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New bounds on the Length of Finite Pierce and Engel Series.

par P. ERDŐS AND J.O. SHALLIT*

ABSTRACT. Every real number x, $0 < x \leq 1$, has an essentially unique expansion as a Pierce series:

$$x = \frac{1}{x_1} - \frac{1}{x_1 x_2} + \frac{1}{x_1 x_2 x_3} - \cdots$$

where the x_i form a strictly increasing sequence of positive integers. The expansion terminates if and only if x is rational. Similarly, every positive real number y has a unique expansion as an Engel series:

$$y = \frac{1}{y_1} + \frac{1}{y_1 y_2} + \frac{1}{y_1 y_2 y_3} + \cdots$$

where the y_i form a (not necessarily strictly) increasing sequence of positive integers. If the expansion is infinite, we require that the sequence y_i be not eventually constant. Again, such an expansion terminates if and only if y is rational. In this paper we obtain some new upper and lower bounds on the lengths of these series on rational inputs a/b. In the case of the Engel series, this answers an open question of Erdös, Rényi, and Szüsz. However, our upper and lower bounds are widely separated.

1. Introduction.

Let a, b be integers with $1 \leq a \leq b$, and define

$$a_1 = a \qquad \text{and} \qquad a_{i+1} = b \bmod a_i \qquad \text{for } i \geq 0. \tag{1}$$

Since $a_{i+1} < a_i$, eventually we must have $a_{n+1} = 0$. Put $P(a,b) = n$. We ask: how big can $P(a,b)$ be as a function of a and b?

This question seems to be much harder than it first appears. Shallit [11] proved that $P(a,b) < 2\sqrt{b}$; also see Mays [6].

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In this paper we improve the bound to $P(a, b) = O(b^{1/3+\epsilon})$ for every $\epsilon > 0$. (This is still a weak result, as we believe that $P(a, b) = O((\log b)^2)$.)

We can also ask about the *average* behavior of $P(a, b)$. We define

$$Q(b) = \frac{1}{b} \sum_{1 \le i \le b} P(i, b).$$

In this paper we prove $Q(b) = \Omega(\log\log b)$. (Again, this result is rather weak, as it seems likely that $Q(b) = \Omega(\log b)$.)

There is a connection between the algorithm given by (1) and the following expansion, called the Pierce series:

Let $0 < x \le 1$ be a real number. Then x may be expressed uniquely in the form

$$x = \frac{1}{x_1} - \frac{1}{x_1 x_2} + \frac{1}{x_1 x_2 x_3} - \cdots \tag{2}$$

where $1 \le x_1 < x_2 < x_3 < \cdots$. We sometimes abbreviate eq. (2) by

$$x = \langle x_1, x_2, x_3, \ldots \rangle.$$

The expansion terminates if and only if x is rational. If the expansion does terminate, with

$$\frac{(-1)^{n+1}}{x_1 x_2 \cdots x_n}$$

as the last term, then we also must have $x_{n-1} < x_n - 1$.

Let $P'(a, b)$ denote the number of terms in the Pierce series for a/b. Then we have the following

Observation 1.

$$P'(a, b) = P(a, b).$$

This follows easily, as $a_2 = b \bmod a_1$ means $b = q_1 a_1 + a_2$; hence

$$\frac{a_1}{b} = \frac{1}{q_1}\left(1 - \frac{a_2}{b}\right).$$

Similarly, from $b = q_2 a_2 + a_3$, we get

$$\frac{a_2}{b} = \frac{1}{q_2}\left(1 - \frac{a_3}{b}\right).$$

Continuing, we find

$$\frac{a}{b} = \frac{1}{q_1} - \frac{1}{q_1 q_2} + \cdots + \frac{(-1)^{n+1}}{q_1 q_2 \cdots q_n}.$$

In fact, the algorithm (1) has the same relationship with expansions into Pierce series as the Euclidean algorithm for the greatest common divisor has with continued fractions.

A similar algorithm is as follows: let $1 \le a \le b$ and define

$$a_1 = a \qquad \text{and} \qquad a_{i+1} = (-b) \bmod a_i \qquad \text{for } i \ge 0. \tag{3}$$

Again, we must eventually have $a_{n+1} = 0$. Put $E(a,b) = n$. Erdös, Renyi, and Szüsz [4] asked for a nontrivial estimate for $E(a,b)$. In this paper we prove the first such estimate, namely $E(a,b) = O(b^{1/3+\epsilon})$ for all $\epsilon > 0$.

The algorithm (3) is related to expansion in Engel series, as follows:

Let y be a positive real number. Then y may be expressed uniquely in the form

$$y = \frac{1}{y_1} + \frac{1}{y_1 y_2} + \frac{1}{y_1 y_2 y_3} + \cdots \tag{4}$$

where $1 \le y_1 \le y_2 \le y_3 \le \cdots$. If the expansion does not terminate, then we require that the sequence y_i be not eventually constant. Such an expansion terminates if and only if y is rational.

Let $E'(a,b)$ denote the number of terms in the expansion for a/b. As above, it is easy to see that $E(a,b) = E'(a,b)$.

For more information about the Pierce series, see [7,8,11,13,14]. The results in Section 3 were announced previously in [12].

For more information about Engel's series, see [1,2,3,4,9,13].

2. Upper bounds.

We recall the proof from [11] that $P(a,b) < 2\sqrt{b}$. We write $a_1 = a$ and

$$\begin{aligned}
b &= q_1 a_1 + a_2 \\
b &= q_2 a_2 + a_3 \\
&\vdots \\
b &= q_{n-1} a_{n-1} + a_n \\
b &= q_n a_n.
\end{aligned}$$

Note that $a_k q_k \le b$ for $1 \le k \le n$.

Without loss of generality we may assume $q_1 = 1$, for if not, then :

$$P(b-a,b) = 1 + P(a,b).$$

Choose k such that $q_k \le \sqrt{b}$ and $q_{k+1} > \sqrt{b}$. (If no such k exists, then $q_k \le \sqrt{b}$ for $1 \le k \le n$; hence $n \le \sqrt{b}$.)

Then, as the q_i are strictly increasing, we have $k \le \sqrt{b}$. Now since $a_{k+1}q_{k+1} \le b$, we have $a_{k+1} < \sqrt{b}$. Since the a_i are strictly decreasing, we have $n - k < \sqrt{b}$. Hence we find $n < 2\sqrt{b}$.

We now show how to modify this argument to get an improved bound:

THEOREM 2.

We have $P(a,b) = O(b^{1/3+\epsilon})$ *for all* $\epsilon > 0$.

Proof.

We first observe that for any fixed r, we cannot have $a_i - a_{i+1} = r$ too often. For if, say, we have

$$\begin{aligned} b &= q_{i_1}a_{i_1} + a_{i_1} - r \\ b &= q_{i_2}a_{i_2} + a_{i_2} - r \\ &\vdots \\ b &= q_{i_j}a_{i_j} + a_{i_j} - r, \end{aligned}$$

then $b + r$ is divisible by each of $a_{i_1}, a_{i_2}, \ldots, a_{i_j}$. Since the a's are all distinct, we have $j \le d(b+r)$, where $d(m)$ is the number of divisors of n. Now it is well known (see [5]) that $d(m) = O(m^\epsilon)$ for all $\epsilon > 0$, so $j \le d(b+r) = O(b^\epsilon)$.

Now as above we can assume $q_1 = 1$. Choose i such that $q_i < b^{1/3}$ and $q_{i+1} \ge b^{1/3}$. (If no such i exists, then $q_i < b^{1/3}$ for all i and hence $n < b^{1/3}$.) Note that

$$i < b^{1/3} \tag{5}$$

and $a_j \le b^{2/3}$ for $i + 1 \le j \le n$.

Let us count the number of j's, $i+1 \le j \le n$, such that $r = a_j - a_{j+1} \le b^{1/3}$. By the argument above, there are $O(b^\epsilon)$ such j for each r, $1 \le r \le b^{1/3}$. Hence there are a total of $O(b^{1/3+\epsilon})$ such j.

Now let us count the number of j's, $i+1 \leq j \leq n$ such that $a_j - a_{j+1} > b^{1/3}$. Since $a_{i+1} - a_n \leq b^{2/3}$, it is clear that there can be at most $b^{1/3}$ such j.

Hence all together there are $O(b^{1/3+\epsilon})$ j's in the range $i+1 \leq j \leq n$, and we conclude

$$n - i = O(b^{1/3+\epsilon}). \tag{6}$$

Adding (5) and (6), we conclude $P(a,b) = n = O(b^{1/3+\epsilon})$. □

We now show how to modify this argument to get an upper bound for $E(a,b)$.

We write $a_1 = a$ and

$$\begin{aligned} b &= q_1 a_1 - a_2 \\ b &= q_2 a_2 - a_3 \\ &\vdots \\ b &= q_{n-1} a_{n-1} - a_n \\ b &= q_n a_n. \end{aligned}$$

Note that $q_i = \lceil b/a_i \rceil$.

In what follows, we assume $1 \leq a < b$; such a restriction ensures that $q_i \geq 2$ for all $1 \leq i \leq n$.

Note that $a_k q_k \leq 2b$ for $1 \leq k \leq n$. The a_i are strictly decreasing.

In the case of Engel series, the q_i form an increasing sequence that is not necessarily strictly increasing. However, it is not difficult to show that we cannot have too many consecutive quotients that are the same:

LEMMA 3.

Suppose $b = qa_i - a_{i+1}$ *for* $j \leq i \leq k$. *Then* $q^{i-j} | a_i - a_{i+1}$ *for* $j \leq i \leq k$.

Proof.

By induction on i. The result is clearly true when $i = j$. Now assume it true for i; we prove it for $i+1$. We have $b = qa_i - a_{i+1}$ and $b = qa_{i+1} - a_{i+2}$. Subtracting, we find $a_{i+1} - a_{i+2} = q(a_i - a_{i+1})$. As $q^{i-j} | a_i - a_{i+1}$ by induction, we have $q^{i+1-j} | a_{i+1} - a_{i+2}$, and the result follows. □

COROLLARY 4.

Let $1 \leq a < b$ and $q \geq 2$. In the Engel series for a/b, there cannot be more than $1 + \log_q a$ quotients q_i that are equal to q.

We may now apply the same argument used to prove Theorem 2 to get a similar result for $E(a,b)$:

THEOREM 5.

Let $1 \leq a < b$. We have $E(a,b) = O(b^{1/3+\epsilon})$ for all $\epsilon > 0$.

Proof.

Again, we choose i such that $q_i < b^{1/3}$ and $q_{i+1} \geq b^{1/3}$. (If no such i exists, then $q_i < b^{1/3}$ for all i and hence by Corollary 4, $n < b^{1/3}(1+\log_2 b)$.)

Note that $i = O(b^{1/3} \log b)$, by Corollary 4. Since $q_{i+1} \geq b^{1/3}$, and the a_i are strictly decreasing, we have $a_j \leq 2b^{2/3}$ for $i+1 \leq j \leq n$. Now an argument similar to that in the proof of Theorem 2 shows that there can be at most $O(b^{1/3+\epsilon})$ subscripts $j \geq i+1$ such that $a_j - a_{j+1} \leq 2b^{1/3}$. Similarly, there can be at most $O(b^{1/3})$ subscripts $j \geq i+1$ such that $a_j - a_{j+1} \geq 2b^{1/3}$. We conclude that there are $O(b^{1/3+\epsilon})$ subscripts j in the range $i+1 \leq j \leq n$, and hence $n - i = O(b^{1/3+\epsilon})$.

Adding our estimates for i and $n-i$, we conclude that $E(a,b) = n = O(b^{1/3+\epsilon})$. □

3. Lower bounds for $P(a,b)$ and $Q(b)$.

In this section we prove some lower bounds for $P(a,b)$ and $Q(b)$.

In [11], it was proved that

$$P(a,b) > \frac{\log b}{\log \log b}$$

infinitely often. Actually, a very simple argument gives a better result:

THEOREM 6.

There exists a constant $c > 0$ such that $P(a,b) > c \log b$ infinitely often.

Proof.

Let $a = n$ and $b = \text{lcm}(1,2,3,\ldots,n) - 1$. Then it is easy to see that $b \bmod j = j-1$ for $1 \leq j \leq n$; hence $a_j = n+1-j$ for $1 \leq j \leq n+1$, and so $P(a,b) = n$. However,

$$\log b < \log(b+1) = \psi(n) < 1.03883n = 1.03883\,P(a,b),$$

where $\psi(x) = \sum_{p^k \leq x} \log p$ and we have used an estimate from [10]. This proves the theorem with $c = (1.03883)^{-1}$. □

REMARK.

It is trivial to find a similar lower bound for Engel's series, as $E(2^n - 1, 2^n) = n$.

We now prove a result on the average complexity of the algorithm (1).

THEOREM 7.
$Q(b) = \Omega(\log\log b)$.

Proof.

Let $T_b(j)$ be the total number of times that j appears as a term in the Pierce expansions of $1/b$, $2/b, \ldots, (b-1)/b$, 1.

Clearly

$$bQ(b) = \sum_{1\leq i\leq b} P(i,b) = \sum_{j\geq 1} T_b(j). \tag{7}$$

The idea is to find a lower bound for this last sum. More precisely, we find a bound for

$$\sum_{1\leq j<\log b} T_b(j).$$

Fix a j, $1 \leq j \leq \log b$. Now every real number in the open interval

$$I = (\ \langle x_1, x_2, \ldots, x_k, j\rangle,\ \langle x_1, x_2, \ldots, x_k, j+1\rangle\ ) \tag{8}$$

has a Pierce series expansion that begins $\langle x_1, x_2, \ldots, x_k, j, \ldots\rangle$ provided $x_k < j$. (Actually, the endpoints of the open interval given in (8) should be reversed if k is even.)

There are $b|I| + O(1)$ rationals with denominator b contained in the interval I, and the interval I is of size $\frac{1}{x_1 x_2 \cdots x_k j(j+1)}$.

Now let us sum $b|I| + O(1)$ over all possible values for $x_1, x_2, \ldots, x_k$; this gives us an estimate for $T_b(j)$. We find

$$T_b(j) = \sum_{A \subseteq \{1,2,\ldots,j-1\}} \left(\frac{b}{(\prod_{a \in A} a) j(j+1)} + O(1) \right)$$
$$= \left(\frac{b}{j(j+1)} \sum_{A \subseteq \{1,2,\ldots,j-1\}} \frac{1}{\prod_{a \in A} a} \right) + O(2^{j-1})$$

Now, using the observation that

$$\sum_{A \subseteq \{1,2,\ldots,j-1\}} \frac{1}{\prod_{a \in A} a} = (1 + \frac{1}{1})(1 + \frac{1}{2}) \cdots (1 + \frac{1}{j-1}) = j,$$

we get

$$T_b(j) = \frac{b}{j+1} + O(2^{j-1}).$$

Now consider $\sum_{1 \le j < \log b} T_b(j)$. We get

$$\sum_{1 \le j < \log b} T_b(j) = \left(b \sum_{1 \le j < \log b} \frac{1}{j+1} \right) + O(b)$$
$$= b \log\log b + O(b).$$

Thus, using (7), we see

$$bQ(b) \ge \sum_{1 \le j < \log b} T_b(j) = \Omega(b \log\log b),$$

and so $Q(b) = \Omega(\log\log b)$. □

4. Worst cases: numerical results.

In this section we report on some computations done to find the least b such that $P(a,b) = n$ and $E(a,b) = n$, for some small values of n.

The following table gives, for each $n \le 42$, the least b such that there exists an a, $1 \le a \le b$, with $P(a,b) = n$. If there is more than one such a for a particular b, the smallest such a is listed. This table extends one given in Mays [6].

n	a	b	n	a	b
1	1	1	22	2416	3959
2	2	3	23	1925	5387
3	3	5	24	3462	5387
4	4	11	25	2130	5879
5	7	11	26	3749	5879
6	12	19	27	6546	17747
7	22	35	28	11201	17747
8	30	47	29	2159	23399
9	32	53	30	2360	23399
10	61	95	31	5186	23399
11	65	103	32	6071	23399
12	115	179	33	8664	23399
13	161	251	34	14735	23399
14	189	299	35	59745	93596
15	296	503	36	68482	186479
16	470	743	37	117997	186479
17	598	1019	38	175672	278387
18	841	1319	39	268618	442679
19	904	1439	40	135585	493919
20	1856	2939	41	178909	493919
21	2158	3359	42	314752	493919

Table I: Worst Cases for Pierce Expansions

[Note added in proof: the following entry extending Table I has recently been discovered by computer : $n = 43$, $a = 490652$, $b = 830939$.]

The next table reports the results of a similar computation for $E(a, b)$:

n	a	b	n	a	b
1	1	1	28	3050	3053
2	2	3	29	3609	3613
3	4	5	30	3611	3613
4	5	7	31	3612	3613
5	6	7	32	5459	5461
6	12	13	33	5460	5461
7	18	19	34	7976	8011
8	20	23	35	7999	8011
9	30	31	36	8005	8011
10	46	47	37	8008	8011
11	60	61	38	10076	10081
12	62	71	39	16379	16381
13	72	73	40	16380	16381
14	89	121	41	16379	16383
15	105	121	42	16381	16383
16	113	121	43	16382	16383
17	117	121	44	32765	32766
18	119	121	45	65513	65521
19	120	121	46	65517	65521
20	241	242	47	65519	65521
21	483	484	48	65520	65521
22	633	661	49	131041	131042
23	647	661	50	262083	262084
24	654	661	51	516985	517001
25	1074	1093	52	516993	517001
26	1752	1753	53	516997	517001
27	1806	1807	54	516999	517001

Table II: Worst Cases for Engel Expansions

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