\mathbb{P}_{\bar{K}}^2$.

to $P_i, i = 2 \cdots l - 1$ belongs a Σ_i .

to P_0 and $P_l = \gamma(P_0)$ belongs a $\tilde{\Sigma}_l$, i.e. a Σ_l blown up one time.

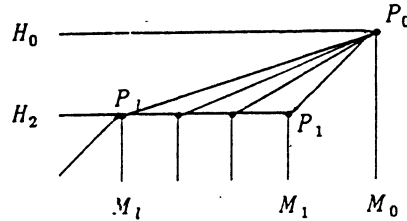
The surfaces belonging to two different extremal points have at most one line in common. They have exactly one line $\mathbb{P}_{\bar{K}}^1$ in common if and only if the extremal points are joined by a line segment that is a part of the boundary of a convex domain. The line they have in common is defined by the monomials that reach their maximum on this rational line. A direct calculation shows that these lines are as in the statement of the proposition.

PROPOSITION 2.4. *Let Γ be generated by a contraction γ such that $|\alpha_2^l| = |\alpha_1|$ for some $l \in \mathbb{Z}_{>1}$. Now there exists a pure affinoid covering as stated in theorem 2.4 above. The reduction is shown in the figure below.*



Proof. The construction of the covering is the same as in proposition 2.3. The only difference is that we have now $\mathcal{Q} = P_0$. Therefore $H_1 = H_0$ and $K = M_0$.

Looking at the figure below it is clear that we have a covering as in theorem 2.4 and that the reduction is as stated above.



Remark. Using the coverings of $K^2 - \{(0,0)\}$ given above, we can find a reduction of the Hopf surface $X = K^2 - \{(0,0)\}/\Gamma$. Since Γ covers $K^2 - \{(0,0)\}$ with the images of the fundamental domain F , it is clear that the reduction of X is as shown in the statements of the propositions 2.3 and 2.4 above with an identification of some lines.

We will now construct another example with $\Gamma = \langle \gamma, \tilde{\zeta} \rangle$ where γ is the contraction $\gamma : (z_1, z_2) \rightarrow (\alpha_1 z_1, \alpha_2 z_2)$ and $\tilde{\zeta}$ generates Γ_{tors} , $\tilde{\zeta}$ is defined by $\tilde{\zeta} : (z_1, z_2) \rightarrow (\zeta z_1, \zeta z_2)$ where ζ is a primitive m -th root of unity. In this case the Hopf surface $X = K^2 - \{(0,0)\}/\Gamma$ has a nice reduction $\Sigma_m/\sim$, where $\sim$ is an equivalence relation identifying two lines of Σ_m .

LEMMA 2.6. Let ζ be a primitive m -th root of unity. Let $\langle \tilde{\zeta} \rangle$ be a group acting on $K^2 - \{(0,0)\}$ where $\tilde{\zeta}$ is defined by $\tilde{\zeta} : (z_1, z_2) \rightarrow (\zeta z_1, \zeta z_2)$.

Let Σ_m be as above with a homogeneous coordinate ring

$$K[x_1, x_2][z, y_0, \dots, y_m]/I,$$

where I is the ideal

$$I = \langle y_i x_1^j x_2^{m-j} - y_j x_1^i x_2^{m-i} \mid 0 \leq i, j \leq m \rangle \quad (\text{see [H]}).$$

Now we have : $K^2 - \{(0,0)\}/\langle \tilde{\zeta} \rangle \simeq \Sigma_m \setminus S$.

Here S consists of the two lines $\mathbb{P}_K^1$ defined by

$$z = 0 \text{ and by } y_0 = y_1 = \dots = y_m = 0.$$

Proof. The group $\langle \tilde{\zeta} \rangle$ acts on the two open subspaces $K^* \times K$ of $K^2 - \{(0,0)\}$ defined by $z_1 \neq 0$ and $z_2 \neq 0$. These two subspaces cover the

whole of $K^2 - \{(0,0)\}$. Now it is sufficient to look at the invariants of the group action on these two open subspaces. The invariants are generated by :

$$\begin{cases} \frac{z_2}{z_1} \text{ and } z_1^m & \text{if } z_1 \neq 0 \\ \frac{z_1}{z_2} \text{ and } z_2^m & \text{if } z_2 \neq 0 \end{cases}$$

We will now glue the spaces defined by these coordinates together to find a description of $K^2 - \{(0,0)\} / \langle \tilde{\zeta} \rangle$. First we take homogeneous coordinates x_1, x_2 such that $\frac{x_1}{x_2} = \frac{z_1}{z_2}$ and $\frac{x_2}{x_1} = \frac{z_2}{z_1}$. Let $y_i, i = 0 \dots m$ be the non-homogeneous coordinates $y_i = z_1^i z_2^{m-i}, i = 0 \dots m$. The y_i are global sections of $K^2 - \{(0,0)\} / \langle \tilde{\zeta} \rangle$. Together with x_1 and x_2 they give a complete description of the coordinate ring of $K^2 - \{(0,0)\} / \langle \tilde{\zeta} \rangle$.

The coordinate ring is given by :

$$R = K[x_1, x_2] \times K[y_0, \dots, y_m] / I, \quad I = \langle y_i x_1^j x_2^{m-j} - y_j x_1^i x_2^{m-i} \mid 0 \leq i, j \leq m \rangle$$

Furthermore we have the condition that y_0 and y_m cannot be both zero, coming from the fact that the point $(0,0)$ is missing in $K^2 - \{(0,0)\}$.

Since the coordinate ring of Σ_m is $K[x_1, x_2] \times K[z, y_0, \dots, y_m] / I$ and $R \simeq K[x_1, x_2] \times K[1, y_0, \dots, y_m] / I$, it is clear that $K^2 - \{(0,0)\} / \langle \tilde{\zeta} \rangle$ is isomorphic to a subspace of Σ_m .

In fact we have : $K^2 - \{(0,0)\} / \langle \tilde{\zeta} \rangle \simeq \Sigma_m \setminus S$.

Here S consists of the two lines $\mathbb{P}_K^1$ defined by :

$$\begin{cases} z = 0 & K[x_1, x_2] \times K[0, y_0, \dots, y_m] / I \\ y_0 = \dots = y_m = 0 & K[x_1, x_2] \times K[1, 0, \dots, 0] / I \end{cases}$$

PROPOSITION 2.5. *Let $\tilde{\zeta}$ be as above. Let Γ be the group $\Gamma = \langle \tilde{\zeta}, \gamma \rangle$ acting on $K^2 - \{(0,0)\}$. Here γ is a contraction defined by*

$$\gamma : (z_1, z_2) \rightarrow (\alpha z_1, \alpha z_2) \text{ with } 0 < |\alpha| < 1.$$

Now there exists a pure Γ -invariant affinoid covering of $K^2 - \{(0,0)\}$ such that the Hopf surface $K^2 - \{(0,0)\} / \Gamma$ has the reduction $\Sigma_m / \sim$. Here $\sim$ is the equivalence relation identifying the two lines $\mathbb{P}_K^1$ defined by $z = 0$ and by $y_0 = y_1 = \dots = y_m = 0$, where $z, y_i, i = 0, \dots, m$ are as in lemma 2.6.

Proof. Since Γ is abelian and $\langle \tilde{\zeta} \rangle$ is finite, it is sufficient to find a fundamental domain F for the action of $\langle \gamma \rangle$ on $K^2 - \{(0,0)\} / \langle \tilde{\zeta} \rangle$ such that the affinoid covering of F together with its $\langle \gamma \rangle$ -images is pure and gives a reduction with the properties mentioned above.

Again we consider the two open subspaces $K^* \times K$ of $K^2 - \{(0,0)\}$ defined by $z_1 \neq 0$ and by $z_2 \neq 0$. It is clear that the action of γ on the $\langle \tilde{\zeta} \rangle$ -invariants is given by :

$$\begin{cases} \gamma : \frac{z_1}{z_2} \rightarrow \frac{z_1}{z_2}, z_2^m \rightarrow \alpha^m z_2^m & \text{if } z_2 \neq 0 \\ \gamma : \frac{z_2}{z_1} \rightarrow \frac{z_2}{z_1}, z_1^m \rightarrow \alpha^m z_1^m & \text{if } z_1 \neq 0. \end{cases}$$

Now we have a fundamental domain $F = F_1 \cup F_2$ where F_1, F_2 are given by :

$$\begin{aligned} F_1 &= \left\{ \left(\frac{z_1}{z_2}, z_2^m \right) \mid |\alpha^m| \leq |z_2^m| \leq 1, \left| \frac{z_1}{z_2} \right| \leq 1 \right\} \\ F_2 &= \left\{ \left(\frac{z_2}{z_1}, z_1^m \right) \mid |\alpha^m| \leq |z_1^m| \leq 1, \left| \frac{z_2}{z_1} \right| \leq 1 \right\}. \end{aligned}$$

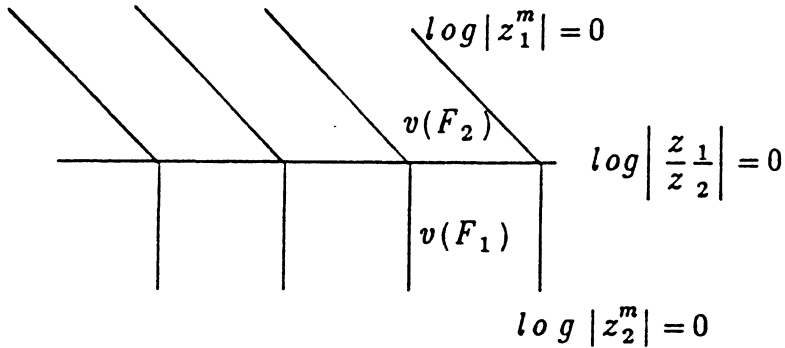
The subspaces F_1 and F_2 are affinoid spaces, they are in fact monomial rational domains. It is easy to see that $\mathcal{C} = \{\gamma^i(F_1), \gamma^j(F_2) \mid i, j \in \mathbb{Z}\}$ is a pure affinoid covering of $K^2 - \{(0,0)\} / \langle \tilde{\zeta} \rangle$ by monomial rational domains.

We will now define maps v' and v'' such that $v'(\gamma^i(F_1))$ and $v''(\gamma^j(F_2))$ are convex domains in $(\mathbb{R} \cup \{\pm\infty\})^2$. The maps are defined by :

$$\begin{aligned} v' : \left\{ \left(\frac{z_1}{z_2}, z_2^m \right) \in K^2 - \{(0,0)\} / \langle \tilde{\zeta} \rangle \mid z_2 \neq 0 \right\} &\rightarrow (\mathbb{R} \cup \{\pm\infty\})^2, \\ v' \left(\frac{z_1}{z_2}, z_2^m \right) &= \left(\log |z_2^m|, \log \left| \frac{z_1}{z_2} \right| \right). \\ v'' : \left\{ \left(\frac{z_2}{z_1}, z_1^m \right) \in K^2 - \{(0,0)\} / \langle \tilde{\zeta} \rangle \mid z_1 \neq 0 \right\} &\rightarrow (\mathbb{R} \cup \{\pm\infty\})^2, \\ v'' \left(\frac{z_2}{z_1}, z_1^m \right) &= \left(m \log \left| \frac{z_2}{z_1} \right| + \log |z_1^m|, -\log \left| \frac{z_2}{z_1} \right| \right). \end{aligned}$$

Since the maps v' and v'' are identical on the subspace defined by $z_1 \neq 0, z_2 \neq 0$, we can glue them together and get a map

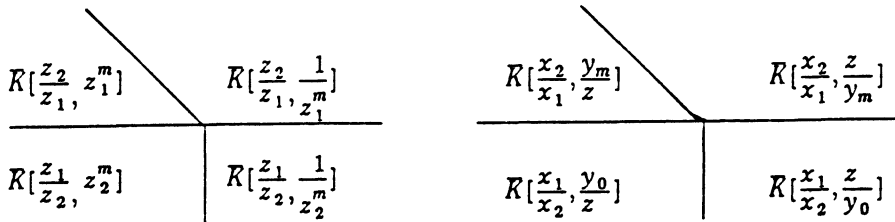
$$v : K^2 - \{(0,0)\} / \langle \tilde{\zeta} \rangle \rightarrow (\mathbb{R} \cup \{\pm\infty\})^2.$$



Note that in this situation $v(F_2)$ does not satisfy condition b of proposition 2.2. This is a consequence of the definition of v used here. But the results of lemma 2.4 remain valid, mutatis mutandis.

Again we have a 1-1 correspondence between extremal points P of the convex domains $v(\gamma^i(F_j)), i \in \mathbb{Z}, j = 1, 2$ and the components of the reduction. Every extremal point P gives a surface in the reduction. Looking at the figure above and using the remark following theorem 2.3 we see that every extremal point P gives a surface Σ_m .

In order to describe the intersections of the surfaces Σ_m we need some more information. We need to know the components in P of the reduction of the four affinoid domains $v(\gamma^i(F_j))$ with $P \in v(\gamma^i(F_j))$ in P . This is a straightforward calculation. The results are shown in the figure below.



The components can be glued together to give a surface Σ_m with homogeneous coordinate ring $\overline{K}[x_1, x_2] \times \overline{K}[z, y_0, \dots, y_m]/I$. The identification is given by :

$$\frac{x_2}{x_1} = \frac{z_2}{z_1}, \frac{x_1}{x_2} = \frac{z_1}{z_2}, \frac{y_m}{z} = \frac{z_1^m}{z}, \frac{y_0}{z} = \frac{z_2^m}{z} \text{ etc.}$$

Now it is clear from the figures above that the surfaces Σ_m belonging to the extremal points P and $\gamma(P)$ have a line in common. This line is defined by $\frac{y_m}{z} = \frac{y_0}{z} = 0$ in the Σ_m belonging to P and by $\frac{z}{y_m} = \frac{z}{y_0} = 0$ in the other Σ_m . So these lines are $\mathbb{P}_{\overline{K}}^1$'s defined by $z = 0$ and by $y_0 = y_1 = \dots = y_m = 0$.

Since $\langle \gamma \rangle$ is transitive on the set of extremal points P it is clear that the reduction of the Hopf surface $K^2 - \{(0, 0)\}/\Gamma$ is $\Sigma_m / \sim$, where $\sim$ identifies the two lines above.

3. Line bundles on a Hopf surface

Let $\Gamma = \langle \gamma \rangle$ be generated by a contraction γ . We will now study the line bundles on the Hopf surface $X = K^2 - \{(0, 0)\}/\Gamma$. We will need some properties of quasi-Stein spaces (see [Ki.2]).

DEFINITION. An analytic space Y is called a *quasi-Stein space* if there exists an admissible covering of Y by open affinoid subspaces $U_1, U_2, U_3, \dots$ such that :

- 1) $U_1 \subseteq U_2 \subseteq U_3 \subseteq \dots$
- 2) the image of $\mathcal{O}_Y(U_{i+1})$ is dense in $\mathcal{O}_Y(U_i)$.

Remark : Let Y be a quasi-Stein space and $\mathcal{F}$ a coherent sheaf on Y . In [Ki.2] the following two properties are proved :

- 1) $H^i(Y, \mathcal{F}) = 0 \forall i > 0$
- 2) The coherent sheaf $\mathcal{F}$ is a sheaf associated with an $\mathcal{O}(Y)$ -module F .

LEMMA 3.1. *The analytic space $K^* \times K$ is a quasi-Stein space.*

Proof. Let U_i be the open affinoid subspace of $K^* \times K$ defined by :

$$R_i^{-1} \leq |z_1| \leq R_i, |z_2| \leq R_i$$

We choose $R_i < R_{i+1}$ and $R_i \rightarrow \infty$ for $i \rightarrow \infty$.

Now the covering $(U_i)_{i \in \mathbb{N}}$ of $K^* \times K$ is admissible and satisfies the definition above. This proves the lemma.

LEMMA 3.2. *Every line bundle $\mathcal{L}$ on $W = K^2 - \{(0,0)\}$ is trivial.*

Proof. We have $W = W_1 \cup W_2$ where $W_1 = \{(z_1, z_2) \in W \mid z_1 \neq 0\}$ and $W_2 = \{(z_1, z_2) \in W \mid z_2 \neq 0\}$. Now $\mathcal{L}|_{W_i} \simeq \mathcal{O}_{W_i} \cdot e_i$, since $W_i \simeq K^* \times K$ is a quasi-Stein space and every line bundle on $K^* \times K$ is trivial.

It is clear that

$$\mathcal{O}(W_1)^* = K^* z_1^{\mathbb{Z}}, \mathcal{O}(W_2)^* = K^* z_2^{\mathbb{Z}} \text{ and } \mathcal{O}(W_1 \cap W_2)^* = K^* z_1^{\mathbb{Z}} z_2^{\mathbb{Z}}.$$

This shows that $e_1 = a e_2$ for some $a \in \mathcal{O}(W_1 \cap W_2)^*$. Furthermore we have $a = a_1^{-1} \cdot a_2$ with $a_i \in \mathcal{O}(W_i^*)$.

Now we take $f_1 = a_1 e_1$ and $f_2 = a_2 e_2$. Clearly we have $\mathcal{L}|_{W_1} = \mathcal{O}_{W_1} \cdot f_1$ and $f|_{W_i} = f_i$ and $\mathcal{L} = \mathcal{O}_{W_i} \cdot f$. This proves the lemma.

Remark : Let u be the map $u : W = K^2 - \{(0,0)\} \rightarrow W/\Gamma$ and let $\mathcal{L}$ be a line bundle on $X = W/\Gamma$. Now $u^* \mathcal{L}$ is a line bundle on W , so we have $u^* \mathcal{L} = \mathcal{O}_W \cdot e$. The action of the contraction γ on $u^* \mathcal{L}$ has the form $\gamma(e) = \alpha \cdot e$ for some $\alpha \in \mathcal{O}_W(W)^*$. Clearly we have : $\mathcal{O}_W(W)^* = K^*$.

DEFINITION. For $\alpha \in K^*$ we denote by $\mathcal{L}_\alpha$ the line bundle on $X = W/\Gamma$ defined by $u^* \mathcal{L}_\alpha = \mathcal{O}_W \cdot e$ with $\gamma(e) = \alpha \cdot e$. Here γ is the contraction generating Γ .

PROPOSITION 3.1. *Let Γ be generated by a contraction γ and let $X = W/\Gamma$. Now we have :*

- a) *Every line bundle $\mathcal{L}$ on X is isomorphic to a unique $\mathcal{L}_\alpha$.*
- b) $\mathcal{L}_\alpha \otimes \mathcal{L}_\beta \simeq \mathcal{L}_{\alpha\beta}$
- c) $\text{Pic}(X) \simeq K^*$

Proof. This is a direct consequence of lemma 3.2 and the remark following that lemma. Another way to prove the proposition is the following. The map $u : W \rightarrow X$ is a local isomorphism of the Grothendieck topology. We have :

$$u^* \mathcal{O}_X^* = \mathcal{O}_W^* \text{ and } H^0(X, \mathcal{O}_X^*) = H^0(W, u^* \mathcal{O}_X^*)^\Gamma.$$

Therefore we have : $H^1(X, \mathcal{O}_X^*) = H^1(\Gamma, H^0(W, \mathcal{O}_W^*)) = H^1(\Gamma, K^*)$. Since Γ has trivial action on K^* we see that $H^1(\Gamma, K^*) = K^*$. This shows that : $\text{Pic}(X) = H^1(X, \mathcal{O}_X^*) = K^*$.

Remark : In the next lemma we shall compute $H^i(W, u^* \mathcal{L}_\alpha) \simeq H^i(W, \mathcal{O}_W)$. We will need this for the calculation of the dimension of the groups $H^i(X, \mathcal{L}_\alpha)$.

LEMMA 3.3. *Let $W = K^2 - \{(0, 0)\}$. The cohomology groups $H^i(W, \mathcal{O}_W)$ are given by :*

$$\begin{aligned} H^0(W, \mathcal{O}_W) &= \left\{ \sum_{n,m \geq 0} a_{n,m} z_1^n z_2^m \mid \text{the powerseries are converging on } W \right\} \\ H^1(W, \mathcal{O}_W) &= \left\{ \sum_{n,m < 0} a_{n,m} z_1^n z_2^m \mid \text{the powerseries are convergent on} \right. \\ &\quad \left. W \setminus \{(z_1 z_2) \mid z_1 = 0 \text{ or } z_2 = 0\} \right\} \\ H^i(W, \mathcal{O}_W) &= 0, \quad i \geq 2. \end{aligned}$$

Proof. Let $W_i \simeq K^* \times K$ be the subspace of W given by $z_i \neq 0$ for $i = 1, 2$. We have $W = W_1 \cup W_2$. Since W_i is a quasi-Stein space, we have $H^j(W_i, \mathcal{L}) = 0$ for $j > 0$ and every coherent sheaf $\mathcal{L}$ on W_i . Therefore we can use Leray's theorem.

Let d be the natural map $d : \mathcal{L}(W_1) \otimes \mathcal{L}(W_2) \rightarrow \mathcal{L}(W_1 \cap W_2)$. Now Leray's theorem gives us : $H^0(W, \mathcal{L}) = \ker d$, $H^1(W, \mathcal{L}) = \text{coker } d$ and

$\dim H^i(W, \mathcal{L}) = 0, i \geq 2$. Now take $\mathcal{L} = \mathcal{O}_W$. It is clear that we have :

$$\begin{aligned}\mathcal{L}(W_1) &= \left\{ \sum_{n \in \mathbb{Z}, m \geq 0} a_{n,m} z_1^n z_2^m \mid \text{the powerseries converges on } W_1 \right\}, \\ \mathcal{L}(W_2) &= \left\{ \sum_{\substack{n \geq 0, \\ m \in \mathbb{Z}}} b_{n,m} z_1^n z_2^m \mid \text{the powerseries converges on } W_2 \right\}, \\ \mathcal{L}(W_1 \cap W_2) &= \left\{ \sum_{n,m \in \mathbb{Z}} c_{n,m} z_1^n z_2^m \mid \text{the powerseries converges on } W_1 \cap W_2 \right\}.\end{aligned}$$

Now the lemma is proved by applying Leray's theorem.

Remark : Let M be a Γ -module and let Γ be $\Gamma = \langle \gamma \rangle \simeq \mathbb{Z}$. Let $d : M \rightarrow M$ be the map given by $d(m) = \gamma(m) - m, m \in M$. Then the groups $H^i(\Gamma, M)$ are given by :

$$H^0(\Gamma, M) = \ker d$$

$$H^1(\Gamma, M) = \operatorname{coker} d$$

$$H^i(\Gamma, M) = 0, i \geq 2.$$

PROPOSITION 3.2. Let $X = W/\Gamma, \Gamma \simeq \mathbb{Z}$ and $\mathcal{L}_\alpha$ a line bundle on the Hopf surface X . In this situation we have :

- a) $H^0(X, \mathcal{L}_\alpha) \simeq H^0(\Gamma, H^0(W, u^* \mathcal{L}_\alpha))$
- b) $0 \rightarrow H^1(\Gamma, H^0(W, u^* \mathcal{L}_\alpha)) \rightarrow H^1(X, \mathcal{L}_\alpha) \rightarrow H^0(\Gamma, H^1(W, u^* \mathcal{L}_\alpha)) \rightarrow 0$ is exact.
- c) $H^2(X, \mathcal{L}_\alpha) \simeq H^1(\Gamma, H^1(W, u^* \mathcal{L}_\alpha))$

Proof. It is clear that $H^0(X, \mathcal{L}_\alpha) = H^0(W, u^* \mathcal{L}_\alpha)^\Gamma = H^0(\Gamma, H^0(W, u^* \mathcal{L}_\alpha))$. We will use spectral sequences to determine the other groups $H^i(X, \mathcal{L}_\alpha)$.

The left exact functor $H^*(X, -)$ is the composition of the two left exact functors $H^0(\Gamma, -)$ and $H^0(W, -)$ and the exact functor $\mathcal{L}_\alpha \rightarrow u^* \mathcal{L}_\alpha$. We can determine the right derived functors $H^i(X, -)$ of $H^0(X, -)$ by using the right derived functors $H^i(\Gamma, -)$ and $H^i(W, -)$ of $H^0(\Gamma, -)$ and $H^0(W, -)$.

Let T, U be covariant functors in one variable. Now [CE] p.376 gives us for the composite functor $V = TU$ a spectral sequence $\Pi_r^{p,q} \Rightarrow \mathcal{R}^n TU(-)$.

Here $\mathcal{R}^n$ is the n -th right derived functor. In this spectral sequence we have $\Pi_2^{p,q} = \mathcal{R}^q T(\mathcal{R}^p U(-))$. In our case we have $V = H^0(X, -)$, $T = H^0(\Gamma, -)$ and $U = H^0(W, -)$.

This gives us : $\Pi_r^{p,q} \implies H^n(X, \mathcal{L}_\alpha)$, $n = p + q$
 $\Pi_2^{p,q} = H^q(\Gamma, H^p(W, u^* \mathcal{L}_\alpha))$.

Furthermore we have $H^q(\Gamma, -) = 0$, $q \neq 0, 1$ and $H^p(W, -) = 0$, $p \neq 0, 1$. Therefore $\Pi_2^{p,q} = 0$, $p, q \neq 0, 1$. Now we have $\Pi_2^{p,q} = \Pi_r^{p,q} \forall r \geq 2$, since $d_r : \Pi_r^{p,q} \rightarrow \Pi_r^{p-r+1, q+r}$ is trivial for $r \geq 2$, i.e. $d_r \equiv 0$, and the spectral sequence is defined by $H(\Pi_r) = \Pi_{r+1}$.

Since $\Pi_2^{p,q} = 0$, $p, q \neq 0, 1$ we have an exact sequence (See [CE] p.333) :

$$0 \rightarrow \Pi_2^{0,1} \rightarrow H^1(X, \mathcal{L}_\alpha) \rightarrow \Pi_2^{1,0} \rightarrow 0.$$

This is the exact sequence in part b of the proposition. Furthermore we have : $H^2(X, \mathcal{L}_\alpha) \cong \Pi_2^{1,1} = H^1(\Gamma, H^1(W, u^* \mathcal{L}_\alpha))$. This proves part c of the proposition.

LEMMA 3.4. Let γ be a contraction given by $\gamma(z_1, z_2) = (\alpha_1 z_1, \alpha_2 z_2)$, $0 < |\alpha_1| \leq |\alpha_2| < 1$ and $\alpha_1^k \neq \alpha_2^l \forall k, l \in \mathbb{Z}$.

Let $\mathcal{L}_\beta$ be a line bundle on $X = W/\Gamma$, where $\Gamma = \langle \gamma \rangle$. Then we have :

$$\begin{aligned} \dim H^0(X, \mathcal{L}_\beta) &= \begin{cases} 1 & \text{if } \beta = \alpha_1^a \alpha_2^b, a, b \in \mathbb{Z}_{\leq 0} \\ 0 & \text{in all other cases} \end{cases} \\ \dim H^1(X, \mathcal{L}_\beta) &= \begin{cases} 1 & \text{if } \beta = \alpha_1^a \alpha_2^b, a, b \in \mathbb{Z} \\ 0 & \text{in all other cases} \end{cases} \\ \dim H^2(X, \mathcal{L}_\beta) &= \begin{cases} 1 & \text{if } \beta = \alpha_1^a \alpha_2^b, a, b \in \mathbb{Z}_{>0} \\ 0 & \text{in all other cases.} \end{cases} \end{aligned}$$

Proof. The element γ multiplies $a_{n,m} z_1^n z_2^m e \in \mathcal{L}_\beta$ by a constant :

$$\gamma(a_{n,m} z_1^n z_2^m e) = \alpha_1^n \alpha_2^m \beta \cdot a_{n,m} z_1^n z_2^m e.$$

Since $\alpha_1^k \neq \alpha_2^l \forall k, l \in \mathbb{Z}$ we have :

$$fe \in H^i(W, u^* \mathcal{L}_\beta), \gamma(fe) = fe \implies f = a_{n,m} z_1^n z_2^m, \alpha_1^n \alpha_2^m \beta = 1.$$

From $H^0(X, \mathcal{L}_\beta) = H^0(\Gamma, H^0(W, u^*\mathcal{L}_\beta))$ we can directly conclude that :

$$\dim H^0(X, \mathcal{L}_\beta) = 1 \Leftrightarrow \beta = \alpha_1^{-a} \alpha_2^{-b}, a, b \in \mathbb{Z}_{\geq 0}.$$

Indeed all monomials in $H^0(W, u^*\mathcal{L}_\beta)$ are of the form $z_1^n z_2^m$, $n, m \in \mathbb{Z}_{\geq 0}$.

In a similar way we can determine $\dim H^0(\Gamma, H^1(W, u^*\mathcal{L}_\beta))$. From the action of γ on $H^i(W, u^*\mathcal{L}_\beta)$ we can see that :

$$\dim H^0(\Gamma, H^i(W, u^*\mathcal{L}_\beta)) = \dim H^1(\Gamma, H^i(W, u^*\mathcal{L}_\beta)).$$

Now we can calculate

$$\dim H^2(X, \mathcal{L}_\beta) = \dim H^1(\Gamma, H^1(W, u^*\mathcal{L}_\beta)) = \dim H^0(\Gamma, H^1(W, u^*\mathcal{L}_\beta))$$

as before.

Now proposition 3.2b shows us that :

$$\begin{aligned} \dim H^1(X, \mathcal{L}_\beta) &= \dim H^0(\Gamma, H^1(W, u^*\mathcal{L}_\beta)) + \dim H^1(\Gamma, H^0(W, u^*\mathcal{L}_\beta)) \\ &= \dim H^2(X, \mathcal{L}_\beta) + \dim H^0(X, \mathcal{L}_\beta). \end{aligned}$$

This gives us : $\dim H^1(X, \mathcal{L}_\beta) = 1 \Leftrightarrow \beta = \alpha_1^a \alpha_2^b$, $a, b \in \mathbb{Z}$

LEMMA 3.5. *Let γ be a contraction given by $\gamma(z_1, z_2) = (\alpha_1 z_1, \alpha_2 z_2)$, $0 < |\alpha_1| \leq |\alpha_2| < 1$ and $\alpha_1^k = \alpha_2^l$ for some $k, l \in \mathbb{Z}_{>0}$ with $\text{g.c.d.}(k, l) = 1$. Let $\mathcal{L}_\beta$ be a line bundle on the Hopf surface $X = W/\Gamma$, where $\Gamma = \langle \gamma \rangle$. Now we have :*

$$\begin{aligned} \dim H^0(X, \mathcal{L}_\beta) &= \begin{cases} b+1 & \text{if } \beta = \alpha_1^{-r} \alpha_2^{-bl-s}, 0 \leq r < k, 0 \leq s < l, \\ & b \in \mathbb{Z}_{\geq 0} \\ 0 & \text{in all other cases} \end{cases} \\ \dim H^1(X, \mathcal{L}_\beta) &= \begin{cases} b+1 & \text{if } \beta = \alpha_1^{-r} \alpha_2^{-bl-s}, 0 \leq r < k, 0 \leq s < l, \\ & b \in \mathbb{Z}_{\geq 0} \\ b+1 & \text{if } \beta = \alpha_1^r \alpha_2^{bl+s}, 0 < r \leq k, 0 < s \leq l, \\ & b \in \mathbb{Z}_{\geq 0} \\ 0 & \text{in all other cases} \end{cases} \\ \dim H^2(X, \mathcal{L}_\beta) &= \begin{cases} b+1 & \text{if } \beta = \alpha_1^r \alpha_2^{bl+s}, 0 < r \leq k, 0 < s \leq l, \\ & b \in \mathbb{Z}_{\geq 0} \\ 0 & \text{in all other cases.} \end{cases} \end{aligned}$$

Proof. As in the previous lemma we have for monomials $a_{nm}z_1^n z_2^m e \in H^i(W, u^* \mathcal{L}_\beta)$:

$$\gamma(a_{nm}z_1^n z_2^m e) = \alpha_1^n \alpha_2^m \beta \cdot a_{nm}z_1^n z_2^m e.$$

So an element $f \cdot e \in H^i(W, u^* \mathcal{L}_\beta)$ is γ -invariant if and only if fe is the sum of monomials $a_{nm}z_1^n z_2^m \cdot e$ that are γ -invariant.

Now we can calculate $H^0(\Gamma, H^i(W, u^* \mathcal{L}_\beta))$. Since

$$H^0(X, \mathcal{L}_\beta) = H^0(\Gamma, H^0(W, u^* \mathcal{L}_\beta)),$$

we have $\dim H^0(X, \mathcal{L}_\beta) \neq 0$ if and only if $\beta = \alpha_1^{-m} \alpha_2^{-n}$, $m, n \in \mathbb{Z}_{\geq 0}$. If $\beta = \alpha_1^{-m} \alpha_2^{-n}$ then we can write β uniquely in the form $\beta = \alpha_1^{-r} \alpha_2^{-bl-s}$ with $0 \leq r < k$, $0 \leq s < l$ since $\alpha_1^k = \alpha_2^l$.

Now we have $\beta = \alpha_1^{-m} \alpha_2^{-n}$, $m, n \in \mathbb{Z}_{\geq 0}$ for the following values of m and n :

$$\begin{cases} m = r + jk \\ n = s + bl - jl, \quad 0 \leq j \leq b \end{cases}$$

This proves that $\dim H^0(X, \mathcal{L}_\beta) = b + 1$.

In a similar way we can calculate $\dim H^0(\Gamma, H^1(W, u^* \mathcal{L}_\beta))$. Since γ acts on the space $H^i(W, u^* \mathcal{L}_\beta)$ by multiplying the monomials with a constant, we have : $\dim H^0(\Gamma, H^i(W, u^* \mathcal{L}_\beta)) = \dim H^1(\Gamma, H^i(W, u^* \mathcal{L}_\beta))$.

Now using proposition 3.2. it is straightforward to calculate $\dim H^1(X, \mathcal{L}_\beta)$ and $\dim H^2(X, \mathcal{L}_\beta)$.

Remark : Let γ be a contraction of the form $\gamma(z_1, z_2) = (\alpha^m(z_1 + \lambda z_2^m), \alpha z_2)$, $0 < |\alpha| < 1$, $\lambda \neq 0$. In this case $\gamma(W_1) \neq W_1$. So the action of γ on $H^1(W, u^* \mathcal{L}_\beta)$ as given in lemma 3.3 is not well-defined. Therefore we need another description of $H^1(W, u^* \mathcal{L}_\beta)$. This will be done in the next lemma.

LEMMA 3.6. Let $W'_1 \subset W$ be the subspace

$$W'_1 = \{(z_1, z_2) \in W \mid |z_1| > |z_2^m|\}.$$

Then W'_1 is a quasi-Stein space and $W = W'_1 \cup W_2$. Let γ be a contraction of the form

$$\gamma(z_1, z_2) = (\alpha^m(z_1 + \lambda z_2^m), \alpha z_2), \quad 0 < |\alpha| < 1, \quad 0 < |\lambda| \leq 1, \quad m \in \mathbb{Z}_{>0}.$$

We have

$$H^1(W, \mathcal{O}_W) \simeq \left\{ \sum_{n,m < 0} a_{n,m} z_1^n z_2^m \mid \text{the powerseries converges on } W'_1 \cap W_2 \right\}$$

The action of γ on $H^1(W, \mathcal{O}_W)$ is now well-defined.

Proof. We can see that W'_1 is a quasi-Stein space by using the following admissible affinoid covering $(U_i)_{i \in \mathbb{Z}_{\geq 0}}$ of W'_1 . Here U_i is defined by :

$$R_{i,2} \geq |z_1| \geq R_{i,1} > 0, \quad |z_1| \geq R_{i,3} \cdot |z_2^m|.$$

Here we have $R_{i,2} \rightarrow \infty$, $R_{i,1} \rightarrow 0$ and $R_{i,3} \downarrow 1$ as $i \rightarrow \infty$.

Since $W = W'_1 \cup W_2$ and W'_1, W_2 are quasi-Stein spaces, we can use Leray's theorem to calculate the $H^i(W, \mathcal{O}_W)$. We see that $H^i(W, \mathcal{O}_W), i \neq 1$ is as in lemma 3.3, only $H^1(W, \mathcal{O}_W)$ is different. In fact we have :

$$H^1(W, \mathcal{O}_W) = \left\{ \sum_{n,m < 0} a_{n,m} z_1^n z_2^m \mid \text{the powerseries converges on } W'_1 \cap W_2 \right\}$$

So only the convergency condition of the powerseries has changed.

Now the action of γ on $H^1(W, \mathcal{O}_W)$ is well-defined. We have :

$$\begin{aligned} \gamma \left(\frac{1}{z_1^k z_2^l} \right) &= \alpha^{-mk-l} \left(1 + \lambda \frac{z_2^m}{z_1} \right)^{-k} z_1^{-k} z_2^{-l} \\ &= \alpha^{-mk-l} z_1^{-k} z_2^{-l} \sum_{i \geq 0} \binom{-k}{i} \left(\lambda \frac{z_2^m}{z_1} \right)^i. \end{aligned}$$

This powerseries is convergent on $W'_1 \cap W_2$ since $\binom{-k}{i} \in \mathbb{Z}$ and therefore $|\binom{-k}{i}| \leq 1$. In fact we may forget about z_i -powers ≥ 0 , since in $H^1(W, \mathcal{O}_W)$ we are looking at powerseries $\sum_{k,l < 0} a_{k,l} z_1^k z_2^l$ modulo monomials having a

z_i -power ≥ 0 . So the series $\gamma \left(\frac{1}{z_1^k z_2^l} \right)$ stops as soon as $mi - l \geq 0$.

LEMMA 3.7. *Let γ be a contraction of the form*

$$\gamma(z_1, z_2) = (\alpha^m(z_1 + \lambda z_2^m), \alpha z_2), 0 < |\alpha| < 1, 0 < |\lambda| \leq 1.$$

Let $\mathfrak{L}_\beta$ be a line bundle on the Hopf surface W/Γ , where $\Gamma = \langle \gamma \rangle$. Now we have :

$$\dim H^0(X, \mathfrak{L}_\beta) = \begin{cases} 1 & \text{if } \beta = \alpha^{-k}, k \in \mathbb{Z}_{\geq 0}, \text{char}(K) = 0 \\ s+1 & \text{if } \beta = \alpha^{-k}, k \in \mathbb{Z}_{\geq 0}, p(s+1) > \frac{k}{m} \geq ps, \\ & \text{char}(K) = p > 0 \\ 0 & \text{in all other cases} \end{cases}$$

$$\dim H^1(X, \mathfrak{L}_\beta) = \begin{cases} 1 & \text{if } \beta = \alpha^k, k \leq 0 \vee k \geq m+1, \text{char}(K) = 0 \\ s & \text{if } \beta = \alpha^k, k \geq m+1, ps < \frac{k}{m} \leq p(s+1), \\ & [k/m] = ps, \text{char}(K) = p > 0 \\ s+1 & \text{if } \beta = \alpha^k, k \geq m+1, ps < \frac{k}{m} \leq p(s+1), \\ & [k/m] \neq ps, \text{char}(K) = p > 0 \\ s+1 & \text{if } \beta = \alpha^k, k \in \mathbb{Z}_{\leq 0}, p(s+1) > \frac{k}{m} \geq ps, \\ & \text{char}(K) = p > 0 \\ 0 & \text{in all other cases} \end{cases}$$

$$\dim H^2(X, \mathfrak{L}_\beta) = \begin{cases} 1 & \text{if } \beta = \alpha^k, k \geq m+1, \text{char}(K) = 0 \\ s & \text{if } \beta = \alpha^k, ps < \frac{k}{m} \leq p(s+1), k \geq m+1 \\ & \text{and } [k/m] = ps, \text{char}(K) = p > 0 \\ s+1 & \text{if } \beta = \alpha^k, ps < \frac{k}{m} \leq p(s+1), k \geq m+1 \\ & \text{and } [k/m] \neq ps, \text{char}(K) = p > 0 \\ 0 & \text{in all other cases} \end{cases}$$

Proof. We may replace the coordinate z_1 by $\lambda^{-1}z_1$, so we can assume that $\lambda = 1$ and that γ has the form : $\gamma(z_1, z_2) = (\alpha^m(z_1 + z_2^m), \alpha z_2)$.

We also replace the monomials $z_1^k z_2^l$ by $x^k z_2^{l+mk}$ where $x = \frac{z_1}{z_2^m}$, so $\gamma(x) = x + 1$.

So now we have :

$$H^0(W, \mathcal{O}_W) = \left\{ \sum_{k \geq 0, l-km \geq 0} a_{k,l} x^k z_2^l \mid \text{the powerseries converges on } W \right\}$$

$$H^1(W, \mathcal{O}_W) = \left\{ \sum_{k < 0, l-km < 0} a_{k,l} x^k z_2^l \mid \text{the powerseries converges on } W'_1 \cap W_2 \right\}$$

$$H^i(W, \mathcal{O}_W) = 0, \quad i \geq 2.$$

We will first calculate $H^0(X, \mathcal{L}_\beta)$. We have :

$$\gamma(x^k z_2^l) = (x+1)^k \alpha^l z_2^l = \alpha^l \sum_{i=0}^k \binom{k}{i} x^i z_2^l, \quad k \geq 0, l - km \geq 0.$$

Now take $f \cdot e \in \mathcal{L}_\beta$ such that $\gamma(f \cdot e) = f \cdot e = \gamma(f) \cdot \beta \cdot e$.

So we must have : $f = \beta \cdot \gamma(f)$. We can write :

$$f = \sum f_i(x) z_2^i \Rightarrow \gamma(f) = \sum \alpha^i \gamma(f_i(x)) \cdot z_2^i, \quad \gamma(f_i(x)) = f_i(x+1).$$

So we have : $\gamma(f) = \beta^{-1} \cdot f \Rightarrow f = f_i(x) z_2^i, \quad \beta = \alpha^{-i}, \quad i - m \cdot \deg(f_i(x)) \geq 0$
and $\gamma(f_i(x)) = f_i(x)$.

Let $f_i(x)$ be given by $f_i(x) = \sum_{j=0}^s a_j x^j$. This gives us :

$$\begin{aligned} \gamma(f_i) = f_i &\Leftrightarrow \sum_{j=0}^s a_j x^j = \sum_{j=0}^s a_j (x+1)^j, \quad s = \deg(f_i) \\ &\Rightarrow s \cdot a_s + a_{s-1} = a_{s-1} \\ &\Rightarrow s = 0 \vee a_s = 0. \end{aligned}$$

Since $a_s \neq 0$, we have $s = 0$. When $\text{char}(K) = 0$ we find $s = 0$ and $f_i \in K$.

Therefore we have : $f = f_i z_2^i = c z_2^i, \quad c \in K, \quad \gamma(f \cdot e) = \alpha^i \cdot \beta \cdot f e$

$$\Rightarrow \begin{cases} \dim H^0(X, \mathcal{L}_\beta) = 1 & \text{if } \beta = \alpha^{-i}, \quad i \geq 0 \\ \dim H^0(X, \mathcal{L}_\beta) = 0 & \text{otherwise.} \end{cases}$$

If $\text{char}(K) = p > 0$ then we find $p|s$. It is easy to see that the polynomials $(x^p - x)^j$ are γ -invariant. Now we have :

$$f_i = \sum_{j=0}^s a_j (x^p - x)^j, \deg(f_i) = ps.$$

$$f = f_i z_2^i, i - m \cdot \deg(f_i) \geq 0 \text{ so } i - mps \geq 0, \beta = \alpha^{-i}.$$

This gives us :

$$\begin{cases} \dim H^0(X, \mathcal{L}_\beta) = s + 1 & \text{if } \beta = \alpha^{-i}, i \geq 0 \text{ and } p(s + 1) > \frac{i}{m} \geq ps \\ \dim H^0(X, \mathcal{L}_\beta) = 0 & \text{otherwise.} \end{cases}$$

Since there is for every $l \geq 0$ only a finite number of polynomials $x^k z_2^l$ with $k \geq 0$ and $l - km \geq 0$ and $\gamma(x^k z_2^l) = \alpha^l (x + 1)^k z_2^l$ and since there exist for at most one value of l invariant polynomials $f_i(x) z_2^l$ such that $\gamma(f_i(x) z_2^l) = f_i(x) z_2^l$, we have again :

$$\dim H^0(\Gamma, H^0(W, u^* \mathcal{L}_\beta)) = \dim H^1(\Gamma, H^0(W, u^* \mathcal{L}_\beta)).$$

We will now study the action of γ on $H^1(W, u^* \mathcal{L}_\beta)$. We have :

$$\gamma(x^{-k} z_2^{-l}) = \alpha^{-l} (x + 1)^{-k} z_2^{-l} = \alpha^{-l} \sum_{i=0}^{\infty} \binom{-k}{i} x^{-k-i} z_2^{-l}, k > 0, l > km.$$

We can forget the terms $x^{-k-i} z_2^{-l}$ with $-l + (k + i)m \geq 0$, so we only have to look at a finite sum. Now take an element $f \cdot e \in \mathcal{L}_{\beta|W'_1 \cap W_2}$ such that $\gamma(fe) = \beta \gamma(f) \cdot e = f \cdot e$. So we have : $f = \beta \cdot \gamma(f)$.

We can write $f = \sum f_i(x^{-1}) z_2^i$ where f_i is a polynomial such that $i - \deg(f_i) \cdot m < 0$. Therefore

$$\gamma(f) = \sum \alpha^i \gamma(f_i(x^{-1}) z_2^i) = \sum \alpha^i \gamma(f_i((x + 1)^{-1}) z_2^i).$$

So we have : $\gamma(f) = \beta^{-1} f \Rightarrow f = f_i(x^{-1}) z_2^i, \beta = \alpha^{-i}, i - \deg(f_i) \cdot m < 0$ and $\gamma(f_i(x^{-1})) z_2^i = f_i(x^{-1}) z_2^i$ modulo monomials $x^k z_2^i$ with $i - km \geq 0$.

Let f be

$$\begin{aligned} f = f_i(x^{-1}) z_2^i &= \sum_{\substack{j > 0 \\ i+jm < 0}} a_j x^{-j} z_2^i \\ &= a_s x^{-s} z_2^i + \sum_{\substack{j > s \\ i+jm < 0}} a_j x^{-j} z_2^i, a_s \neq 0. \end{aligned}$$

Now $\gamma(f) = \beta^{-1}f$ implies that :

$$\begin{aligned} \binom{-s}{1} a_s + a_{s+1} &= a_{s+1}, \quad \beta = \alpha^{-i} \\ \Leftrightarrow \binom{-s}{1} a_s &= 0 \quad \forall ms + m + i \geq 0 \\ \Leftrightarrow \binom{-s}{1} &= -s = 0 \quad \forall ms + m + i \geq 0. \end{aligned}$$

If $\text{char}(K) = 0$ then $s = 0$ cannot occur since in $x^{-s}z_2^i$ we have $s > 0$. So we must have $ms + m + i \geq 0$ and $sm + i < 0$, since otherwise $f \equiv 0$.

This gives us :

$$\begin{cases} \dim H^0(\Gamma, H^1(W, u^* \mathcal{L}_\beta)) = 1 & \text{if } \beta = \alpha^{-i}, i < -m \\ \dim H^0(\Gamma, H^1(W, u^* \mathcal{L}_\beta)) = 0 & \text{otherwise.} \end{cases}$$

If $\text{char}(K) = p > 0$ then we have $p|s$ or $ms + m + i \geq 0$, $sm + i < 0$.

It is easy to see that the powerseries $\left(\frac{1}{x^p - x}\right)^r$ is γ -invariant :

$$\frac{1}{(x^p - x)^r} = \frac{1}{x^{pr}} \cdot \frac{1}{(1 - x^{1-p})^r} = x^{-pr} \left(\sum_{j=0}^{\infty} x^{(1-p)j} \right)^r.$$

So we can use the polynomials :

$$z_2^i \cdot x^{-pr} \left(\sum_{j=0}^{j_0} x^{(1-p)j} \right)^r, \quad i + mpr < 0, \quad r > 0.$$

Here j_0 is taken such that $i + m(pr + j_0(p-1)) \geq 0$.

Furthermore we find an extra γ -invariant monomial $x^{-r}z_2^i$ if $-(r+1) \leq \frac{i}{m} < -r < 0$ and $r \neq p \cdot s$ for some $s \in \mathbb{Z}_{>0}$. So we now have :

$$\begin{cases} \dim H^0(\Gamma, H^1(W, u^* \mathcal{L}_\beta)) = s & \text{if } -p(s+1) \leq \frac{i}{m} < -ps, \beta = \alpha^{-i}, \\ & -i > m \quad \text{and} \quad -(r+1) \leq \frac{i}{m} < r \Rightarrow r = ps, \\ \dim H^0(\Gamma, H^1(W, u^* \mathcal{L}_\beta)) = s+1 & \text{if } -p(s+1) \leq \frac{i}{m} < -ps, \beta = \alpha^{-i}, \\ & -i > m \quad \text{and} \quad -(r+1) \leq \frac{i}{m} < r \Rightarrow r \neq ps \\ \dim H^0(\Gamma, H^1(W, u^* \mathcal{L}_\beta)) = 0 & \text{otherwise.} \end{cases}$$

Again we have : $\dim H^1(\Gamma, H^1(W, u^* \mathcal{L}_\beta)) = \dim H^0(\Gamma, H^1(W, u^* \mathcal{L}_\beta))$.

So we have : $\dim H^2(X, \mathcal{L}_\beta) = \dim H^0(\Gamma, H^1(W, u^* \mathcal{L}_\beta))$.

Furthermore we have : $\dim H^1(X, \mathcal{L}_\beta) = \dim H^0(X, \mathcal{L}_\beta) + \dim H^2(X, \mathcal{L}_\beta)$.

THEOREM 3.1. *Let $\Gamma = \langle \gamma \rangle$ be generated by a contraction γ .*

Let $\mathcal{L}_\beta$ be a line bundle on the Hopf surface $X = W/\Gamma$. We now have :

- a) $\chi(\mathcal{L}_\beta) = 0$
- b) *There exists an unique line bundle $\mathcal{L}$ such that :*

$$\forall \mathcal{L}_\beta \quad H^{2-i}(X, \mathcal{L} \otimes \mathcal{L}_\beta^{-1}) = H^i(X, \mathcal{L}_\beta).$$

- c) *We have $\mathcal{L} = \mathcal{L}_{\alpha_1 \cdot \alpha_2} = \mathcal{L}(K)$.*

Proof. The Euler characteristic $\chi(\mathcal{L}_\beta)$ is defined by :

$$\chi(\mathcal{L}_\beta) = \sum_{i=0}^2 (-1)^i \dim H^i(X, \mathcal{L}_\beta).$$

In the lemmas 3.4, 3.5 and 3.7 we have used the following fact :

$$\dim H^1(X, \mathcal{L}_\beta) = \dim H^0(X, \mathcal{L}_\beta) + \dim H^2(X, \mathcal{L}_\beta).$$

This proves statement *a* of the theorem.

The Serre duality in part *b* can be found by direct verification. We shall only determine the only possible line bundle $\mathcal{L}$. We have :

$$\begin{cases} \dim H^0(X, \mathcal{L}) = \dim H^2(X, \mathcal{L} \otimes \mathcal{L}^{-1}) = \dim H^2(X, \mathcal{L}_1) = 0 \\ \dim H^2(X, \mathcal{L}) = \dim H^0(X, \mathcal{L} \otimes \mathcal{L}^{-1}) = \dim H^0(X, \mathcal{L}_1) = 1 \end{cases}$$

Since $\dim H^2(X, \mathcal{L}_1) = 1$, we have $\mathcal{L} = \mathcal{L}_a$ with $a = \alpha_1^k \alpha_2^l$, $k, l \in \mathbb{Z}_{>0}$.

Now take $\beta = \alpha_1^r \alpha_2^s$, $r, s > 0$, then we have :

$$0 \neq \dim H^2(X, \mathcal{L}_\beta) = \dim H^0(X, \mathcal{L}_{\alpha_1^{r-k} \alpha_2^{s-l}}) \Rightarrow -r + k \geq 0, -s + l \geq 0.$$

So only $k = l = 1$ can satisfy *b*, therefore $\mathcal{L} = \mathcal{L}_{\alpha_1 \alpha_2}$.

The canonical divisor K on X is given by

$$K = \begin{cases} z_1^{-1} z_2^{-1} \cdot dz_1 \wedge dz_2 & \text{if } \gamma(z_1, z_2) = (\alpha_1 z_1, \alpha_2 z_2) \\ z_2^{-(m+1)} \cdot dz_1 \wedge dz_2 & \text{if } \gamma(z_1, z_2) = (\alpha^m(z_1 + z_2^m), \alpha z_2) \end{cases}$$

Clearly we have : $\mathfrak{L}(K) = \mathfrak{L}_{\alpha_1 \alpha_2}$. This proves part c of the theorem.

Remark : Theorem 3.1 gives us a Riemann-Roch theorem on the Hopf surface X .

$$\begin{aligned} \text{Indeed we have : } l(D) - s(D) + l(K - D) &= \frac{1}{2} D(D - K) + 1 + p_a \\ &\Rightarrow \chi(\mathfrak{L}(D)) = \frac{1}{2} D(D - K) + 1 + p_a \end{aligned}$$

Taking $D = K$ we have $0 = 1 + p_a \Rightarrow p_a = -1$.

Taking $D = n \cdot K$ we have $\frac{1}{2}(nK^2 - n^2K^2) = 0 \Rightarrow K^2 = 0$.

So we have :

$$D^2 = D \cdot K \text{ for every } D.$$

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Max Planck Institut für Mathematik
Gottfried-Claren-Strasse 26
5300 Bonn 3
Germany .