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The k -dimensional Duffin and Schaeffer conjecture.

par A.D. POLLINGTON AND R.C. VAUGHAN

Résumé — Nous montrons que la conjecture de Duffin et Schaeffer est vraie en toute dimension supérieure à 1.

Abstract — We show that the Duffin and Schaeffer conjecture holds in all dimensions greater than one.

In 1941 Duffin and Schaeffer [1] made the following conjecture:

CONJECTURE. Let $\{\alpha_n\}$ denote a sequence of real numbers with

$$0 \leq \alpha_n < \frac{1}{2}$$

then the inequalities

$$(1) \quad |nx - a| < \alpha_n, \quad (a, n) = 1,$$

have infinitely many solutions for almost all x if and only if

$$(2) \quad \sum_{n=1}^{\infty} \frac{\alpha_n \varphi(n)}{n}$$

diverges.

If (2) converges then it easily follows from the Borel-Cantelli lemma that the set of x 's satisfying infinitely many of the inequalities (1) has Lebesgues measure zero. Duffin and Schaeffer gave conditions on the α_n for which the conjecture is true and showed that the condition $(a, n) = 1$ is necessary. In 1970 Erdős [2] showed that the conjecture holds if $\alpha_n = \frac{\epsilon}{n}$ or 0. This was later extended by Vaaler [6] who showed that $\alpha_n = O(\frac{1}{n})$ is sufficient. In his book on metric number theory [5] Sprindzuk considers a k -dimensional analogue of the conjecture in which (1) is replaced by

$$(3) \quad \max(|x_1 n - a_1|, \dots, |x_k n - a_k|) < \alpha_n \quad (a_i, n) = 1, \quad i = 1, \dots, k$$

and (2) by

$$(4) \quad \sum_{n=1}^{\infty} \left(\frac{\alpha_n \varphi(n)}{n} \right)^k$$

where the measure is now k -dimensional Lebesgues measure. He states that the study of such approximations subject to the conditions $(a_1, n) = \cdots = (a_k, n) = 1$ is probably a problem of the same degree of complexity as the case $n = 1$. This appears not to be the case. For we can now prove the k -dimensional analogue of the Duffin and Schaeffer conjecture.

We prove the following result:

THEOREM. *Let $k > 1$ and let $\{\alpha_n\}$ denote a sequence of real numbers with*

$$0 \leq \alpha_n < \frac{1}{2}$$

and suppose that

$$\sum_{n=1}^{\infty} \left(\frac{\alpha_n \varphi(n)}{n} \right)^k$$

diverges. Then the inequalities

$$\max(|x_1 n - a_1|, \dots, |x_k n - a_k|) < \alpha_n \quad (a_i, n) = 1, \quad i = 1, \dots, k$$

have infinitely many solutions for almost all $x \in R^k$.

Unfortunately our method does not readily extend to the case $k = 1$. We are able find some more sequences $\{\alpha_n\}$ for which the conjecture holds, for example if $\alpha_n = 0$ or $1 \ll \alpha_n$, but not to settle the dimension one case. In the k -dimensional case, $k \geq 2$, Vilchinski has previously shown that we may take $\alpha_n = O(n^{-\gamma})$ for any $\gamma > 0$.

Put

$$E_n = E_n^{(1)} \times \cdots \times E_n^{(k)}$$

where

$$(5) \quad E_n^{(i)} = \bigcup_{\substack{1 \leq a_i \leq n \\ (a_i, n) = 1}} \left(\frac{a_i - \alpha_n}{n}, \frac{a_i + \alpha_n}{n} \right).$$

Then E_n is the set counted in (3) and

$$(6) \quad \lambda_k(E_n) = \left(\frac{2\alpha_n \varphi(n)}{n}\right)^k.$$

Thus (4) becomes

$$(7) \quad \sum_{k=1}^{\infty} \lambda_k(E_n)$$

diverges.

We are interested in

$$(8) \quad \lambda_k \left(\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} E_n \right).$$

By an ergodic theorem of Gallagher [3], see Sprindzuk [5], this is either zero or one. Gallagher proves his result for dimension one. The corresponding k -dimensional result is proved by Vilchinski [7]. In order to prove the theorem it therefore suffices to show that (8) is not zero.

Since (7) diverges, and $\lambda_k(E_n) \rightarrow 0$ as $n \rightarrow \infty$, given any $1 > \eta > 0$, for every N we can find a finite set Z so that if $z \in Z$ then $z > N$, and

$$(9) \quad \eta^2 < \Lambda(Z) = \sum_{n \in Z} \lambda_k(E_n) < \eta.$$

By the Cauchy-Schwarz inequality

$$(10) \quad \lambda_k \left(\bigcup_{n \in Z} E_n \right) \geq \frac{(\sum_n \lambda_k(E_n))^2}{(\sum_n \sum_m \lambda_k(E_n \cap E_m))},$$

this is Lemma 5 of Sprindzuk [5]. So provided there is some absolute constant c for which

$$(11) \quad \sum_{\substack{n \neq m \\ m, n \in Z}} \lambda_k(E_n \cap E_m) \leq c \sum_{n \in Z} \lambda_k(E_n)$$

the theorem is proved.

Thus we need to bound

$$\sum_{\substack{n \neq m \\ m, n \in \mathbb{Z}}} \lambda_k(E_n \cap E_m).$$

In the subsequent discussion constants in the Vinogradov \ll symbol depend at most on the dimension k .

Note that

$$(12) \quad \lambda_k(E_n) = \prod_{i=1}^k \lambda(E_n^{(i)})$$

and

$$(13) \quad \lambda_k(E_n \cap E_m) = \prod_{i=1}^k \lambda(E_n^{(i)} \cap E_m^{(i)}).$$

We now concentrate on $k = 1$ and use (12) and (13) to obtain a bound for $k > 1$.

We wish to obtain an estimate for

$$\lambda(E_n \cap E_m) \quad \text{with } n > m.$$

Put

$$(14) \quad \delta = \min\left(\frac{\alpha_n}{n}, \frac{\alpha_m}{m}\right) \quad \text{and} \quad \Delta = \max\left(\frac{\alpha_n}{n}, \frac{\alpha_m}{m}\right),$$

and

$$d = (m, n) \quad m' = \frac{m}{d} \quad n' = \frac{n}{d}.$$

Then

$$(15) \quad \lambda(E_n \cap E_m) \leq 2\delta \sum_{\substack{|\frac{a}{n} - \frac{b}{m}| < 2\Delta \\ (a, n) = 1 \\ (b, m) = 1}} 1.$$

By estimating

$$(16) \quad \sum_{\substack{|\frac{a}{n} - \frac{b}{m}| < 2\Delta \\ (a, n) = 1 \\ (b, m) = 1}} 1$$

we obtain

LEMMA 1. *With E_n as above and $d = (m, n)$ there is an absolute constant c so that*

$$(17) \quad \lambda(E_n \cap E_m) \leq c\lambda(E_n)\lambda(E_m) \prod_{\substack{p|m'n' \\ p > \frac{2mn\Delta}{d}}} \left(1 - \frac{1}{p}\right)^{-1}.$$

Let $g(n)$ be defined as the least number so that

$$\sum_{\substack{p|n \\ p > g(n)}} \frac{1}{p} < 2$$

Put

$$t = \max(g(n'), g(m')).$$

Then by Lemma 1

$$(18) \quad \lambda(E_n \cap E_m) \ll \lambda(E_n)\lambda(E_m) \prod_{\substack{p|m'n' \\ \frac{2mn\Delta}{d} < p < t}} \left(1 - \frac{1}{p}\right)^{-1}.$$

In particular if $\frac{2mn\Delta}{d} \geq t$ then

$$(19) \quad \lambda(E_n \cap E_m) \ll \lambda(E_n)\lambda(E_m).$$

Now

$$(20) \quad \prod_{\substack{p|\frac{nm}{d^2} \\ p < t}} \left(1 - \frac{1}{p}\right)^{-1} \leq \prod_{p < t} \left(1 - \frac{1}{p}\right)^{-1} \ll \log t.$$

We now return to the k -dimensional case. By (12) and (13)

$$(21) \quad \sum_{\substack{n \neq m \\ \log t \leq \Lambda(Z)^{-1/k}}} \lambda_k(E_n \cap E_m) \ll \Lambda(Z)^{-1} \sum_{n \neq m} \lambda_k(E_n)\lambda_k(E_m) \\ \ll \Lambda(Z).$$

From now on we shall assume that

$$\log t > \Lambda(Z)^{-1/k}.$$

We distinguish four cases:

$$(a) \quad \Delta = \frac{\alpha_m}{m}, \quad \alpha_n < \alpha_m$$

$$(b) \quad \Delta = \frac{\alpha_m}{m}, \quad \alpha_n \geq \alpha_m$$

$$(c) \quad \Delta = \frac{\alpha_n}{n}, \quad \alpha_n < \alpha_m$$

$$(d) \quad \Delta = \frac{\alpha_n}{n}, \quad \alpha_n \geq \alpha_m.$$

Recall that $m < n$ so (c) is impossible. We will consider the other three cases separately. The first case corresponds to the situation considered by Erdős .

Case (a).

We have

$$\frac{mn\Delta}{d} = n'\alpha_m.$$

We need to consider pairs m, n with

$$(22) \quad 1 < n'\alpha_m < t$$

Let $A_{u,v}$ denote that part of the sum

$$\sum_{\substack{m \neq n \\ \text{case (a)}}} \lambda_k(E_n \cap E_m)$$

for which $g(m') = u$ and $g(n') = v$. Then, by Lemma 1 and (22)

$$A_{u,v} \ll \log^k t \sum_m \lambda_k(E_m) \sum_{\substack{(u) \\ m' | m \\ 1 \leq m' < t\alpha_m^{-1}}} \sum_{\substack{(v) \\ \alpha_m^{-1} \leq n' < t\alpha_m^{-1} \\ n = n'd}} \lambda_k(E_n).$$

Where $\sum_{(u)}$ means the sum of those m' with $g(m') = u$ and $m = m'd$. Since

$$\lambda_k(E_n) = \left(2\alpha_n \frac{\varphi(n)}{n} \right)^k \ll \alpha_m^k$$

then

$$A_{u,v} \ll \log^k t \sum_m \lambda_k(E_m) \alpha_m^k \sum_{\substack{(u) \\ m'|m \\ 1 \leq m' < t\alpha_m^{-1}}} \sum_{\substack{(v) \\ \alpha_m^{-1} \leq n' < t\alpha_m^{-1} \\ n=n'd}} 1.$$

To estimate the inner sum we will use the following result due to Erdős which also appears in Vaaler [6]. Let $N(\xi, v, x)$ be the number of integers $n \leq x$ which satisfy

$$\sum_{\substack{p|n \\ p \geq v}} \frac{1}{p} \geq \xi.$$

Then

LEMMA 2. For any $\epsilon > 0$ and any $\xi > 0$ there exists a positive integer $v_0 = v_0(\xi, \epsilon)$ such that for all $x > 0$ and all $v > v_0$,

$$N(\xi, v, x) \leq x \exp\{-v^{\beta(1-\epsilon)}\}$$

where $\log \beta = \xi$.

Applying this result with $\xi = 1$ and $\beta(1 - \epsilon) = \frac{3}{2}$ we have

COROLLARY. Given $x > 0$ and $v \geq 1$

$$(23) \quad N(1, v, x) \ll x \exp(-v^{\frac{3}{2}}).$$

Using (23) and partial summation

$$\sum_{\substack{(v) \\ 1 \leq n' \leq t\alpha_m^{-1}}} \ll t\alpha_m^{-1} \exp(-v^{\frac{3}{2}}).$$

Applying (23) again

$$A_{u,v} \ll t^2 \log^k t \exp(-u^{\frac{3}{2}}) \exp(-v^{\frac{3}{2}}) \sum_m \lambda_k(E_m),$$

since $k \geq 2$. Then summing over u and v we have

$$(24) \quad \sum_{\substack{m \neq n \\ \text{case}(a)}} \lambda_k(E_n \cap E_m) \ll \Lambda(Z)$$

if η is sufficiently small.

The other cases are similar and we have

$$(25) \quad \sum_{n \neq m} \lambda_k(E_n \cap E_m) \leq c \sum_n \lambda_k(E_n)$$

where c is a constant depending only on k . This completes the proof of the Theorem .

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Mots clefs: Diophantine approximation, k - dimensional, Lebesgues measure, Duffin and Schaeffer conjecture.

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