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On drawings composed of uniform straight lines

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On drawings composed of uniform straight lines;

By George D. BIRKHOFF.

1. The Problem. — Suppose that one traces on a flat sheet of white paper a large number of indefinitely extended straight lines. These lines are supposed to be of the same constant width and intensity throughout. If it be assumed that the surface density of lead (or ink) thereby deposited is additive, there arises (in the limiting case of a smooth distribution of infinitely many lines) a certain *density function* F of position such that the quantity of lead in any infinitesimal area dA is given by $F\,dA$ where F is evaluated at some point of this area. Similarly there is a *distribution function* f dépendent not only on position but also on the angular coordinate φ of the line considered, such that $f\,d\varphi\,dA$ yields the amount of lead deposited in the same small area dA, by lines l whose direction lies between φ and $\varphi + d\varphi$. In this way one obtains the fundamental relation

$$F = \int_0^{2\pi} f\,d\varphi$$

between the density function F and the distribution function f, which holds at any point of the plane.

In what follows we shall choose some point O of the plane, as origin of polar coordinates, and consider only continuous functions F and f within a circle of radius $0 < R \leqq +\infty$ having O as center. If we adopt the coordinates indicated in the adjoining figure it is clear that the density function at P may be written $F(r, \theta)$ with

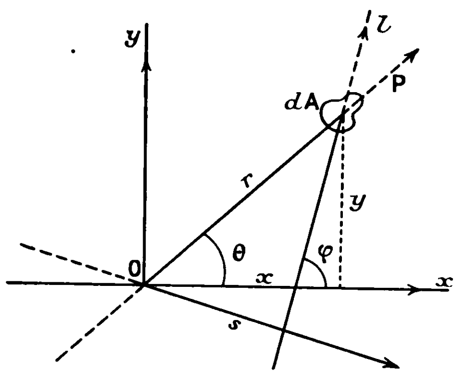
$$F(-r, \theta + \pi) = F(r, \theta),$$

while the distribution function may be written $f(s, \varphi)$ where

$$s = r \sin(\varphi - \theta).$$

Furthermore if we make the convention that the distribution function has the same numerical value in whichever sense a line l is taken, we

Fig. 1.



have similarly $f(-s, \varphi + \pi) = f(s, \varphi)$. With these notations the fundamental linear integral equation above takes the more explicit form

$$\text{(1)} \qquad F(r, \theta) = \int_0^{2\pi} f[r \sin(\varphi - \theta), \varphi] d\varphi.$$

The direct problem, namely, given the distribution function $f(s, \varphi)$, to obtain the density function $F(r, \theta)$, is completely solved by the equation just written. It is the inverse problem of determining $f(s, \varphi)$ when $F(r, \theta)$ is given to which we shall devote attention. Interpreted for the application to rectilinear drawings, this is the question of ascertaining whether or not a given drawing is possible by such rectilinear means; and, if so, of détermining just how the drawing is composed of the constituent lines.

Perhaps the simplest case is that afforded by a family of parallel straight lines, with $F = F^*[r \sin(\varphi_0 - \theta)]$ where φ_0 is the angle which these straight lines make with a fixed direction, and F is an arbitrary positive continuous function of its argument. This function may also be written

$$F^*(x\cos\varphi_0 - y\sin\varphi_0),$$

that is as an arbitrary function of a general homogeneous linear expression in x and y. Here we evidently require a *singular* distribution function f which vanishes for $\varphi \neq \varphi_0$. Bearing this special type of solution in mind we easily obtain *formal* solutions of the inverse problem by means of familiar identities such as the two following.

$$(2)\quad p(x,y) = \frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F(t,w)\cos[u(t-x)+v(w-y)]\,dt\,dw\,du\,dv,$$

$$(2')\quad p(x,y) = \sum_{m,n=0}^{\infty} \frac{1}{m!\,n!}\,\frac{d^{m+n}p(0,0)}{dx^m\,dy^n}\left[\frac{d^{m+n}}{da^m\,db^n}(e^{ax+by})_{x=y=0}\right].$$

For, the terms on the right formally express an arbitrary $p(x, y)$ as the limit of sums or differences of functions of a linear expression of rectangular coordinates x and y, such as arise in the special case just referred to. Indeed the first of these is the familiar Fourier integral identity for $p(x, y)$ and the second yields the Taylor's expansion of p in a power series in x and y. It is difficult to see, however, how such forms do more than show how an approximation to the desired drawing may be accomplished by means of an extremely complicated set of straight lines and rectilinear erasures. They do not show in the least how to solve the problem *without any erasures*. Furthermore they do not tell how to determine the distribution function over a circle when the density function is merely given over the same circle. It is these more specific problems which are discussed here.

It is obvious that the general type of problem under consideration admits of generalization in various directions, as for instance to more dimensions than two, and to other types of geometric figures than straight lines.

In our approach we will assume that a certain further *hypothesis* (H) be satisfied, namely that $F(r, \theta)$ and $f(s, \varphi)$ are asymptotically representable in a power series in the corresponding rectangular coordinates

$$F(r,\theta) \sim a_{00} + (a_{10}x + a_{01}y) + \frac{1}{2}(a_{20}x^2 + 2a_{11}xy + a_{02}y^2) + \dots$$

$$f(s,\varphi) \sim b_{00} + (b_{10}x + b_{01}y) + \frac{1}{2}(b_{20}x^2 + 2b_{11}xy + b_{02}y^2) + \dots$$

in the neighborhood of the origin. When this condition is satisfied we may write for instance

$$F(r, \theta) = a_{00} + r(a_{10}\cos\theta + a_{10}\sin\theta) + \dots$$
$$+ r^{m-1}(a_{m-10}\cos^{m-1}\theta + \dots + a_{0m-1}\sin^{m-1}\theta) + r^m Q_m(r, \theta)$$

where $[Q_m(r, \theta)] \leqq Q_m$ a fixed constant, for r small. Hence we find that

$$\int_0^{2\pi} F(r, \theta)\cos m\theta\, d\theta = r^m \int_0^{2\pi} Q_m(r, \theta)\cos m\theta\, d\theta,$$
$$\int_0^{2\pi} F(r, \theta)\sin m\theta\, d\theta = r^m \int_0^{2\pi} Q_m(r, \theta)\sin m\theta\, d\theta.$$

Consequently the coefficients $F_m(r)$ and $G_m(s)$ of $\cos m\theta$ and $\sin m\theta$ in the Fourier series for $F(r, \theta)$ will vanish to at least the m-th order with r; the similar coefficients $f_m(s)$ and $g_m(s)$ for $f(s, \varphi)$ must likewise vanish to the m-th order in s. Furthermore it is apparent from (1) that if $f(s, \varphi)$ satisfies the condition (H) so must $F(r, \theta)$ also.

2. Reduction of the problem. — Evidently both F and f, being by hypothesis continuous, admit of unique formal expansion in Fourier's series in θ and φ respectively :

$$(3) \quad \begin{cases} F(r, \theta) = \frac{1}{2}F_0(r) + \sum_{m=1}^{\infty}[F_m(r)\cos m\theta + G_m(r)\sin m\theta], \\ f(s, \varphi) = \frac{1}{2}f_0(s) + \sum_{m=1}^{\infty}[f_m(r)\cos m\varphi + g_m(r)\sin m\varphi] \end{cases}$$

where $F_m(r)$ and $G_m(r)$ are even or odd in r according as m is even or odd, and $f_m(s)$ and $g_m(s)$ are likewise even or odd in s according as m is even or odd. Conversely, given these Fourier coefficients of such continuous functions it is of course easy to obtain the corresponding functions in explicit terms.

Now we can at once deduce certain necessary relations betwen $F_m(r)$, $G_m(r)$ and $f_m(s)$, $g_m(s)$ in the following manner. The well known explicit values of these coefficients are

$$(4)\quad \begin{cases} F_m(r) = \frac{1}{\pi}\int_0^{2\pi} F(r,\theta)\cos m\theta\, d\theta, & G_m(r) = \frac{1}{\pi}\int_0^{2\pi} G_m(r,\theta)\sin m\theta\, d\theta, \\ f_m(r) = \frac{1}{\pi}\int_0^{2\pi} f(s,\varphi)\cos m\varphi\, d\varphi, & g_m(r) = \frac{1}{\pi}\int_0^{2\pi} g_m(s,\varphi)\sin m\varphi\, d\varphi, \end{cases}$$

where we write in particular $G_0(r) = g_0(s) = 0$. Consequently if we multiply the equation (1) through by $\cos m\theta$ and by $\sin m\theta$ for $m = 0, 1, \ldots,$ and integrate from 0 to 2π, there results the two sets of equations

$$(5)\quad \begin{cases} F_m(r) = \frac{1}{\pi}\int_0^{2\pi}\left\{\int_0^{2\pi} f[r\sin(\varphi-\theta),\varphi]\, d\varphi\right\}\cos m\theta\, d\theta, \\ G_m(r) = \frac{1}{\pi}\int_0^{2\pi}\left\{\int_0^{2\pi} f[r\sin(\varphi-\theta),\varphi]\, d\varphi\right\}\sin m\theta\, d\theta \end{cases}$$

which may be replaced by the single equivalent set :

$$(5')\qquad H_m(r) = \frac{1}{\pi}\int_0^{2\pi}\left\{\int_0^{2\pi} f[r\sin(\varphi-\theta),\varphi]\, d\varphi\right\} e^{im\theta} d\theta$$

where we write

$$(6)\qquad H_m(r) = F_m(r) + i\, G_m(r), \qquad h_m(s) = f_m(s) + i\, g_m(s).$$

Conversely if (5) or (5′) holds for $m = 0, 1, 2, \ldots,$ the Fourier series for the left-and right-hand members of (1) are identical, and therefore the equation (1) will hold. Thus we may replace the equation (1) by the sequence (5′) of simpler type.

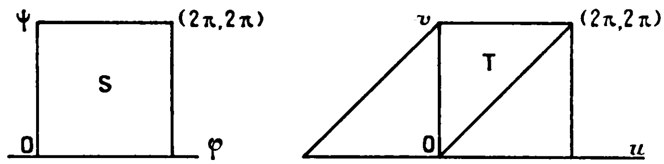
These équations (5′) may be further simplified. Replace the variables φ, θ by u, v where

$$(7)\qquad \varphi = v, \qquad \theta = v - u,$$

so that $\frac{\partial(\theta,\varphi)}{\partial(u,v)} = 1$. The square region S of integration appearing in (5′) is then transformed into the parallelogram T of the u, v plane bounded by $v = 0$, $v = 2\pi$, $v - u = 0$, $v - u = 2\pi$ (see the adjoining figure). But inasmuch as $f(r\sin u, v)$ is periodic of period 2π in u it is clear that the integral in the transformed variables has the same value over T as over the square T′ bounded by $u = 0$, $u = 2\pi$, $v = 0$, $v = 2\pi$. Hence equations (5′) may be rewritten in the form

$$(5'') \qquad H_m(s) = \frac{1}{\pi}\int_0^{2\pi}\left[\int_0^{2\pi} f(r\sin u,\, v)\, e^{im(v-u)}\, du\right] dv.$$

Fig. 2.



Integrating first as to v and recalling the formulas for $h_m(s)$ we obtain

$$(8) \qquad H_m(s) = \int_0^{2\pi} h_m(r\sin u)\, e^{-imu}\, du.$$

Here the functions $H_m(r)$ appear as given continuous function of (r), and the functions $h_m(s)$ are to be determined.

Thus we are led to the following preliminary result :

In order that there exists a continuous distribution function $f(s, \varphi)$ *corresponding to a continuous density function* $F(r, \theta)$ *it is necessary and sufficient that : (a) the linear integral equations* (8) *admit of continuous solutions* $h_m(s)$ *for* $m = 0, 1$ *with* $H_m(r) = F_m(r) + iG_m(r)$ *where* $F_m(r)$ *and* $G_m(r)$ *are the Fourier coefficients of* $F(r, \theta)$ *as shown in* (3); *(b) these functions* $h_m(s) = f_m(s) + ig_m(s)$ *must correspond to the Fourier series of a continuous function* $f(s, \varphi)$ *with coefficients* $f_m(s)$ *and* $g_m(s)$ *as shown in* (3); *(c) the function* $f(s, \varphi)$ *must be positive or zero for* $r < R$ (1).

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(1) There is no essential restriction in taking $f(s, \varphi)$ such that

$$f(-s, \varphi + \pi) = f(s, \varphi)$$

since we can always write

$$f(s, \varphi) = \frac{1}{2}\left[f(s, \varphi) + f(-s, \varphi + \pi)\right].$$

The hypothesis (H) is not made in this first result.

3. THE CASE OF CIRCULAR SYMMETRY. — When $F(r, \theta)$ depends only on r, we have the case of circular symmetry. Clearly if $f(s, \varphi)$ depends only on s, then this symmetric case will always arise. However, even if a non-symmetric $f(s, \varphi)$ could give rise to a symmetric $F(r)$, we see at once that

$$f(s) = \frac{1}{2\pi}\int_0^{2\pi} f(s, \varphi)\, d\varphi.$$

would yield a symmetric $f(s)$ leading to the same $F(r)$. Hence there is no limitation in restricting $f(s, \varphi)$ to be of symmetric type also.

But under the circumstances the Fourier series of F and f reduce to the first terms only and the theorem just stated leads to the conclusion that in the case of circular symmetry there will exist a corresponding distribution function $f(s)$ for the given density $F(r)$ if and only if the integral equation

$$F(r) = \int_0^{2\pi} f(r \sin u)\, du, \tag{9}$$

admits a non-negative continuous solution $f(s)$. But here we may take $F(r)$ and $f(s)$ to be even functions of r and s respectively and restrict attention to $r > 0$, $s > 0$. The integral on the right is then four times the same integral taken between the limits 0 and $\frac{\pi}{2}$. Hence if we employ the variable $r \sin u = s$, the equation (9) becomes

$$F(r) = 4\int_0^{\frac{\pi}{2}} \frac{f(s)\, ds}{\sqrt{r^2 - s^2}}. \tag{9'}$$

This is essentially an integral equation first treated by Abel (1828) with unique continuous solution given by

$$f(s) = \frac{1}{2\pi}\frac{d}{ds}\left[\int_0^s \frac{rF(r)\, dr}{\sqrt{s^2 - r^2}}\right], \tag{10}$$

provided that the integral on the right, which is a continous function of s necessarily vanishing for $s = 0$, admits a continuous derivative for $r \geqq 0$. There is no continuous solution when these conditions are not fulfilled ([1]).

([1]) See, for example, BÔCHER, *Introduction to the Study of Linear Integral Equations*, 1909, pp. 6-11.

In the case of circular symmetry, therefore, a given continuous density function $F(r)$ *admits of a corresponding distribution function* $f(s)$ *if and only if the integral appearing in* (10) *revresents a continuous function of s with continuous non-negative derivative, in which case the unique corresponding* $f(s)$ *is that specified by the same equation.*

It is interesting to remark upon one particular discontinuous case, namely

$$F(r) = \begin{cases} 0 & (r < r_0), \\ 1 & (r > r_0). \end{cases}$$

Here we have

$$f(s) = \begin{cases} 0 & (r < r_0), \\ \dfrac{1}{2\pi} \dfrac{s}{\sqrt{s^2 - r_0^2}} & (r > r_0), \end{cases}$$

since for $s > r_0$ we have

$$f(s) = \frac{1}{2\pi} \frac{d}{ds} \int_{r_0}^{s} \frac{r\,dr}{\sqrt{s^2 - r^2}} = \frac{1}{2\pi} \frac{s}{\sqrt{s^2 - r_0^2}} \geqq 0.$$

This shows that we may shade the plane uniformly except over a circle $r \leqq r_0$ by the means allowed; and that for this case the distribution function must be proportional to the secant of half the angle subtended from the point $(s, 0)$ by the given circle.

4. THE GENERAL CASE. FIRST SOLUTION. — Turning now to the non-symmetric case we observe first that for $m = 0$ we have

$$H_0(r) = \int_0^{2\pi} h_0(r \sin u)\,du = 4 \int_0^{\frac{\pi}{2}} \frac{h_0(s)\,ds}{\sqrt{r^2 - s^2}},$$

so that, as in the case of circular symmetry, we have

$$h_0(s) = \frac{1}{2\pi} \frac{d}{ds} \left[\int_0^{r} \frac{H_0(r)\,r\,dr}{s^2 - r^2} \right].$$

More generally, for $m > 0$ we may rewrite (8) in the form

$$H_m(r) = \int_0^{2\pi} h_m(r \sin u)(\cos u - i \sin u)e^{-i(m-1)u}\,du.$$

But, if we introduce the function

$$h_m^{(-1)}(s) = \int_0^s h_m(s)\,ds,$$

we obtain at once

$$h_m(r\sin u)\cos u = \frac{1}{r}\frac{d}{du} h_m^{(-1)}(r\sin u),$$

$$h_m(r\sin u)\sin u = \frac{d}{dr} h_m^{(-1)}(r\sin u),$$

and thus the relation (8) is seen to be essentially the same as

$$\mathrm{H}_m(r) = \int_0^{2\pi}\left[\frac{1}{r}\frac{d}{dr} h_m^{(-1)}(r\sin u) - i\frac{d}{dr} h_m^{(-1)}(r\sin u)\right] e^{-i(m-1)u}\,du.$$

Integrating the first term under the integral sign here by parts and observing that $h_m^{(-1)}(r\sin u)$ is periodic of period 2π in u we obtain as the equivalent of (8)

$$\mathrm{H}_m(r) = \frac{i(m-1)}{r}\int_0^{2\pi} h_m^{(-1)}(r\sin u)e^{-i(m-1)u}\,du$$
$$- i\frac{d}{dr}\int_0^{2\pi} h_m^{(-1)}(r\sin u)e^{-i(m-1)u}\,du.$$

Multiplying through by ir^{-m+1} this takes the form

$$\frac{i\mathrm{H}_m(r)}{r^{m-1}} = \frac{d}{dr}\left[\frac{1}{r^{m-1}}\int_0^{2\pi} h_m^{(-1)}(r\sin u)\right] e^{-i(m-1)u}\,du.$$

Integrating from 0 to r as is possible because of the hypothesis (H) we obtain immediately the formula

$$(8') \qquad ir^{m-1}\int_0^r \frac{\mathrm{H}_m(t)}{t^{m-1}}\,dt = \int_0^{2\pi} h_m^{(-1)}(r\sin u)e^{-i(m-1)u}\,du.$$

We note that the constant of integration must be disposed of as indicated since otherwise the right-hand member would vanish to the order $m+1$ in r while the left-hand member would vanish only to the order $m-1$.

But (8′) is essentially of the form (8) with m replaced by $m-1$, and we may repeat again the type of integration just performed. Thus by successive steps, m in number, we are led finally to

$$(11)\quad i^m \int_0^r \left[\int_0^{t_m} t_{m-1}\cdots\left(\int_0^{t_1} \frac{H_m(t_1)}{t_1^{m-1}}\,dt_1\right)\cdots\right] dt_m = \int_0^{2\pi} h_m^{(-m)}(r\sin u).$$

If we replace the m-fold integral on the left by a simple integral (1) we may rewrite this equation as

$$(11')\quad \frac{2\left(\frac{i}{2}\right)^m}{(m-1)!}\int_0^r \frac{H_m(t)}{t^{m-1}}(r^2-t^2)^{m-1}\,dt = \int_0^{2\pi} h_m^{(-m)}(r\sin u)\,du.$$

But $h_m^{(-m)}(r\sin u)$ is clearly an even function of r, so that we may assume $r \geqq 0$ in (11′) and replace this integral by four times the integral with modified limits of integration o and $\frac{\pi}{2}$. On rewriting $s = r\sin u$ on the right, this member therefore becomes

$$4\int_\delta^r h_m^{(-m)}(s)\frac{ds}{\sqrt{r^2-s^2}},$$

so that (11) is to be regarded as an integral equation for $h_m^{(-m)}(s)$ of Abel type. Solving, we obtain at once

$$(12)\quad h_m(s) = \frac{1}{\pi}\frac{\left(\frac{i}{2}\right)^m}{(m-1)!}\frac{d^{m+1}}{ds^{m+1}}\left[\int_0^s r\left(\int_0^r \frac{H_m(t)}{t^{m-1}}(r^2-t^2)^{m-1}\,dt\right)\frac{dr}{\sqrt{s^2-r^2}}\right].$$

Conversely if $h_m(s)$ given by (12) exists and is continuous, we find by retracing steps that the corresponding equation (8) holds.

Now the equation (12) admits of a simplifying transformation. In the first place it is readily seen that the double integral on the right

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(1) Note that the integral in (11) may be written

$$\frac{1}{2^m}\int_0^{r^2}\left[\int_0^{\tau_m}\cdots\int_0^{\tau_1}\left(\frac{H_m(\sqrt{\tau_1})}{\tau_1^{\frac{m}{2}}}\,d\tau_1\right)\cdots\right]d\tau_m$$
$$= \frac{1}{2^m}\frac{1}{(m-1)!}\int_0^{r^2}\frac{H_m(\sqrt{\tau})}{\tau^{\frac{m}{2}}}(r^2-\tau)^{m-1}\,d\tau,$$

where we set $\tau_i = t_i^2$ for $i = 1, \ldots, m$.

may be expressed in inverse order as

$$\int_\delta^s \frac{H_m(t)}{t^{m-1}}\left[\int_t^s \frac{(r^2-t^2)^{m-1}}{\sqrt{s^2-r^2}}\, r\, dr\right] dt.$$

Setting $r^2 = t^2 + (s^2 - t^2)\sin^2\sigma$ and using the familiar value of $\int_0^{\frac{\pi}{2}} \sin^{2n}\sigma\, d\sigma$ this becomes

$$\frac{2.4\ldots 2m-2}{1.3\ldots 2m-1}\int_0 \frac{H_m(t)}{t^{m-1}}(s^2-t^2)^{m-\frac{1}{2}}\, dt.$$

Thus we have

$$h_m(s) = \frac{1}{2\pi}\,\frac{i^m}{1.3\ldots 2m-1}\,\frac{d^{m+1}}{ds^{m+1}}\left[\int_0^s \frac{H_m(t)}{t^{m-1}}(s^2-t^2)^{m-\frac{1}{2}}\, dt\right].$$

But since the integrand on the right and its first $m-1$ derivatives as to s vanish for $t=s$, we may differentiate m times as to s *under the integral sign*. However if we write $s = tu$, we see that

$$\frac{d^m}{ds^m}(s^2-t^2)^{m-\frac{1}{2}} = t^{m-1}\,\frac{d^m(u^2-1)^{m-\frac{1}{2}}}{du^m},$$

so that we may write the preceding equation for $h_m(s)$ in the essentially equivalent form

$$(12')\quad h_m(s) = \frac{1}{2\pi}\,\frac{i^m}{1.3\ldots 2m-1}\,\frac{d}{ds}\int_0^s H_m(t)\frac{d^m}{du^m}(u^2-1)^{m-\frac{1}{2}}\, dt,\ u = \frac{s}{t} \geqq 1.$$

Now let us establish that for all integral m, we have (¹)

$$(13)\quad \varphi_m(u) = \frac{d^{m-1}}{du^{m-1}}(u^2-1)^{m-\frac{1}{2}} = \frac{1.3\ldots 2m-1}{2m}\left[(u+\sqrt{u^2-1})^m - (u-\sqrt{u^2-1})^m\right].$$

It is immediately seen that $\varphi_m(1) = 0$ for all positive integral m, and that the stated formula holds for $m = 1, 2$. Hence it suffices to show that if the formula holds out to a certain m, then we have

$$\varphi'_{m+1}(u) = \frac{1.3\ldots 2m+1}{2}\left[\frac{(u+\sqrt{u^2-1})^{m+1} + (u-\sqrt{u^2-1})^{m+1}}{\sqrt{u^2-1}}\right]$$

as it must be if the above formula (13) is to hold.

(¹) This formula is doubtless a well known one.

But we have

$$\varphi'_{m+1}(u) = \frac{d^{m+1}}{du^{m+1}}(u^2-1)^{m+\frac{1}{2}} = (2m+1)\frac{d^m}{du^m}\left[u(u^2-1)^{m-\frac{1}{2}}\right]$$
$$= (2m+1)[u\varphi'_m(u) + m\varphi_m(u)]$$
$$= \frac{1.3\ldots 2m+1}{2}\left[u\frac{(u+\sqrt{u^2-1})^m+(u-\sqrt{u^2-1})^m}{\sqrt{u^2-1}} + (u+\sqrt{u^2-1})^m - (u-\sqrt{u^2-1})^m\right]$$

which yields the required expression for $\varphi'_{m+1}(u)$.

By use of (13) then we obtain the final explicit formula

$$(14)\quad h_m(s) = \frac{i^m}{4\pi}\frac{d}{ds}\left\{\int_0^s \mathrm{H}_m(t)\left[\frac{(u+\sqrt{u^2-1})^m+(u-\sqrt{u^2-1})^m}{\sqrt{u^2-1}}\right]dt\right\}$$
$$\left(u = \frac{s}{t} > 1\right).$$

It is interesting to note that this formula also holds for $m = 0$.

Now it is clear that the m-th term of the Fourier series for $f(s, \varphi)$ is $\mathcal{R}[h_m(s)e^{-im\varphi}]$ where $\mathcal{R}$ indicates « the real part of ». Recalling the definition of $\mathrm{H}_m(r)$ and using (14) we find that the m-th term of this series may be written

$$\mathcal{R}\left\{\frac{1}{4\pi^2}i^m\frac{d}{ds}\int_0^{2\pi}\int_0^s \mathrm{F}(t, \theta)\left[\frac{(u+\sqrt{u^2-1})^m+(u-\sqrt{u^2-1})}{\sqrt{u^2-1}}\,dt\right]e^{-im(\varphi-\theta)}d\theta\right\}.$$

Thus we obtain the following result :

Under the special hypothesis (H) *upon* $\mathrm{F}(r, \theta)$ *the only possible distribution function* $f(s, \varphi)$ *corresponding to the given density function* $\mathrm{F}(r, \theta)$ *is given by the Fourier series*

$$(15)\quad \mathcal{R}\left[\frac{1}{4\pi^2}i^m\frac{d}{ds}\sum_{m=0}^{\infty}\int_0^{2\pi}\left\{\int_0^s \mathrm{F}(r,\theta)\left[\frac{(u+\sqrt{u^2-1})^m+(u-\sqrt{u^2-1})^m}{\sqrt{u^2-1}}\right]dr\right\}e^{-im(\varphi-\theta)}d\theta\right]$$
$$\left(u = \frac{s}{r} > 1\right).$$

Here the operation $\mathcal{R}$ *(the real part of) is to be applied to each term separately. If and only if the coefficients of* $\cos m\varphi$, $\sin m\varphi$ *so obtained are continuous, and form the Fourier coefficients of a non-negative continuous function will there exist a (unique) solution* $f(s, \varphi)$, *namely that given by* (15).

It is interesting that there should exist at most one solution. At first sight this seems paradoxical when it is observed that one and the same density function, everywhere equal to 1, arises if we take $f(s, \varphi) = f(\varphi)$ with $\int_0^{2\pi} f(\varphi)\,d\varphi = 1$. Hence in this special case we find many distribution functions for the same continuous density function $F(r, \theta) = 1$. However, such functions $f(\varphi)$ will not be continuous at the origin unless $f(\varphi)$ is independent of φ. Consequently it appears that the uniqueness property flows from the continuity assumption.

5. Extension of the above result. — It is more or less evident that the hypothesis (H) which we have employed can be eliminated without essentially altering the formal solution which we have given. We shall only make a single remark in this connection.

If for a continuous density function $F(r, \theta)$, there is a corresponding continuous distribution function $f(s, \varphi)$, then it is obvious that for any continuous $f^\star(s, \varphi)$ near to $f(s, \varphi)$ *and satisfying* (H) the corresponding $F^\star(r, \theta)$ will be continuous and near to $F(r, \theta)$ and will also necessarily satisfy (H), in view of (1). But this $f^\star(s, \varphi)$ may be expressed in terms of $F^\star(s, \varphi)$ as indicated above, whence the following result :

If $F(r, \theta) \geqq 0$ *is continuous but does not satisfy* (H), *and there exists a corresponding continuous* $f(s, \varphi) \geqq 0$, *there will be arbitrarily nearly functions* $F^\star(r, \theta) \geqq 0$ *satisfying* (H) *and yielding functions* $f^\star(s, \varphi)$ *satisfying* (H), *where* $f^\star(s, \varphi)$ *is obtained from* $F^\star(s, \varphi)$ *as in* (15). *Furthermore a sequence of these functions* $F^\star(r, \varphi)$ *with* $\lim F^\star(r, \theta) = F(r, \theta)$ *may be found so that* $\lim f^\star(s, \varphi)$ *approaches a continuous limit function* $f(s, \varphi)$. *This will then yield the unique solution of the problem.*

In fact we infer at once that $f(s, \varphi)$ as thus defined satisfies (1). The stated uniquenes follows from the fact that the difference $f_1^\star(s, \varphi) - f_2^\star(s, \varphi) = \Delta(s, \varphi)$ betwen two distinct limits $f_1^\star(s, \varphi)$ and $f_2^\star(s, \varphi)$ would satisfy the equation [compare with (1)]

$$0 = \int_0^{2\pi} \Delta[r \sin(\varphi - \theta), \varphi]\,d\varphi,$$

whence it would follow that

$$0 = \int_0^{2\pi} \Delta_m(r \sin\varphi)\, e^{-im\varphi}\, d\varphi,$$

where $\Delta_m(s)$ is related to $\Delta(s, \varphi)$ just as $h_m(s)$ is to $f(s, \varphi)$. As before this leads by successive integration [*see* (11)] to

$$P_m(r^2) = \int_0^{2\pi} \Delta_m^{(-m)}(r \sin u)\, du = 4 \int_0^r \frac{\Delta^{(-m)}(t)\, dt}{\sqrt{r^2 - t^2}},$$

where $P_m(r^2)$ is polynomial of degree $m - 1$ at most in r^2. Solving this integral equation of Abel type we find

$$\Delta^{(-m)}(s) = \frac{1}{2\pi} \frac{d}{ds} \left[\int_0^s \frac{r P_m(r^2)}{\sqrt{s^2 - r^2}} dr \right] = \frac{1}{2\pi} \frac{d}{ds} \left[Q_m(r^2) \sqrt{s^2 - r^2} \Big|_0^s \right]$$

where $Q_m(r^2)$ is also a polynomial in r^2 of degree at most $m - 1$. Thus we find

$$\Delta^{(-m)}(s) = \frac{1}{2\pi} Q_m(0),$$

a constant, and this is impossible since $\Delta^{(-m)}(s)$ cannot reduce to a mere constant.

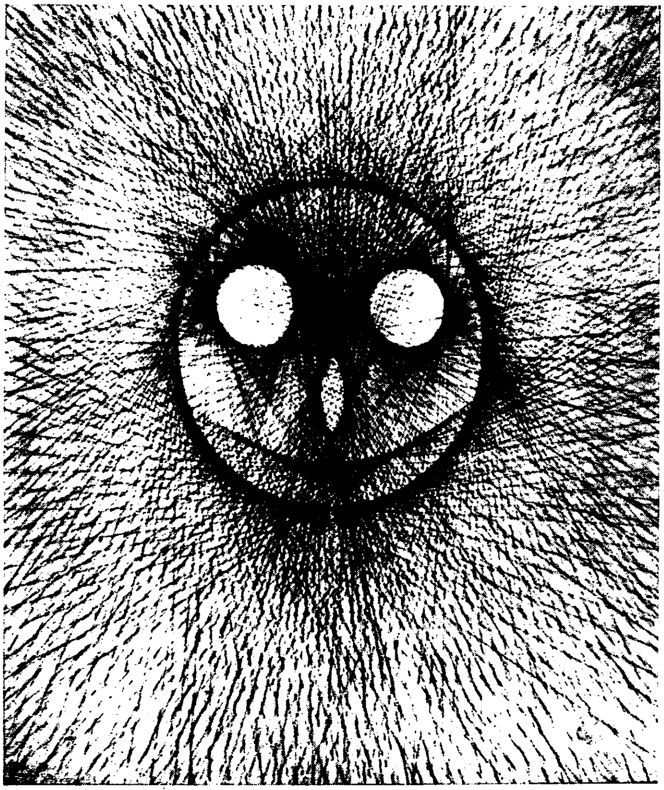
6. On a general geometric condition. — If A is any area and A_k is an area enclosing A such that any line cutting A in a length l cuts A_k in a length at least kl, then it is obvious that

$$\int_{A_k}\int F(x, y)\, dx\, dy \geqq k \int_A \int F(x, y)\, dx\, dy.$$

For example, if A is a unit circle and A_2 is a concentric circle of double the radius, we see that at least twice as much lead is deposited within the larger circle as within the smaller circle. Thus the average density is at least $1/3$ as great in the ring formed by the two circles as within the smaller circle.

It would be very interesting to determine just to what extent this general geometric condition upon $F(r, \theta)$ is completely characteristic. Clearly this condition strongly limits the kind of drawings which can be made by the rectilinear method. The adjoining simple drawing by Mr. David Middleton, a student at Harvard University, shows that this medium is not without amusing possibilities.

Fig. 3.



In this connection one further remark may be made. Imagine a *negative* of an arbitrary given drawing to be taken, and then subinited to uniform additional exposure. *Our rectilinear means enable us to draw such a faint negative of any drawing whatsoever.*

To see this, we note first that, as seen above, we can draw a white circular spot surrounded by a uniform gray background. Hence we could stipple the given drawing against a uniform nearly black background.

7. REMARK ON TWO RELATED PROBLEMS. — We commenced with a $F(r, \theta) \geqq 0$, and looked for a $f(s, \varphi) \geqq 0$ such that (1) was satisfied. It is, however, clear that this condition of nonnegativeness upon $F(r, \theta)$ and $f(s, \varphi)$ is largely irrelevant to the analytic problem. Thus we are able to deal similarly with the following more general question : suppose that not only are uniform straight lines allowed in the drawing but that subsequent uniform rectilinear erasure is allowed, which does not erase lines already drawn. Then $F(r, \theta)$ is positive as before and $f(s, \varphi)$ may be positive or negative. In this case there is a unique continuous solution (obtained as before), or none at all. If we write

$$f(s, \varphi) = f_1(s, \varphi) - f_2(s, \varphi),$$

where $f_1(s, \varphi)$ is the positive part of $f(s, \varphi)$ and $f_2(s, \varphi)$ is the negative part of $f(s, \varphi)$, then we have only to make the drawing with the positive distribution function $f_1(s, \varphi)$ and then make subsequent rectilinear erasures with the distribution function $f_2(s, \varphi)$, in order to make the drawing associated with $F(r, \theta)$. The following concluding observation shows that to all intents and purposes one can make any drawing if a single uniform erasure all over the figure be allowed at the end. In fact the given $F(r, \theta)$ may be approximated by a polynomial in x and y, corresponding to a terminating Fourier series for $F(r, \varphi)$. Our formulas show that then $f(s, \varphi)$ is given by a terminating Fourier series also which may be written

$$f^*(s, \varphi) + \min f$$

where $\min f$ designates the minimum of $f(s, \varphi)$ for $r < R$. Thus $f^*(s, \varphi)$ is positive or zero. Consequently if we use the distribution function $f^*(s, \varphi)$, and then erase uniformly a constant density $\min f$, the desired drawing with density $f(s, \varphi)$ will be left.

Evidently the two problems we have just referred to are those of *minimum rectilinear erasure* and of *minimum uniform erasure* respectively, and are to be regarded as solved by the explicit formulas derived above for the problem without any erasure.