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An application of the bracketed number to the summation of a certain type of series

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An application of the bracketed number to the summation of a certain type of series;

By I. J. SCHWATT.

The author has found nowhere in mathematical literature any method which would enable him to find the summation of the series given in this paper. The principles and methods applied in the solution of the problem are believed to be new.

If in the given series

$$(1) \qquad S = \sum_{k=1}^{n} (-1)^{k-1} \cos k \frac{\pi}{g},$$

the first h terms are retained and the following v terms removed; again, the next h terms retained and the v following them removed; and if this processis carried throughout the series, to find the sum of the terms retained.

Let $n = m(h+v) + d$, where $m = \left[\frac{n}{h+v}\right]$, then the required sum may be represented thus

$$(2) \qquad S_1 = \sum_{k=1}^{mh+d} (-1)^{k-1+\left[\frac{k-1}{h}\right]v} \cos\left(k + \left[\frac{k-1}{h}\right]v\right)\frac{\pi}{g},$$

for $d < h$, if $d \geqq h$, then h takes the place of d.

Let $n' = mh + d$, $0 \leqq d < h$ and $h \overline{\gtrless} 1$, then (2) may be written

$$(3) \qquad S_1 = \sum_{k=1}^{h} (-1)^{k-1} \cos k \frac{\pi}{g} + (-1)^{\nu} \sum_{k=h+1}^{2h} (-1)^{k-1} \cos (k+\nu) \frac{\pi}{g}$$

$$+ (-1)^{2\nu} \sum_{k=2h+1}^{3h} (-1)^{k-1} \cos (k + 2\nu) \frac{\pi}{g} + \ldots$$

$$+ (-1)^{\overline{m-1}\nu} \sum_{k=\overline{m-1}h+1}^{mh} (-1)^{k-1} \cos\left(k + \overline{m-1}\nu\right) \frac{\pi}{g}$$

$$+ (-1)^{m\nu} \sum_{k=mh+1}^{mh+d} (-1)^{k-1} \cos (k + m\nu) \frac{\pi}{g},$$

$$(4) \qquad = \sum_{\alpha=0}^{m-1} (-1)^{\alpha\nu} \sum_{k=\alpha h+1}^{(\alpha+1)h} (-1)^{k-1} \cos (k + \alpha\nu) \frac{\pi}{g} + S_2,$$

where s_2 is the last single summation in (3).

Letting $K = \alpha h = K'$ in the double summation in (4) and $K - mh = K'$ in S_2, we have

$$(5) \qquad S_1 = \sum_{\alpha=0}^{m-1} (-1)^{\alpha(h+\nu)} \sum_{k=1}^{h} (-1)^{k-1} \cos\left(k + \alpha\overline{h+\nu}\right) \frac{\pi}{g}$$

$$+ (-1)^{m(h+\nu)} \sum_{k=0}^{d} (-1)^{k-1} \{ k + m(h+\nu) \} \frac{\pi}{g}.$$

On place of (4) we may write, letting $h + \nu = n$

$$S_1 = \sum_{k=1}^{h} (-1)^{k-1} \cos k \frac{\pi}{g} + \sum_{k=w+1}^{w+h} (-1)^{k-1} \cos k \frac{\pi}{g} + \ldots$$

$$+ \sum_{k=\overline{m-1}w+1}^{\overline{m-1}w+h} (-1)^{k-1} \cos k \frac{\pi}{g} + \sum_{k=mw+1}^{mw+d} (-1)^{k-1} \cos k \frac{\pi}{g},$$

$$= \sum_{\alpha=0}^{m-1} \sum_{k=\alpha w+1}^{\alpha w+h} (-1)^{k-1} \cos k \frac{\pi}{g} + S_2,$$

letting in the double summation $K - \alpha w = K'$ and $K - mw = K'$ in S_2, gives (5).

Now if let $q = 6$, then from (5)

$$(6)\quad S_1 = \sum_{\alpha=0}^{m-1} (-1)^{\alpha w} \cos \alpha w \frac{\pi}{6} \sum_{k=1}^{h} (-1)^{k-1} \cos k \frac{\pi}{6}$$
$$- \sum_{\alpha=0}^{m-1} (-1)^{\alpha w} \sin \alpha w \frac{\pi}{6} \sum_{k=1}^{k} (-1)^{k-1} \sin k \frac{\pi}{6}$$
$$+ (-1)^{mw} \cos mw \frac{\pi}{6} \sum_{k=1}^{d} (-1)^{k-1} \cos k \frac{\pi}{6}$$
$$- (-1)^{mw} \sin mw \frac{\pi}{6} \sum_{k=1}^{d} (-1)^{k-1} \sin k \frac{\pi}{6},$$

$$(7)\quad = \frac{1}{4 \sin w \frac{\pi}{6}} \Bigg[2(-1)^{w} \left(\frac{\sqrt{3}}{2} - 1 \right) + 2 \left\{ \cos \overline{w-1} \frac{\pi}{6} - \cos w \frac{\pi}{6} \right\}$$
$$+ 2(-1)^{h} \left\{ \cos \overline{w-h} \frac{\pi}{6} - \cos \overline{w-h-1} \frac{\pi}{6} \right\}$$
$$- 2(-1)^{h+w} \left\{ \cos (h+1) \frac{\pi}{6} - \cos h \frac{\pi}{6} \right\}$$
$$- A_1 \left\{ \cos (mw+1) \frac{\pi}{6} - \cos mw \frac{\pi}{6} \right\}$$
$$- A_2 \left\{ \cos (\overline{m-1}\, w+1) \frac{\pi}{6} - \cos \overline{m-1}\, w \frac{\pi}{6} \right\}$$
$$+ (-1)^{h} A_1 \left\{ \cos (mw+h+1) \frac{\pi}{6} - \cos (mw+h) \frac{\pi}{6} \right\}$$
$$+ (-1)^{h} A_2 \left\{ \cos (\overline{m-1}\, w+h+1) \frac{\pi}{6} - \cos (\overline{m-1}\, w+h) \frac{\pi}{6} \right\} \Bigg]$$
$$+ \Bigg[(-1)^{n''} \left\{ \sin n'' \frac{\pi}{6} - \sin (n''+1) \frac{\pi}{6} \right\}$$
$$- (-1)^{mw} \left\{ \sin mw \frac{\pi}{6} - \sin (mw+1) \frac{\pi}{6} \right\} \Bigg],$$

where

$$A_1 = \{ 1 - (-1)^m + (-1)^w + (-1)^{m+w} \},$$
$$A_2 = \{ 1 + (-1)^m + (-1)^w + (-1)^{m+w} \}$$

and

$$n'' = mw + d.$$

Then

$$(8)\quad S_1 = \sum_{\alpha=0}^{m-1} \cos\alpha w\frac{\pi}{6} \sum_{k=1}^{h} (-1)^{k-1} \cos k\frac{\pi}{6} - \sum_{\alpha=0}^{m-1} \sin\alpha w\frac{\pi}{6} \sum_{k=1}^{h} (-1)^{k-1} \sin k\frac{\pi}{6}$$
$$+ \cos mw\frac{\pi}{6} \sum_{k=1}^{d} (-1)^{k-1} \cos k\frac{\pi}{6} - \sin mw\frac{\pi}{6} \sum_{k=1}^{d} (-1)^{k-1} \sin k\frac{\pi}{6}$$

(if w is even),

$$(9)\quad = \frac{1}{2\sin w\frac{\pi}{6}} \Bigg[\left(\frac{\sqrt{3}}{2} - 1 \right) - \left\{ \cos w\frac{\pi}{6} - \cos \overline{w-1}\frac{\pi}{6} \right\}$$
$$+ (-1)^h \Big\{ \cos \overline{w-h}\frac{\pi}{6} - \cos \overline{w-h-1}\frac{\pi}{6}$$
$$- \cos(h+1)\frac{\pi}{6} + \cos h\frac{\pi}{6} \Big\}$$
$$- \left\{ \cos(mw+1)\frac{\pi}{6} - \cos mw\frac{\pi}{6} \right\}$$
$$- \left\{ \cos(\overline{m-1}+1)\frac{\pi}{6} - \cos \overline{m-1}\,w\frac{\pi}{6} \right\}$$
$$+ (-1)^h \left\{ \cos(mw+h+1)\frac{\pi}{6} - \cos(mw+h)\frac{\pi}{6} \right\}$$
$$+ (-1)^h \left\{ \cos(\overline{m-1}\,w+h+1)\frac{\pi}{6} - \cos(\overline{m-1}\,w+h)\frac{\pi}{6} \right\} \Bigg]$$
$$+ (-1)^{n''} \left\{ \sin n''\frac{\pi}{6} - \sin(n''+1)\frac{\pi}{6} \right\} - \left\{ \sin mw\frac{\pi}{6} - \sin(mw+1)\frac{\pi}{6} \right\}$$

(if w is even),

and

$$(10)\quad S_1 = \sum_{\alpha=0}^{m-1}(-1)^{\alpha}\cos\alpha w\frac{\pi}{6}\sum_{k=1}^{h}(-1)^{k-1}\cos k\frac{\pi}{6}$$

$$-\sum_{\alpha=0}^{m-1}(-1)^{\alpha}\sin\alpha w\frac{\pi}{6}\sum_{k=1}^{h}(-1)^{k-1}\sin k\frac{\pi}{6}$$

$$+(-1)^m\cos mw\frac{\pi}{6}\sum_{k=1}^{d}(-1)^{k-1}\cos k\frac{\pi}{6}-(-1)^m\sin mw\frac{\pi}{6}\sum_{k=1}^{d}(-1)^{k-1}\sin k\frac{\pi}{6}$$

(if w is odd),

$$(11)\quad = \frac{1}{2\sin w\frac{\pi}{6}}\Bigg[-\left(\frac{\sqrt{3}}{2}-1\right)-\left\{\cos w\frac{\pi}{6}-\cos\overline{w-1}\frac{\pi}{6}\right\}$$

$$+(-1)^h\Big\{\cos\overline{w-h}\frac{\pi}{6}-\cos\overline{w-h-1}\frac{\pi}{6}$$

$$+\cos(h+1)\frac{\pi}{6}-\cos h\frac{\pi}{6}\Big\}$$

$$+(-1)^m\left\{\cos(mw+1)\frac{\pi}{6}-\cos mw\frac{\pi}{6}\right\}$$

$$-(-1)^m\left\{\cos(\overline{m-1}\,w+1)\frac{\pi}{6}-\cos\overline{m-1}\,w\frac{\pi}{6}\right\}$$

$$-(-1)^{m+h}\left\{\cos(mw+h+1)\frac{\pi}{6}-\cos(mw+h)\frac{\pi}{6}\right\}$$

$$+(-1)^{h+m}\left\{\cos(\overline{m-1}\,w+h+1)\frac{\pi}{6}-\cos(\overline{m-1}\,w+h)\frac{\pi}{6}\right\}\Bigg]$$

$$-(-1)^{n''}\left\{\sin n''\frac{\pi}{6}-\sin(n''+1)\frac{\pi}{6}\right\}-(-1)^m\left\{\sin mw\frac{\pi}{6}-\sin(mw+1)\frac{\pi}{6}\right\}$$

(if w is odd);

(8) and (9) hold only when w is even, and (10) and (11) only time when w is odd.

The result (7) is obtained from (5) by the applications of the principles

$$(12)\quad \begin{cases} \displaystyle\sum_{k=1}^{h}(-1)^{k-1}f(k)=\sum_{k=0}^{\left[\frac{h-1}{2}\right]}f(2k+1)-\sum_{k=1}^{\left[\frac{h}{2}\right]}f(2k) \\ \text{and} \\ \displaystyle\sum_{\alpha=0}^{m-1}(-1)^{\alpha w}f(\alpha)=\sum_{\alpha=0}^{\left[\frac{m-1}{2}\right]}f(2\alpha)+(-1)^{w}\sum_{\alpha=0}^{\left[\frac{m-2}{2}\right]}f(2\alpha+1), \end{cases}$$

and by means of

$$(13)\quad \begin{cases} \displaystyle\sum_{k=0}^{h}\cos(a+k\theta)=\frac{\sin\frac{1}{2}\overline{h+1}\theta\cos\left(a+\frac{h}{2}\theta\right)}{\sin\frac{1}{2}\theta} \\ \text{and} \\ \displaystyle\sum_{k=0}^{h}\sin(a+k\theta)=\frac{\sin\frac{h+1}{2}\theta\,\sin\left(a+\frac{h}{2}\theta\right)}{\sin\frac{1}{2}\theta}, \end{cases}$$

$$(14)\quad \begin{cases} \displaystyle\sum_{k=1}^{h}\cos(a+k\theta)=\frac{\sin\frac{1}{2}h\theta\cos\left(a+\frac{h-1}{2}\theta\right)}{\sin\frac{1}{2}\theta} \\ \text{and} \\ \displaystyle\sum_{k=1}^{h}\sin(a+k\theta)=\frac{\sin\frac{h+1}{2}\theta\,\sin\left(a+\frac{h}{2}\theta\right)}{\sin\frac{1}{2}\theta}, \end{cases}$$

and successively letting w even and odd we obtain (9) and (11) respectively. Applying the same formulae to (6), (8) and (10) we arrive at the same results.

Now each of (7), (9) and (11) is composed of terms similar to

$$(15)\quad \begin{cases} S_3=(-1)^h\left\{\cos(h+1)\frac{\pi}{6}-\cos h\frac{\pi}{6}\right\} \\ \text{and} \\ S_4=(-1)^h\left\{\sin h\frac{\pi}{6}-\sin(h+1)\frac{\pi}{6}\right\}. \end{cases}$$

We shall next calculate the numerical values of these expressions, for which purpose we first find

$$(16) \qquad \cos h\frac{\pi}{6}, \qquad \sin h\frac{\pi}{6} \qquad \text{and} \qquad \operatorname{tang} h\frac{\pi}{6},$$

if $h = 6p$, then $\cos p\pi = (-1)^p = (-1)^{\frac{h}{6}} = (-1)^{\left[\frac{h+2}{6}\right]}$,

if $h = 6p+1$, then $\cos\left(p + \frac{1}{6}\right)\pi = \frac{\sqrt{3}}{2}(-1)^p = \frac{\sqrt{3}}{2}(-1)^{\frac{h-1}{6}} = \frac{\sqrt{3}}{2}(-1)^{\left[\frac{h+2}{6}\right]}$,

if $h = 6p+2$, then $\cos\left(p + \frac{1}{3}\right)\pi = \frac{1}{2}(-1)^p = \frac{1}{2}(-1)^{\frac{h-2}{6}} = \frac{1}{2}(-1)^{\left[\frac{h+2}{6}\right]}$,

if $h = 6p+3$, then $\cos\left(p + \frac{1}{2}\right)\pi = 0$,

if $h = 6p+4$, then $\cos\left(p + \frac{2}{3}\right)\pi = \frac{1}{2}(-1)^{p+1} = \frac{1}{2}(-1)^{\frac{h+2}{6}} = \frac{1}{2}(-1)^{\left[\frac{h-2}{6}\right]}$,

if $h = 6p+5$, then $\cos\left(p + \frac{5}{6}\right)\pi = \frac{\sqrt{3}}{2}(-1)^{p-1}$

$$= \frac{\sqrt{3}}{2}(-1)^{\left[\frac{h+1}{6}\right]} = \frac{\sqrt{3}}{2}(-1)^{\left[\frac{h+2}{6}\right]}.$$

Therefore

$$(17) \qquad \cos h\frac{\pi}{6} = \frac{(-1)^{\left[\frac{h+2}{6}\right]}}{4}\left\{3 + (-1)^{\left[\frac{h+1}{3}\right]}\right\} = \frac{1}{4}\left\{3(-1)^{\left[\frac{h+2}{6}\right]} + (-1)^{\left[\frac{h}{2}\right]}\right\}$$

(if h is even);

$$(18) \qquad = \frac{(-1)^{\left[\frac{h+2}{6}\right]}}{4}\sqrt{3}\left\{1 + (-1)^{\left[\frac{h+1}{3}\right]}\right\} = \frac{\sqrt{3}}{4}\left\{(-1)^{\left[\frac{h+2}{3}\right]} + (-1)^{\left[\frac{h+3}{8}\right]}\right\}$$

(if h is odd);

$$(19) \qquad = \frac{\sqrt{3}}{4}\left\{(-1)^{\left[\frac{h+2}{6}\right]} - (-1)^{\left[\frac{5h+2}{6}\right]}\right\}$$

$$+ \frac{3}{8}\left\{(-1)^{\left[\frac{h+2}{6}\right]} + (-1)^{\left[\frac{5h+3}{6}\right]}\right\} + \frac{1}{8}\left\{(-1)^{\left[\frac{h}{2}\right]} + (-1)^{\left[\frac{h+1}{2}\right]}\right\}$$

(whether h is even or odd).

Similarly,

$$(20)\qquad \sin h\frac{\pi}{6}=(-1)^h\frac{\sqrt{3}}{4}\left\{1-(-1)^{\left[\frac{h+1}{3}\right]}\right\}=\frac{\sqrt{3}}{4}\left\{(-1)^{\left[\frac{h}{2}\right]}-(-1)^{\left[\frac{h+4}{6}\right]}\right\}$$

(if h is even);

$$(21)\qquad =\frac{(-1)^h}{4}\left\{3-(-1)^{\left[\frac{h+1}{6}\right]}\right\}=\frac{1}{4}\left\{3(-1)^{\left[\frac{h}{6}\right]}-(-1)^{\left[\frac{h}{2}\right]}\right\}$$

(if h is odd);

$$(22)\qquad =\frac{\sqrt{3}}{4}\left\{(-1)^{\left[\frac{h}{6}\right]}-(-1)^{\left[\frac{5h}{6}\right]}\right\}$$

$$+\frac{3}{8}\left\{(-1)^{\left[\frac{h}{6}\right]}-(-1)^{\left[\frac{5h+5}{2}\right]}\right\}-\frac{1}{8}\left\{(-1)^{\left[\frac{h}{6}\right]}-(-1)^{\left[\frac{h+1}{2}\right]}\right\}$$

(whether h is even or odd),

and

$$(23)\qquad \operatorname{tang} h\frac{\pi}{2}=\frac{\sqrt{3}}{2}\left\{(-1)^{\left[\frac{h}{3}\right]}-(-1)^{\left[\frac{2h}{3}\right]}\right\}\qquad \text{(if } h \text{ is even);}$$

$$(24)\qquad =\frac{2}{\sqrt{3}\left\{(-1)^{\left[\frac{h}{3}\right]}+(-1)^{\left[\frac{2h}{3}\right]}\right\}}\qquad \text{(if } h \text{ is odd),}$$

$$(25)\qquad =\frac{\sqrt{3}}{4}\left\{(-1)^{\left[\frac{h}{3}\right]}-(-1)^{\left[\frac{h+2}{2}\right]}+(-1)^{\left[\frac{2h+2}{3}\right]}-(-1)^{\left[\frac{2h}{3}\right]}\right\}$$

$$+\frac{1}{\sqrt{3}}\,\frac{1-(-1)^h}{1+2(-1)^{\left[\frac{h}{3}\right]}+(-1)^{\left[\frac{2h+1}{3}\right]}}$$

(whether h is oven or odd).

We can now evaluate S_3 and S_4.
From (9)

$$S_3=(-1)^h\left\{\cos(h+1)\frac{\pi}{6}-\cos h\frac{\pi}{6}\right\}$$

$$=\frac{\sqrt{3}}{4}\left\{(-1)^{\left[\frac{5h+2}{6}\right]}+(-1)^{\left[\frac{h+2}{6}\right]}-(-1)^{\left[\frac{5h+3}{6}\right]}+(-1)^{\left[\frac{h+3}{6}\right]}\right\}$$

$$+\frac{3}{8}\left\{(-1)^{\left[\frac{5h+2}{6}\right]}-(-1)^{\left[\frac{h+2}{6}\right]}-(-1)^{\left[\frac{5h+3}{6}\right]}-(-1)^{\left[\frac{h+3}{6}\right]}\right\}+\frac{1}{4}(-1)^{\left[\frac{h+1}{2}\right]},$$

but

$$(-1)^{\left[\frac{5h+2}{6}\right]}-(-1)^{\left[\frac{5h+3}{6}\right]}+(-1)^{\left[\frac{h+3}{6}\right]}=\quad(-1)^{\left[\frac{h+4}{6}\right]}$$

and

$$(-1)^{\left[\frac{5h+2}{6}\right]}-(-1)^{\left[\frac{5h+3}{6}\right]}-(-1)^{\left[\frac{h+2}{6}\right]}=-(-1)^{\left[\frac{h+3}{6}\right]},$$

therefore

$$(26)\quad S_3=\frac{\sqrt{3}}{4}\left\{(-1)^{\left[\frac{h+2}{6}\right]}+(-1)^{\left[\frac{h+4}{6}\right]}\right\}-\frac{1}{4}\left\{3(-1)^{\left[\frac{h+3}{6}\right]}+(-1)^{\left[\frac{h+1}{2}\right]}\right\}.$$

Similarly

$$(27)\quad S_4=(-1)^h\left\{\sin h\frac{\pi}{6}-\sin(h+1)\frac{\pi}{6}\right\}$$

$$=\frac{\sqrt{3}}{4}\left\{(-1)^{\left[\frac{h+1}{6}\right]}-(-1)^{\left[\frac{h+5}{6}\right]}\right\}-\frac{1}{4}\left\{3(-1)^{\left[\frac{h}{6}\right]}-(-1)^{\left[\frac{h}{2}\right]}\right\}.$$

In the same way the following results are obtained,

$$(28)\quad S_5=\sum_{k=1}^{h}(-1)^{k-1}\cos k\frac{\pi}{6}$$

$$=\frac{1}{2}+(-1)^h\left\{\sin h\frac{\pi}{6}-\sin(h+1)\frac{\pi}{6}\right\}$$

$$=\frac{1}{2}+\frac{\sqrt{3}}{4}\left\{(-1)^{\left[\frac{h+1}{2}\right]}-(-1)^{\left[\frac{h+5}{2}\right]}\right\}-\frac{1}{4}\left\{(-1)^{\left[\frac{h}{6}\right]}-(-1)^{\left[\frac{h}{2}\right]}\right\};$$

$$(29)\quad S_6=\sum_{k=1}^{h}(-1)^{k-1}\sin k\frac{\pi}{6}$$

$$=1-\frac{\sqrt{3}}{2}+(-1)^h\left\{\cos(h+1)\frac{\pi}{6}-\cos h\frac{\pi}{6}\right\}$$

$$=1-\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{4}\left\{(-1)^{\left[\frac{h+2}{6}\right]}+(-1)^{\left[\frac{h+4}{6}\right]}\right\}-\frac{1}{4}\left\{(-1)^{\left[\frac{h+3}{6}\right]}+(-1)^{\left[\frac{h+1}{2}\right]}\right\};$$

$$(30)\quad S_7=\sum_{\alpha=0}^{m-1}(-1)^{\alpha w}\cos\alpha w\frac{\pi}{6}=\frac{1}{2}+\frac{1}{4}A_1\frac{\sin mw\frac{\pi}{6}}{\sin w\frac{\pi}{6}}+\frac{1}{4}A_2\frac{\sin(m-1)w\frac{\pi}{6}}{\sin w\frac{\pi}{6}},$$

where

$$A_1=\{1-(-1)^m+(-1)^w+(-1)^{m+w}\}$$

and

$$A_2 = \{ 1 + (+1)^m + (-1)^w + (-1)^{m+w} \}.$$

$$(31)\quad S_8 = \sum_{\alpha=0}^{m-1} (-1)^{\alpha w} \sin \alpha w \frac{\pi}{6}$$

$$= \frac{1}{2} \cot w \frac{\pi}{6} + \frac{(-1)^w}{2 \sin w \frac{\pi}{6}} - \frac{1}{4} A_1 \frac{\cos m w \frac{\pi}{6}}{\sin w \frac{\pi}{6}} - \frac{1}{4} A_2 \frac{\cos (m-1) w \frac{\pi}{6}}{\sin w \frac{\pi}{6}},$$

$$(32)\quad S_9 = \sum_{\alpha=0}^{m-1} \cos \alpha w \frac{\pi}{6} = \frac{1}{2} + \frac{1}{2} \frac{\sin m w \frac{\pi}{6} + \sin (m-1) w \frac{\pi}{6}}{\sin w \frac{\pi}{6}}$$

(if w is even);

$$(33)\quad = \frac{1}{2} + \frac{1}{8} \left\{ (-1)^{\left[\frac{mw}{6}\right]} - (-1)^{\left[\frac{mw+4}{6}\right]} \right\}$$

$$\times \frac{4 + 3(-1)^{\left[\frac{w+2}{6}\right]} + (-1)^{\left[\frac{w}{2}\right]}}{(-1)^{\left[\frac{w}{6}\right]} - (-1)^{\left[\frac{w+4}{6}\right]}} - \frac{1}{8} \left\{ 3(-1)^{\left[\frac{mw+2}{6}\right]} + (-1)^{\left[\frac{mw}{2}\right]} \right\}$$

(if w is even);

$$(34)\quad S_{10} = \sum_{\alpha=0}^{m-1} \sin \alpha w \frac{\pi}{6} = \frac{1}{2 \sin w \frac{\pi}{6}} + \frac{1}{2} \cot w \frac{\pi}{6} - \frac{1}{2} \frac{\cos m w \frac{\pi}{6} + \cos \overline{m-1}\, w \frac{\pi}{6}}{\sin w \frac{\pi}{6}}$$

(if w is even);

$$(35)\quad = \frac{1}{8\sqrt{3}} \left\{ 4 - 3(-1)^{\left[\frac{mw+2}{6}\right]} - (-1)^{mw} \right\}$$

$$\times \frac{4 + 3(-1)^{\left[\frac{w+2}{6}\right]} + (-1)^{\left[\frac{w}{2}\right]}}{(-1)^{\left[\frac{w}{6}\right]} - (-1)^{\left[\frac{w+4}{6}\right]}} - \frac{\sqrt{3}}{8} \left\{ (-1)^{\left[\frac{mw}{6}\right]} - (-1)^{\left[\frac{mw+4}{6}\right]} \right\}$$

(if w is even);

$$(36)\quad S_{11} = \sum_{\alpha=0}^{m-1} (-1)^{\alpha} \cos \alpha w \frac{\pi}{6} = \frac{1}{2} - \frac{1}{2} (-1)^m \left\{ \frac{\sin m w \frac{\pi}{6} - \sin \overline{m-1}\, w \frac{\pi}{6}}{\sin w \frac{\pi}{6}} \right\}.$$

(if w is odd);

$$(37)\qquad = \frac{1}{2} - \left\{ \frac{(-1)^{\left[\frac{mw}{6}\right]+m}}{4\sqrt{2}} \sqrt{4 - 3(-1)^{\left[\frac{mw+1}{3}\right]} - (-1)^{mw}} \right\}$$

$$\times \frac{4 - \sqrt{3}\left\{(-1)^{\left[\frac{w+2}{6}\right]} + (-1)^{\left[\frac{w+3}{6}\right]}\right\}}{3(-1)^{\left[\frac{w}{6}\right]} - (-1)^{\left[\frac{w}{2}\right]}}$$

$$- \frac{(-1)^{\left[\frac{mw+2}{6}\right]+m}}{4\sqrt{2}} \sqrt{4 + 3(-1)^{\left[\frac{m+1}{3}\right]} + (-1)^{mw}}$$

(if w is odd),

$$(38)\qquad = \frac{1}{2} - \frac{\sqrt{3}}{8}\left\{(-1)^{\left[\frac{mw}{6}\right]} + (-1)^{\left[\frac{mw+4}{6}\right]}\right\}$$

$$\times \frac{4 - \sqrt{3}\left\{(-1)^{\left[\frac{w+2}{6}\right]} + (-1)^{\left[\frac{w+3}{6}\right]}\right\}}{3(-1)^{\left[\frac{w}{2}\right]} - (-1)^{\left[\frac{w}{2}\right]}} - \frac{1}{8}\left\{3(-1)^{\left[\frac{mw+2}{6}\right]} + (-1)^{\left[\frac{mw}{2}\right]}\right\}$$

(if w is odd and m is even),

$$(39)\qquad = \frac{1}{2} + \frac{1}{8}\left\{3(-1)^{\left[\frac{mw}{6}\right]} - (-1)^{\left[\frac{mw}{2}\right]}\right\}$$

$$\times \frac{4 - \sqrt{3}\left\{(-1)^{\left[\frac{w+2}{6}\right]} + (-1)^{\left[\frac{w+3}{6}\right]}\right\}}{3(-1)^{\left[\frac{w}{6}\right]} - (-1)^{\left[\frac{w}{2}\right]}} + \frac{\sqrt{3}}{8}\left\{(-1)^{\left[\frac{mw+2}{6}\right]} + (-1)^{\left[\frac{mw+3}{6}\right]}\right\}$$

(if w is odd and also m odd);

$$(40)\qquad S_{12} = \sum_{\alpha=0}^{m-1} (-1)^{\alpha} \sin \alpha w \frac{\pi}{6}$$

$$= \frac{1}{2}\cot w\frac{\pi}{6} - \frac{1}{2\sin w\frac{\pi}{6}} + \frac{1}{2}(-1)^{m}\left\{\frac{\cos mw\frac{\pi}{6} - \cos \overline{m-1}\, w\frac{\pi}{6}}{\sin w\frac{\pi}{6}}\right\}$$

(if w is odd);

$$(41)\qquad = \frac{1}{4\sqrt{2}}\left[2\sqrt{2} - \left\{(-1)^{\left[\frac{mw+2}{6}\right]+m}\sqrt{4 + 3(-1)^{\left[\frac{mw+1}{3}\right]} + (-1)^{mw}}\right\}\right]$$

$$\times \frac{\sqrt{3}\left\{(-1)^{\left[\frac{w+2}{6}\right]} + (-1)^{\left[\frac{w+2}{6}\right]}\right\} - 4}{3(-1)^{\left[\frac{w}{6}\right]} - (-1)^{\left[\frac{w}{2}\right]}}$$

$$- \frac{1}{4\sqrt{2}}(-1)^{\left[\frac{mw}{6}\right]+m}\sqrt{4 - 3(-1)^{\left[\frac{mw+1}{3}\right]} - (-1)^{mw}}$$

(if w is odd);

$$(42)\qquad =\frac{1}{8}\left\{4-3(-1)^{\left[\frac{mw+2}{6}\right]}-(-1)^{\left[\frac{mw}{2}\right]}\right\}$$

$$\times\frac{\sqrt{3}\left\{(-1)^{\left[\frac{w+2}{6}\right]}+(-1)^{\left[\frac{w+3}{6}\right]}\right\}-4}{3(-1)^{\left[\frac{w}{6}\right]}-(-1)^{\left[\frac{w}{2}\right]}}-\frac{\sqrt{3}}{8}\left\{(-1)^{\left[\frac{mw}{6}\right]}-(-1)^{\left[\frac{mw+4}{6}\right]}\right\}$$

(if w is odd and m even);

$$(43)\qquad =\frac{1}{8}\left[4+\sqrt{3}\left\{(-1)^{\left[\frac{mw+2}{6}\right]}+(-1)^{\left[\frac{mw+3}{6}\right]}\right\}\right]$$

$$\times\frac{\sqrt{3}\left\{(-1)^{\left[\frac{w+2}{6}\right]}+(-1)^{\left[\frac{w+3}{6}\right]}\right\}-4}{3(-1)^{\left[\frac{w}{6}\right]}+(-1)^{\left[\frac{w}{2}\right]}}+\frac{1}{8}\left\{3(-1)^{\left[\frac{mw}{6}\right]}-(-1)^{\left[\frac{mw}{2}\right]}\right\}$$

(if both w and m are odd);

$$(44)\quad S_{13}=\sum_{k=1}^{d}(-1)^{k-1}\cos k\frac{\pi}{6}=\frac{1}{2}+\frac{\sqrt{3}}{3}\left\{(-1)^{\left[\frac{n''-mw+1}{6}\right]}-(-1)^{\left[\frac{n''-mw+5}{6}\right]}\right\}$$

$$-\frac{1}{4}\left\{(-1)^{\left[\frac{n''-mw}{6}\right]}-(-1)^{\left[\frac{n''-mw}{2}\right]}\right\};$$

$$(45)\quad S_{14}=\sum_{k=1}^{d}(-1)^{k-1}\sin k\frac{\pi}{6}=1-\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{4}\left\{(-1)^{\left[\frac{n''-mw+2}{6}\right]}+(-1)^{\left[\frac{n''-mw+4}{6}\right]}\right\}$$

$$+\frac{1}{4}\left\{(-1)^{\left[\frac{n''-mw+3}{6}\right]}+(-1)^{\left[\frac{n''-mw+1}{2}\right]}\right\}.$$

In this way the desired result is obtained.