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On the greatest prime divisor of quadratic sequences

par J. POMYKAŁA

1. Introduction. Let P_x be the greatest prime divisor of $\prod_{n\leq x}(n^2-a)$ where a is not a perfect square. In 1967 C. Hooley (see[4]) showed that $P_x > x^{\gamma-\varepsilon}$, where $\gamma = \frac{11}{10}$ and ε is any positive number. In 1982 J.-M. Deshouillers and H. Iwaniec applied the new estimates of exponential sums of Kloosterman type (see[2]) to obtain the improvement in the exponent (see[1]) $\gamma = \gamma_0 = 1.2024...$ (in fact they considered the case $a = -1$). Assuming Selberg's eigenvalue conjecture their method implies the value $\gamma = \sqrt{\frac{3}{2}} = 1.2247...$ One can avoid Selberg's conjecture dealing with additional averaging of Kloosterman sums over the levels of the Hecke congruence subgroups involved here. This motivates the investigation of the greatest prime divisor of

$$\prod_{n\leq x}\prod_{\substack{q\in\mathbb{B}\\ q\leq x^{\Theta}}}(n^2+q^2) \tag{1}$$

where $\mathbb{B}$ is a rather thin set of primes and very general. Assuming the density condition

$$\mathbb{B}(y) := \operatorname{card}\{b\in\mathbb{B} \mid b\leq y\} \geq y^{\beta}, \qquad \text{as} \qquad y\to\infty$$

we denote by $P_x(\beta,\Theta)$ the greatest prime divisor of (1). In such circumstances the progress in the exponent $\gamma \geq \gamma_0$ is obtained for the values

$$\frac{1}{2}\leq\beta<1,\ \ 0<\Theta\leq\sqrt{3/2}-1.$$

Moreover the limit of the method is attained for $\Theta = \sqrt{\frac{3}{2}}-1$ *as* $\beta\to 1$. In the above range of variables β, Θ we define the function

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(*)

$$F_{\beta,\Theta}(\alpha) = \left\{ \frac{(1+\Theta\beta)^2 - 1}{1+\Theta(\beta-1)} + 4\left([1+\Theta(\beta-1/2)] \log \left| \frac{1+\Theta(\beta-1)}{2+\Theta(2\beta-1)-\alpha} \right| - \frac{\alpha-1-\Theta\beta}{2} \right) \right\}.$$

We shall prove the following

THEOREM. *Let* $\frac{1}{2} \leq \beta < 1$, *and* $\alpha = \alpha(\beta,\Theta)$ *be a (unique) solution of* $F_{\beta,\Theta}(\alpha) = \frac{1}{2}$. *Then for* $\gamma = \gamma(\beta,\Theta)$ *satisfying the equality* $\gamma = \sup_{0<\Theta'\leq\Theta} \alpha(\beta,\Theta')$ *and any* $\varepsilon > 0$ *we have*

$$P_x(\beta,\Theta) > x^{\gamma-\varepsilon}, \quad \text{as} \quad x \to \infty.$$

Remark.

The uniqueness of $\alpha(\beta,\Theta)$ is implied by the integral representation (20) of $F_{\beta,\Theta}(\alpha)$. Moreover comparing the values of $D(x,P,\beta,\Theta)$ with the values $D = x^{1-\varepsilon}/\sqrt{P}$ and $D = x^{\frac{1}{2}-\varepsilon}$ (see[1]) we obtain

$$\lim_{\Theta\to 0} \alpha(\beta,\Theta) = \gamma_0, \quad (\beta - \text{fixed})$$

$$\lim_{\beta\to 1} \alpha(\beta, \sqrt{\frac{3}{2}} - 1) = \sqrt{\frac{3}{2}}$$

2. Notation. Throughout the paper ε will be arbitrary sufficiently small positive constant not necessarily the same in each occurrence. Moreover we take the following notation:

$\mathrm{e}(z)$ - the additive character $e^{2\pi i z}$

$\hat{f}$ – the Fourier transform of f, i.e.,

$$\hat{f}(\eta) = \int_{-\infty}^{+\infty} f(\xi)\mathrm{e}(\xi\eta)\, d\xi$$

$\eta \equiv a(m)$ – means $\eta \equiv a \pmod m$

$m \sim M$ – means $M \leq m < 2M$

$||f||$ – means the sup norm of f

$\frac{\overline{d}}{c}$– means $\frac{a}{c}$ (mod 1) where $ad \equiv 1(c)$.

$S(a, b; c)$ – means the Kloosterman sum i.e.

$$S(a,b;c) = \sum_{\substack{m \pmod{c} \\ (m,c)=1}} \mathrm{e}((am + b\overline{m})/c)$$

$\Sigma^*_{a \pmod{s}}$ - means the summation over residue classes $a \pmod{s}$ with $(a, s) = 1$

$| X |$ - means the cardinality of the set X

$\vartheta := \vartheta \pmod{m}$– stands for a solution of the congruence $\vartheta^2 + 1 \equiv 0(m)$

$\varrho(m) := \{\vartheta \pmod{m} : \vartheta^2 + 1 \equiv 0(m)\} \mid$

p, q - denote always the prime numbers and $q \in \mathbb{B}$.

3. Tchebyshev's method.

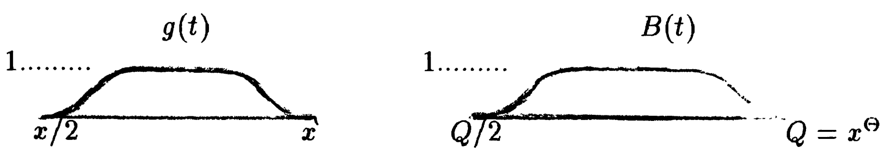
Given $\Theta \in (0, 1)$ we consider the sequence

$$\mathcal{A} = \{n^2 + q^2 : n \in \mathbb{N},\ n \leq x,\ q \in \mathbb{B},\ q \leq x^\Theta,\ (n,q) = 1\},$$

the elements of which are counted which the smooth functions $g(n), B(q)$ (drawn below) with derivatives

$$g^{(\nu)}(t) \ll t^{-\nu},\quad B^{(\nu)}(t) \ll t^{-\nu},\quad \nu = 0, 1, 2, \ldots$$

where the constant implied in the symbol $\ll$ depends on ν only.



We start from the evaluation

$$T(x) = \sum_p |\mathcal{A}_p| \log\, p =$$

$$= \sum_d |\mathcal{A}_d| \Lambda(d) + O\left(\sum_{l \geq 2} \sum_{\substack{p \\ p^l \leq 2x^2}} |\mathcal{A}_{p^l}| \log\, p\right)$$

where $\Lambda(d)$ is von Mangoldt function. Since $2 \leq l \leq \log(2x^2)/\log p$ in the above and

$$|\mathcal{A}_{p^l}| = \sum_{q\in\mathbb{B}} B(q) \sum_{\substack{(n,q)=1\\ n^2+q^2\equiv 0(p^l)}} g(n) \ll B\varrho(p^l)\left(\frac{x}{p^l}+1\right) \ll B\left(\frac{x}{p^l}+1\right)$$

where

$$B = \sum_{q\in\mathbb{B}} B(q),$$

we obtain that the error term $O(\ .\ )$ contributes to $T(x)$ at most

$$\ll B\sum_{l\geq 2}\sum_{p\leq x\sqrt{2}} \frac{x\log p}{p^l} + B\sum_{l\leq 3\log x/\log p}\ \sum_{p\leq x\sqrt{2}} \log p \ll Bx.$$

Hence

$$\begin{aligned} T(x) &= \sum_d \Lambda(x)|\mathcal{A}_d| + O(Bx) \\ &= \sum_{(n,q)=1} B(q)g(n) \sum_{d|n^2+q^2} \Lambda(d) + O(Bx) \\ &= \sum_q B(q)\sum_n g(n)\log(n^2+q^2) + O(x^{1+\varepsilon}+Bx). \end{aligned}$$

By the Poisson summation formula and the inequality $q < x^{\Theta}$ it is equal to

$$\begin{aligned} &\sum_q B(q)\int g(\xi)\log(\xi^2+q^2)d\xi + O(x^{1+\varepsilon}+Bx) \\ (2)\qquad &= \sum_q B(q)\int g(\xi)\log\xi^2 d\xi(1+O(x^{-\varepsilon})) + O(x^{1+\varepsilon}+Bx) \\ &= 2B\hat{g}(0)\log x(1+O(\frac{1}{\log x})) + O(x^{1+\varepsilon}+Bx) \\ &= 2B\hat{g}(0)\log x + O(x^{1+\varepsilon}+Bx). \end{aligned}$$

To estimate the partial sum $T^*(x) = \Sigma_{p\leq x}|\mathcal{A}_p|\log p$ we write (using the

Möbius function)

$$
\begin{aligned}
|\mathcal{A}_m| &= \sum_q B(q) \sum_{\substack{(q,n)=1\\ n^2+q^2\equiv 0(m)}} g(n) = \sum_{(q,m)=1} B(q) \sum_{\substack{(n,q)=1\\ n^2+q^2\equiv 0(m)}} g(n)\\
&= \sum_{e=0}^{1} \mu(q^e) \sum_{(q,m)=1} B(q) \sum_{(q^e n)^2+q^2\equiv 0(m)} g(q^e n)\\
&= \sum_{e=0}^{1} (-1)^e \sum_{(q,m)=1} B(q) \sum_{n^2+(q^{1-e})^2\equiv 0(m)} g(q^e n)\\
&= \sum_{e=0}^{1} (-1)^e \sum_{(q,m)=1} B(q) \sum_{\vartheta(m)} \sum_{n\equiv \vartheta q^{1-e}(m)} g(q^e n).
\end{aligned}
\tag{3}
$$

Letting $m = p$ we obtain that for $e = 1$ the corresponding contribution to $T^*(x)$ is bounded by

$$
\ll \sum_{p\geq x} Bx\varrho(p)\log p/(qp) + Bx \ll Bx \quad (\text{since}\Theta > 0). \tag{4}
$$

In case $e = 0$ applying the Poisson summation formula for the inner sum in (3) we get

$$
\begin{aligned}
\sum_{n\equiv \vartheta q(p)} g(n) &= \frac{1}{p} \sum_{h=-\infty}^{\infty} \hat{g}(\frac{h}{p}) e(\frac{-hq\vartheta}{p}) =\\
&= \frac{\hat{g}(0)}{p} + O(\frac{p}{x})
\end{aligned}
$$

when integrating $\int g(\xi)\mathrm{e}(\frac{h\xi}{p})d\xi$ (twice) by parts. Hence the main contribution to $T^*(x)$ is equal to

$$
\begin{aligned}
&\sum_{p\leq x} \log p \sum_{(q,p)=1} B(q) \sum_{\vartheta(p)} \left(\frac{\hat{g}(0)}{p} + O(\frac{p}{x})\right)\\
&= x \sum_{p\leq x} \varrho(p) \frac{\log p}{p} \sum_{(q,p)=1} B(q) + O(xB) =\\
&= B\hat{g}(0)\log x + O(x^{1+\varepsilon} + xB)
\end{aligned}
\tag{5}
$$

by the prime ideal theorem.

Gathering (2), (4), (5) together we obtain

$$S(x) = \sum_{x \le p < P_x} |\mathcal{A}_p| \log p = \hat{g}(0) B \log x + O(x^{1+\varepsilon} + xB). \tag{6}$$

Now the main goal is to find the largest possible value of P_x such that the upper bound for $S(x)$ be less than $\hat{g}(0)B\log x$.

4. Application of the sieve method.

Like Iwaniec and Deshouillers in [1] we use the smooth partition of unity to split the sum $S(x)$ into $J \le 2\log x$ sums of the type

$$S(x, P_j) = \sum_p |\mathcal{A}_p| C_j(p) \log p$$

where

$$\operatorname{supp} C_j \subset [P_j, 2P_j] \ , \ P_j \ge x$$

$$C_j^{(\nu)}(t) \ll P_j^{-\nu} \ , \nu = 0, 1, 2, \ldots$$

and such that

$$S(x) = \sum_{1 \le j \le J} S(x, P_j) + O(xB).$$

We estimate the typical sum

$$S(x, P) = \sum_p |\mathcal{A}_p| C(p) \log p$$

using an upper bound sieve $\{\lambda_d\}_{d \le \Delta}$ of level Δ $(1 \le \Delta \le x)$. By (3) and the Poisson formula we obtain

$$S(x, P) \le \sum_d \lambda_d \sum_{m \equiv O(d)} |\mathcal{A}_m| C(m) \log m \tag{7}$$

$$= \sum_{e=0}^{1} (-1)^e \sum_d \lambda_d \sum_{m \equiv O(d)} C(m) \log m \sum_{(q,m)=1} B(q) \sum_{\vartheta} \sum_{n \equiv \vartheta q^{1-e}(m)} g(q^e n)$$

$$= \sum_{h=-\infty}^{\infty} \sum_{e=0}^{1} (-1)^e \sum_d \lambda_d \sum_{m \equiv O(d)} \frac{C(m) \log m}{m} \sum_{(q,m)=1} \frac{B(q)}{q^e} \sum_{\vartheta} \mathrm{e}\left(-\frac{hq^{1-e}\vartheta}{m}\right) \hat{g}\left(\frac{h}{mq^e}\right).$$

The value $h = 0$ contributes to the main term, namely

$$\sum_{e=0}^{1}(-1)^e \sum_d \lambda_d \sum_{m\equiv O(d)} \frac{\varrho(m)C(m)\log m}{m} \sum_{(q,m)=1} B(q)q^{-e}\hat{g}(0) =$$

$$= \sum_d \lambda_d \sum_{m\equiv O(d)} \sum_{(q,m)=1} \frac{\varrho(m)C(m)\log m}{m} B(q)\hat{g}(0)\left(1+O(x^{-\varepsilon})\right) =$$

$$= B\hat{g}(0)\sum_d \lambda_d \sum_{m\equiv 0(d)} C(m)\log m\varrho(m)m^{-1}\left(1+O(x^{-\varepsilon})\right).$$

Denoting

$$V(x,P) = \sum_d \lambda_d \sum_{m\equiv 0(d)} m^{-1}\varrho(m)C(m)\log m$$

the above sum is equal to

$$(8) \quad B\hat{g}(0)V(x,P) + O(Bx^{1-\varepsilon}(1+D/P)) = B\hat{g}(0)V(x,P) + O(Bx^{1-\varepsilon})$$

provided $D < x$ which we henceforth assume.

Therefore we obtain

$$S(x,P) \le B\hat{g}(0)V(x,P) + O(Bx^{1-\varepsilon}) + \text{Remainder term}.$$

The remainder term (corresponding to the values $h \neq 0$) will be considered in the next sections.

5. Transformation of the remainder term.

With the aid of the Möbius function (cf.(3)) we translate the condition $(q,m) = 1$ in the right-hand side of (7) to obtain the sum

$$(9) \qquad \sum_{e=0}^{1}\sum_{\nu=0}^{1}(-1)^{e+\nu}\sum_d \lambda_d \sum_q R^{(e)}(x, dq^\nu, P)$$

where

$$R^{(e)}(x, dq^\nu, P) =$$

$$= \sum_{h\neq 0}\sum_{m\equiv O(dq^\nu)}\sum_{\vartheta(\bmod m)} e\left(-hq^{1-e}\frac{\vartheta}{m}\right)\hat{g}\left(\frac{h}{mq^e}\right)C(m)m^{-1}\log mB(q)q^{-e}.$$

Applying the smooth partition of unity we may assume that the ranges of d and h are controlled by the smooth functions with their supports contained in $[D, 2D]$ and $[H, 2H]$ respectively $(D, H \geq 1)$. Since the partial integration allows us to truncate the series $R^{(e)}(x, dq^\nu, P)$ at $|h| \leq PQ^e x^{-1+\varepsilon}$ we shall assume that $H \leq PQ^e x^{-1+\varepsilon}$. The oscillatory character of the inner sum in $R^{(e)}(x, dq^\nu, P)$ is to be expressed by means of the following

LEMMA 1 (GAUSS). *If the congruence*

$$\vartheta^2 + 1 \equiv O(m) \quad (m > 1) \tag{10}$$

is soluble then m is represented properly as a sum of two squares

$$m = r^2 + s^2 \qquad (r, s) = 1, \quad r, s > 0. \tag{11}$$

There is a one to one correspondence between the incongruent solutions of (10) and the solutions (r, s) of (11) given by

$$\frac{\vartheta}{m} = \frac{\overline{r}}{s} - \frac{r}{s(r^2 + s^2)}.$$

PROOF. - see [6] and [3], p.34, eq. (68).

We use Lemma 1 to express $R^{(e)}(x, dq^\nu, P)$ in terms of r and s. Applying the smooth partition of unity we conclude that the typical sum to be considered is the following

$$\sum_{h \sim H} \sum_{\substack{s \sim S \\ (r,s)=1}} \sum_{\substack{r \sim R \\ r^2+s^2 \equiv 0(dq^\nu)}} e\left(-hq^{1-e}\frac{\overline{r}}{s}\right) f(d, q, h, r, s) \tag{12}$$

where

$$f(d, q, h, r, s)$$
$$= \frac{C(r^2 + s^2)}{r^2 + s^2} \log(r^2+s^2) \hat{g}\left(\frac{h}{q^{e(r^2+s^2)}}\right) e\left(\frac{hq^{1-e}r}{s(r^2 + s^2)}\right) B(q) q^{-e} w(h, d, r, s),$$

and $w(h,d,r,s)$ is the suitable smooth function supported in

$$[H,2H]\times[D,2D]\times[R,2R]\times[S,2S]$$

with $R,S\ll\sqrt{P}$. In fact we may require that $S\le R$ since otherwise by the congruence

$$\frac{\overline{r}}{s}\equiv-\frac{\overline{s}}{r}+\frac{1}{sr}\ (mod 1)$$

we would obtain

$$\frac{\vartheta}{m}\equiv-\frac{\overline{s}}{r}+\frac{1}{sr}-\frac{r}{s(r^2+s^2)}\ (mod 1).$$

Therefore the change of the roles of r and s involves the additional factor $\mathrm{e}\left(-\frac{hq^{1-\epsilon}}{rs}\right)$ which contributes to the weighted function $f(d,q,h,s,r)$. One can verify directly that for $0\le v_i\le 2, i=1,2,3,4,5$ it holds

(13)

$$\frac{\partial^{v_1+\ldots v_5}}{\partial d^{v_1}\partial q^{v_2}\partial h^{v_3}\partial r^{v_4}\partial s^{v_5}}f(d,q,h,r,s)/\parallel f\parallel\ll D^{-v_1}Q^{-v_2}H^{-v_3}S^{-v_4}R^{-v_5}x^{\varepsilon}$$

provided

$$HQ^{1-\epsilon}/(SR)\le 1$$

which is satisfied whenever $P\le(\frac{x}{Q})^{2-\epsilon}$. We shall therefore assume it in the sequel.

By the Poisson summation formula the inner sum over r in (12) is equal to

$$\sum_{(r,s)=1}\mathrm{e}(-hq^{1-\epsilon}\tfrac{\overline{r}}{s})f(d,q,h,r,s)=$$

$$=\sum_{\substack{\vartheta(mod\ dq^{\nu})\\ \vartheta^2+1\equiv 0(dq^{\nu})}}\sum\nolimits^{*}_{a\ (\mathrm{mod}\ s)}\mathrm{e}(-hq^{1-\epsilon})\sum_{\substack{r\equiv a(s)\\ r\equiv\omega s(dq^{\nu})}}f(d,q,h,r,s)$$

$$=(sdq^{\nu})^{-1}\sum_{\vartheta}\sideset{}{^*}\sum_{a\ (\mathrm{mod}\ s)}\mathrm{e}(-hq^{1-\epsilon}\frac{\overline{a}}{s})\sum_{k}\mathrm{e}(-k\frac{adq^{\nu}\overline{dq^{\nu}}+\vartheta s}{sdq^{\nu}})$$

$$\int f(d,q,h,\zeta,s)\mathrm{e}(\frac{k\zeta}{sdq^{\nu}})d\zeta$$

$$=\sum_{k}\sum_{\vartheta}\mathrm{e}(-k\tfrac{\vartheta}{dq^{\nu}})\sum\nolimits^{*}_{a\ (\mathrm{mod}\ s)}\mathrm{e}(-hq^{1-\epsilon}\tfrac{\overline{a}}{s}-k\overline{dq^{\nu}}\tfrac{a}{s})f_1(d,q,h,k,s)$$

$$= \sum_{\vartheta} \sum_{k} e(-k\tfrac{\vartheta}{dq^{\nu}}) S(h\overline{dq^{\nu}}, kq^{1-e}; s) f_1(d,q,h,k,s),$$

where f_1 is the Fourier transform of f (taken for $\eta = \frac{k}{sdq^{\nu}}$) divided by sdq^{ν}. Therefore

$$\| f_1 \| \ll \| f \| R(SDQ^{\nu})^{-1} \ll (PSDQ^{e+\nu})^{-1} R x^{1+\varepsilon}.$$

For $k = 0$ the Kloosterman sum $S(h\overline{dq^{\nu}}, kq^{1-e}; s)$ reduces to a Ramanujan sum for which it holds

$$|S(h\overline{dq^{\nu}}, 0; s)| \leq (h, s)$$

hence the corresponding contribution to $R^{(e)}(x, dq^{\nu}, P)$ does not exceed

$$\ll HS \| f_1 \| x^{\varepsilon} \ll \frac{Rx^{\varepsilon}}{DQ^{\nu}} \leq \frac{Rx^{\varepsilon}}{D}$$

which by (9) is admissible since $P \leq (\frac{x}{Q})^{2-\varepsilon}$.

For the remaining range of K we apply the smooth partition of unity. Then integrating $f_1(\zeta) = f(d,q,h,\zeta,s)e(\frac{k}{sdq^{\nu}}\zeta)$ by parts several times we away assume that $k \sim K$ with

$$(14) \qquad 1 \leq K \leq SDQ^{\nu} R^{-1} x^{\varepsilon} \leq DQ^{\nu} x^{\varepsilon}, \text{ since } R \geq S.$$

The sum we deal with has the following form

$$R^{(e,\nu)}(x,P) = \sum_{d,q,h,k} \beta_{d,q,k} \sum_{(s,dq^{\nu})=1} S(h\overline{dq^{\nu}}, kq^{1-e}; s) G(d,q,h,k,s)$$

where $\beta_{d,q,k}$ are arbitrary complex coefficients bounded by 1 in absolue value, G is a function satisfying (13) and the above sums is taken over $d \sim D, q \sim Q, h \sim H, k \sim K, s \sim S$.

To estimate it we shall appeal to the method of Deshouillers and Iwaniec (see[2]) which provides the estimates for multilinear forms in Kloosterman sums.

6. Linear forms in Kloosterman sums.

LEMMA 2 (DESHOUILLERS, IWANIEC). *Let $D, H, K, S \geq 1$ and $\Phi(d,h,k,s)$ be a smooth function supported in $[D,2D] \times [H,2H] \times [K,2K] \times [S,2S]$ with partial derivates satisfying (13). Then for any complex numbers $b_{d,k}$ it holds*

$$\sum_{d\sim D}\sum_{h\sim H}\sum_{k\sim K} b_{d,k} \sum_{s\sim S} S(h\bar{d},k;s)\Phi(d,h,k,s) \tag{15}$$

$$\ll (DHKS)^{\epsilon}\left(\sum_{d,k}|b_{d,k}|^2\right)^{\frac{1}{2}} H^{\frac{1}{2}}(\mathcal{L}+\mathcal{M}) \parallel \Phi \parallel$$

where

$$\mathcal{L} = \sqrt{D}\frac{(S\sqrt{D}+\sqrt{HK}+S\sqrt{H})(S\sqrt{D}+\sqrt{HK}+S\sqrt{K})}{S\sqrt{D}+\sqrt{HK}}$$

and

$$\mathcal{M} = S^{\frac{3}{2}}\{D(D+K)\}^{\frac{1}{4}}.$$

PROOF - See Theorem 11 of [2].

To simplify the $\mathcal{L}-$ term suppose that $S\sqrt{H}$ and $S\sqrt{K}$ dominate in the corresponding expressions in brackets. Then

$$\mathcal{L} \ll \sqrt{D}S^2\sqrt{K}H/(S\sqrt{D}) = S\sqrt{HK}.$$

Otherwise

$$\mathcal{L} \ll \sqrt{D}(S\sqrt{D+H+K}+\sqrt{HK}).$$

Therefore in any case

$$\mathcal{L} \ll \sqrt{D}(S\sqrt{D+H+K}+\sqrt{HK}) + S\sqrt{HK}.$$

Hence including the $\mathcal{M}-$ term we obtain

$$\mathcal{L}+\mathcal{M} \ll \sqrt{D}\left(S\sqrt{D+H+K}+\sqrt{HK}+S^{\frac{3}{2}}(1+\frac{K}{D})^{\frac{1}{4}}\right) + S\sqrt{HK}.$$

Let

$$(16)\quad \mathcal{K}^{(e,\nu)}(D,H,K,S,R) == Q^{-e-\nu}\frac{Rx\sqrt{KH}}{PS\sqrt{D}}$$
$$\left\{S\sqrt{D}(K+D+H+HK/D+\sqrt{PK/D+P})^{\frac{1}{2}}+\sqrt{DHK}\right\}.$$

In order to estimate $R^{(e,\nu)}(x,P)$ we distinct four cases :

I. $e=0$, $\nu=0$.

The corresponding Kloosterman sum is equal to $S(h\overline{dq^{\nu}},kq^{1-e};s)$ $=S(h\overline{d},kq;s)$. Hence by Lemma 2 and separation of variables (see [2] p.269) we have

$$(17)\qquad R^{(0,0)}(x,P)\ll\left(\sum_{d,q,k}|\beta_{d,q,k}|^2\right)^{\frac{1}{2}}H^{1/2}(\mathcal{L}+\mathcal{M})\,||\,G\,||\,x^{\varepsilon}$$

with the variable K replaced by KQ in the right-hand side of (15) thus yielding

$$R^{(0,0)}(x,P)\ll\sqrt{\frac{B}{Q}}\mathcal{K}^{(0,0)}(D,H_0,K_0Q,S,R)x^{\varepsilon}$$

where

$$(18)\qquad H_e=Q^eP/x,\qquad K_\nu=DQ^\nu,\qquad e,\nu\in\{0,1\}.$$

Now we shall infer that in the remaining cases $R^{(e\nu)}(x,P)$ admits better estimates than in the above case.

II. $e=0$, $\nu=1$.

The Kloosterman sum in question is equal to $S(h\overline{dq},kq;s)=S(h\overline{d},k;s)$ hence it does not depend on q. Therefore by (16),(17),(18) we obtain

$$R^{(0,1)}(x,P)\ll B\;\mathcal{K}^{(0,1)}(D,H_0,K_1,S,R)x^{\varepsilon}\ll\frac{B}{Q}\mathcal{K}^{0,0}(D,H_0,K_0Q,S,R)x^{\varepsilon}$$

$$\ll \sqrt{\frac{B}{Q}}\mathcal{K}^{(0,0)}(D,H_0,K_0Q,S,R)x^{\varepsilon} \qquad (\text{since} D \le Q).$$

III. $e = 1$, $\nu = 1$.

The Kloosterman sum is equal to $S(h\overline{dq}, k; s)$. Hence by (16)-(18) we obtain

$$\begin{aligned} &R^{(1,1)}(x,P) \\ &\ll \sqrt{\frac{B}{Q}}\mathcal{K}^{(1,1)}(DQ,H_1,K_1,S,R)x^{\varepsilon} \ll \sqrt{\frac{B}{Q}}\sqrt{Q}\mathcal{K}^{(1,1)}(D,H_1,K_1,S,R)x^{\varepsilon} \\ &\ll \sqrt{B}\mathcal{K}^{(0,1)}(D,H_0,K_1,S,R)x^{\varepsilon} \ll \sqrt{B}/Q\ \mathcal{K}^{(0,0)}(D,H_0,K_0Q,S,R)x^{\varepsilon}. \end{aligned}$$

IV. $e = 1$, $\nu = 0$.

The Kloosterman sum is equal to $S(h\overline{d}, k, ; s)$. Since $H_1 \le H_1K_0/D$ it follows that

$$R^{(0,1)}(x,P) \ll B\mathcal{K}^{(1,0)}(D,H_1,K_0,S,R)x^{\varepsilon} \ll B\mathcal{K}^{(0,1)}(D,H_0,K_1,S,R)x^{\varepsilon}$$

$$\ll \frac{B}{Q}\mathcal{K}^{(0,0)}(D,H_0,K_0Q,S,R)x^{\varepsilon} \ll \sqrt{\frac{B}{Q}}\mathcal{K}^{(0,0)}(D,H_0,K_0Q,S,R)x^{\varepsilon}.$$

Therefore it is sufficient to consider the case $R^{(0,0)}(x,P)$.

We observe that $\sqrt{DH_0K_0Q} \le \sqrt{D^2H_0QS/R}x^{\varepsilon} \le S\sqrt{D}(P^{1/4})x^{\varepsilon}$ provided $D \le (\frac{x}{Q})^{1-\varepsilon}$ which we henceforth assume. Since

$$D + H_0 \le K_0Q + H_0K_0Q/D \text{ and } P \le PK_0Q/D$$

we obtain by (16)

$$R^{(0,0)}(x,P) \ll \sqrt{\frac{B}{Q}}\frac{Rx\sqrt{K_0QH_0}}{PS\sqrt{D}}(S\sqrt{D}[\sqrt{K_0Q} + \sqrt{H_0Q} + (PQ)^{\frac{1}{4}}])x^{\varepsilon}$$

$$\ll \sqrt{BQDx}\left(\sqrt{D} + \sqrt{\frac{P}{x}} + \left(\frac{P}{Q}\right)^{\frac{1}{4}}\right)x^{\varepsilon}.$$

Now we have

$$D\sqrt{BQx} < Bx \Longleftrightarrow D < x^{\frac{1}{2}}\sqrt{B/Q} = D_1$$

$$\sqrt{BQDP} < Bx \Longleftrightarrow D < x^2 P^{-1}\frac{B}{Q} = D_2$$

$$\sqrt{BQDx}(P/Q)^{\frac{1}{4}} < Bx \Longleftrightarrow D < xP^{-\frac{1}{2}}\frac{B}{\sqrt{Q}} = D_3.$$

Hence for $D = \min\,(D_1, D_2, D_3)x^{-\varepsilon}$ we obtain

$$R^{(0,0)}(x, P) \ll Bx^{1-\varepsilon}$$

7. Completion of the proof.

Collecting together the results of the previous section we obtain by (8) that $S(x,P) \leq B\hat{g}(0)V(x,P) + O(x^{1+\varepsilon} + Bx^{1-\varepsilon})$ provided $P \leq (\frac{x}{Q})^{2-\varepsilon}$ and $D = \min\,(D_1, D_2, D_3, x/Q)x^{-\varepsilon}$. Since $P \leq x^{\sqrt{3/2}}, Q \in [\frac{1}{2}x^{\Theta}, x^{\Theta}]$ with $\Theta \leq \sqrt{\frac{3}{2}} - 1$ and $B \geq \sqrt{Q}$ we obtain that the above conditions are satisfied provided

$$D = \min\,\left(x^{\frac{1}{2}}\sqrt{B/Q}, \frac{xB}{\sqrt{PQ}}\right)x^{-\varepsilon}.$$

Therefore the main contribution to $S(x)$ (see[1],[5]) is to be evaluated on the basis of equality

$$V(x,P) = \frac{2}{\log D(x,P,\beta,\Theta)}\int C(\zeta)\frac{\log\zeta}{\zeta}d\zeta\left\{1 + O\left(\frac{1}{\log D(x,P,\beta,\Theta)}\right)\right\}$$

with

$$D(x,P,\beta,\Theta) = \min\left(x^{\frac{1}{2}(1+\Theta(\beta-1))}, \frac{x^{1+\Theta(\beta-\frac{1}{2})}}{\sqrt{P}}\right)x^{-\varepsilon}. \tag{19}$$

Hence (cf. [1]) defining $\alpha = \frac{\log P_x(\beta,\Theta)}{\log x}$ we obtain that the total main term is equal to

$$\hat{g}(0)B\sum_{0\leq j\leq J} V(x,P_j) \leq 2(1+\varepsilon)\hat{g}(0)B\int_x^{Px}\frac{\log\zeta}{\zeta\log D(x,\zeta,\beta,\Theta)}d\zeta$$

$$\leq 2(1+2\varepsilon)\hat{g}(0)B\int_x^{Px} \max\left(\frac{2}{(1+\Theta(\beta-1))\log x}, \frac{1}{(1+\Theta(\beta-\frac{1}{2}))\log x-\frac{1}{2}\log\zeta}\right)\frac{\log\zeta}{\zeta}d\zeta$$

$$= 2(1+2\varepsilon)\hat{g}(0)B\log x\int_1^{\alpha}\max\left(\frac{2t}{1+\Theta(\beta-1)}, \frac{t}{1+\Theta(\beta-1/2)-t/2}\right)dt$$

$$= 2(1+2\varepsilon)\hat{g}(0)B\log x\left\{\int_1^{1+\Theta\beta}\frac{2t}{1+\Theta(\beta-1)}dt + \int_{1+\Theta\beta}^{\alpha}\frac{t}{1+\Theta(\beta-1/2)-t/2}dt\right\}$$

$$= 2(1+2\varepsilon)\hat{g}(0)B\log x\ .\ F_{\beta,\Theta}(\alpha)$$

by a direct calculation (see (*)).

Therefore $S(x) \leq \hat{g}(0)B\log x$ if $F_{\beta,\Theta}(\alpha) < 1/2$. On the other hand by an obvious inequality $P_X(\beta,\Theta) \geq P_X(\beta,\Theta')$ whenever $\Theta \geq \Theta'$, we conclude that $Px(\beta,\Theta) > x^{\gamma}$. The proof of the theorem is complete.

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