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On integral equation satisfied by Lamé's functions whose order is half of an odd integer;

BY J.-L. SHARMA.

1. The Lamé's equation

$$\frac{d^2y}{du^2} = [n(n+1)\wp(u) + B]y, \tag{1.1}$$

can be transformed [1] to

$$\frac{d^2z}{dv^2} - 2n\frac{\wp''(v)}{\wp'(v)}\frac{dz}{dv} + 4[n(2n-1)\wp(v) - B]z = 0,$$

by putting $u = 2v$ and $z = y[\wp'(v)]^n$.

Starting from this Crawford [2] has shown that Lamé's equation when $2n$ is an odd integer, possesses the fundamental integrals

$$\mathrm{E}_n^m(u) = \left[\wp'\left(\frac{u}{2}\right)\right]^{-n}\left\{\left[\wp\left(\frac{u}{2}\right)-e_2\right]^{2n} + a_1\left[\wp\left(\frac{u}{2}\right)-e_2\right]^{2n-1} + \ldots + a_{n-\frac{1}{2}}\left[\wp\left(\frac{u}{2}\right)-e_2\right]^{n+\frac{1}{2}}\right\}, \tag{1.2}$$

$$\mathrm{F}_n^m(u) = \left[\wp'\left(\frac{u}{2}\right)\right]^{-n}\left\{\left[\wp\left(\frac{u}{2}\right)-e_2\right]^{n-\frac{1}{2}} + a_1\left[\wp\left(\frac{u}{2}\right)-e_2\right]^{n-\frac{3}{2}} + \ldots + a_{n-\frac{1}{2}}\right\}. \tag{1.3}$$

for $n + \frac{1}{2}$ values of B given by a certain equation.

The object of the present paper is to show that these functions are also solutions of certain homogeneous integral equations. Such inte-

(1) HALPHEN, *Traité des fonctions elliptiques*, Vol. II, p. 471.

(2) Lawrence CRAWFORD, *Quart. Journ. Math.*, 27, 1895, p. 93-98.

gral equations for integral values of n were considered by Whittaker (1), Lambe (2), Ward (3) and Sharma (4).

2. THEOREM I. — *The polynomial* $E_n^m(u)$ *is a solution of the integral equation*

$$E_n^m(u) = \lambda_m \int_{\alpha}^{\alpha + 4\omega_1} \left[p'\left(\frac{u}{2}\right) p'\left(\frac{v}{2}\right) \right]^{-n} K_1(u, v)\, E_n^m(v)\, dv$$

where

$$K_1(u, v) = \left\{ \left[p\left(\frac{u}{2}\right) - e_2 \right] \left[p\left(\frac{v}{2}\right) - e_2 \right] \right\}^{n+\frac{1}{2}} \times F\left(-n + \frac{1}{2}, 1, n + \frac{3}{2}; \frac{\left[p\left(\frac{u}{2}\right) - e_2 \right] \left[p\left(\frac{v}{2}\right) - e_2 \right]}{(e_2 - e_1)(e_2 - e_3)} \right)$$

and α *is any point other than zero.*

It is easy to see that $K_1(x)$ satisfies the differential équation

$$(2.2) \qquad x(1-x)y'' - \left(n - \frac{1}{2}\right)(1 - 3x)y' - 2n\left(n - \frac{1}{2}\right)y = 0,$$

where x stands for $\dfrac{\left[p\left(\frac{u}{2}\right) - e_2 \right] \left[p\left(\frac{v}{2}\right) - e_2 \right]}{(e_2 - e_1)(e_2 - e_3)}$.

Before we prove this theorem, we establish the following Lemma that

$$(2.3) \qquad \Phi(u, v) = \left[p'\left(\frac{u}{2}\right) p'\left(\frac{v}{2}\right) \right]^{-n} K_1(u, v),$$

is annihilated by the operator

$$(2.4) \qquad D_u^2 - D_v^2 = \frac{\partial^2}{\partial u^2} - n(n+1)p(u) - \frac{\partial^2}{\partial v^2} + n(n+1)p(v).$$

Differentiating (2.3) with respect to u, we get,

$$\frac{\Phi'}{\Phi} = -\frac{n}{2} \frac{p''\left(\frac{u}{2}\right)}{p'\left(\frac{u}{2}\right)} + \frac{K_1'}{K_1} \frac{\left[p\left(\frac{v}{2}\right) - e_2 \right] p'\left(\frac{u}{2}\right)}{2(e_2 - e_1)(e_2 - e_3)},$$

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(1) WHITTAKER, *Integral equation for Lamé's functions* [*Proc. Lond. Math. Soc.*, (2), XIV, 1935, p. 260-268].

(2), (3) LAMBE and WARD, *Quart. Journ. Math.* (Oxford Series), Vol. 5, n° 18, 1934, p. 81-97.

(4) SHARMA, *On integral equation satisfied by Lamé's functions* (*Journal de Mathématiques*, Paris, 1937, p. 199-203).

et

$$\frac{\Phi''}{\Phi} = n(n+1)\wp(u) + 2n\left(n-\frac{1}{2}\right)\wp\left(\frac{u}{2}\right)$$

$$+ \frac{K_1''}{K_1}\,\frac{\left[\wp\left(\frac{v}{2}\right)-e_2\right]^2\left[\wp'\left(\frac{u}{2}\right)\right]^2}{4(e_1-e_2)^2(e_3-e_2)^2} - \frac{1}{2}\left(n-\frac{1}{2}\right)\frac{K_1'}{K_1}\,\frac{\left[\wp\left(\frac{v}{2}\right)-e_2\right]\wp''\left(\frac{u}{2}\right)}{(e_2-e_1)(e_2-e_3)}.$$

Substituting this in (2.4) and simplifying, we get

$$\begin{aligned}
&\frac{1}{\Phi}[D_u^2 - D_v^2]\Phi \\
&= 2n\left(n-\frac{1}{2}\right)(p-q) + \frac{K_1''}{K_1}\,\frac{1}{4a^2}[q^2(4p^3+12e_2p^2+4ap) - p^2(4q^3+12e_2q^2+4aq)] \\
&\quad - \frac{1}{2}\left(n-\frac{1}{2}\right)\frac{1}{a}\,\frac{K_1'}{K_1}\,[q(6p^2+12e_2p+2a) - p(6q^2+12e_2q+2a)] \\
&= \frac{p-q}{K_1}\left[2n\left(n-\frac{1}{2}\right)K_1 + \left(n-\frac{1}{2}\right)K_1'(1-3x) - K_1''x(1-x)\right] \\
&= 0 \text{ by virtue of } (2.2)
\end{aligned}$$

where

$$p = \wp\left(\frac{u}{2}\right) - e_2, \qquad q = \wp\left(\frac{v}{2}\right) - e_2, \qquad \wp'^2\left(\frac{u}{2}\right) = 4p^3 + 12e_2p^2 + 4ap,$$

$$\wp''\left(\frac{u}{2}\right) = 6p^2 + 12e_2p + 2a \qquad \text{and} \qquad a = (e_2-e_1)(e_2-e_3).$$

Hence the lemma is proved.

Apply the operator

$$\frac{d^2}{du^2} - n(n+1)\wp(u) - B_m,$$

to the integral

$$I = \int_\alpha^{\alpha+4\omega_1}\left[\wp'\left(\frac{u}{2}\right)\wp'\left(\frac{v}{2}\right)\right]^{-n} K_1(u, v)\,E_n^m(v)\,dv,$$

we get,

$$\begin{aligned}
&\int_\alpha^{\alpha+4\omega_1}\left(\frac{d^2}{du^2} - n(n+1)\wp(u) - B_m\right)\left[\wp'\left(\frac{u}{2}\right)\wp'\left(\frac{v}{2}\right)\right]^{-n} K_1(u, v)\,E_n^m(v)\,dv \\
&= \int_\alpha^{\alpha+4\omega_1}\left\{\left(\frac{\partial^2}{\partial v^2} - n(n+1)\wp(v) - B_m\right)\left[\wp'\left(\frac{u}{2}\right)\wp'\left(\frac{v}{2}\right)\right]^{-n} K_1(u, v)\right\} E_n^m(v)\,dv.
\end{aligned}$$

Integrating by parts we get,

$$\left\{\frac{\partial}{\partial v}\left[\left[p'\left(\frac{u}{2}\right)p'\left(\frac{v}{2}\right)\right]^{-n}K_1(u,v)\right]E_n^m(v)-\left[p'\left(\frac{u}{2}\right)p'\left(\frac{v}{2}\right)\right]^{-n}K_1(u,v)\frac{dE_n^m(v)}{dv}\right\}_\alpha^{\alpha+4\omega_1}$$
$$+\int_\alpha^{\alpha+4\omega_1}\left[p'\left(\frac{u}{2}\right)p'\left(\frac{v}{2}\right)\right]^{-n}K_1(u,v)\left(\frac{d^2}{dv^2}-n(n+1)p(v)-B_m\right)E_n^m(v)\,dv.$$

Now the first expression vanishes because the functions are periodic and the second integral vanishes because $E_n^m(v)$ is a solution of (1.1). Hence it follows that the integral

$$\int_\alpha^{\alpha+4\omega_1}\left[p'\left(\frac{u}{2}\right)p'\left(\frac{v}{2}\right)\right]^{-n}K_1(u,v)\,E_n^m(v)\,dv,$$

as annihilated by the operator

$$\frac{d^2}{du^2}-n(n+1)\,p(u)-B_m,$$

and it is evidently a polynomial in $p\left(\frac{u}{2}\right)-e_2$ of degree $2n$. Since the equation (1.1) has only one solution of this type viz., (1.2), it follows that the integral is a constant multiple of $E_n^m(u)$. Hence

$$E_n^m(u)=\lambda_m\int_\alpha^{\alpha+4\omega_1}\left[p'\left(\frac{u}{2}\right)p'\left(\frac{v}{2}\right)\right]^{-n}K_1(u,v)\,E_n^m(v)\,dv.$$

3. Theorem II. — *The polynomial $F_n^m(u)$ is a solution of the homogeneous integral equation*

$$F_n^m(u)=\lambda_m\int_\alpha^{\alpha+4\omega_1}\left[p'\left(\frac{u}{2}\right)p'\left(\frac{v}{2}\right)\right]^{-n}K_2(u,v)\,F_n^m(v)\,dv$$

where

$$K_2(u,v)=\left\{\left[p\left(\frac{u}{2}\right)-e_2\right]\left[p\left(\frac{v}{2}\right)-e_2\right]\right\}^{n-\frac{1}{2}}\times F\left(-n+\frac{1}{2},\,1,\,n+\frac{3}{2};\,\frac{(e_2-e_1)(e_2-e_3)}{\left[p\left(\frac{u}{2}\right)-e_2\right]\left[p\left(\frac{v}{2}\right)-e_2\right]}\right).$$

It is easy to see that $K_2(x)$ satisfies the differential equation

$$(3.1)\qquad x(1-x)\frac{d^2K}{dx^2}-\left(n-\frac{1}{2}\right)(1-3x)\frac{dK}{dx}-2n\left(n-\frac{1}{2}\right)K=0$$

where

$$x = \frac{\left[\wp\left(\frac{u}{2}\right) - e_2\right]\left[\wp\left(\frac{v}{2}\right) - e_2\right]}{(e_1 - e_2)(e_1 - e_3)}.$$

To prove this we should establish the preliminary lemma that

$$\Phi_1 \equiv \left[\wp'\left(\frac{u}{2}\right)\wp'\left(\frac{v}{2}\right)\right]^{-n} K_2(u, v),$$

is annihilated by the operator (2.4).

Proceeding as in the previous article, we get,

$$\frac{1}{\Phi_1}(D_u^2 - D_v^2)\Phi_1 = \frac{p-q}{K_2}\left[2n\left(n - \frac{1}{2}\right)K_2 + \left(n - \frac{1}{2}\right)K_2'(1 - 3x) - x(1-x)K_2''\right],$$

which vanishes by virtue of (3.1), p, q, x, having the same meaning as in § 2.

The rest of the proof is exactly similar to that given in § 2. Hence we find that

$$\int_{\alpha}^{\alpha + 4\omega_1}\left[\wp'\left(\frac{u}{2}\right)\wp'\left(\frac{v}{2}\right)\right]^{-n} K_2(u, v)\, F_n^m(v)\, dv,$$

satisfies the differential equation (1.1). Evidently it is a polynomial in $\left[\wp\left(\frac{u}{2}\right) - e_2\right]$, and is of the form (1.3). Since (1.1) possesses only one integral of this type, therefore it is a constant multiple of $F_n^m(u)$.

Hence $F_n^m(u)$ is a solution of the integral equation

$$F_n^m(u) = \lambda_m \int_{\alpha}^{\alpha + 4\omega_1}\left[\wp'\left(\frac{u}{2}\right)\wp'\left(\frac{v}{2}\right)\right]^{-n} K_2(u, v)\, F_n^m(v)\, dv.$$

4. The Kernels in both are of the form $\sum_{1}^{n+\frac{1}{2}} a_i(u)\, b_i(v)$.

Therefore the equation in λ will be of degree $n + \frac{1}{2}$ and will have $n + \frac{1}{2}$ roots corresponding to $n + \frac{1}{2}$ values of B. Therefore for each solution of (1.1) there corresponds one and only one value of λ.

5. If $E_n^m(u)$ and $E_n^p(u)$ are two solutions of the equations (1.1) corresponding to B_m and B_p, then we have

$$\int_{\alpha}^{\alpha+4\omega_1} E_n^m(v)\, E_n^p(v)\, dv = \begin{cases} 0 & (m \neq p) \\ \theta_m & (m = p). \end{cases}$$

Proof : Suppose it is not zero but is equal to A, then since from (1.1)

$$B_m E_n^m(u) = \left[\frac{d^2}{du^2} - n(n+1)\,p(u)\right] E_n^m(u).$$

Therefore,

$$B_m A = \int_{\alpha}^{\alpha+4\omega_1} E_n^p(v)\, B_m E_n^m(v)\, dv$$

$$= \int_{\alpha}^{\alpha+4\omega_1} E_n^p(v)\left[\frac{d^2}{dv^2} - n(n+1)\,p(u)\right] E_n^m(v)\, dv.$$

Integrating twice as in § 2, it becomes

$$= \int_{\alpha}^{\alpha+4\omega_1} E_n^m(v)\left[\frac{d^2}{dv^2} - n(n+1)\,p(v)\right] E_n^p(v)\, dv$$

$$= \int_{\alpha}^{\alpha+4\omega_1} E_n^m(v)\, B_p E_n^p(v)\, dv = B_p A.$$

Therefore,

$$B_m A = B_p A,$$

hence $A = 0$, since $B_m \neq B_p$.

Similar property holds for the second function $F_n^m(u)$.

6. If $\lambda_1, \lambda_2, \lambda_3, \ldots$ are the characteristic values of λ, then

$$K_1(u, v) = \left[p'\left(\frac{u}{2}\right)p'\left(\frac{v}{2}\right)\right]^n \sum_{1}^{n+\frac{1}{2}} \frac{E_n^m(v)\, E_n^m(u)}{\lambda_m \theta_m},$$

which follows directly from the orthogonal property of § 5. Similarly for the other function, we get,

$$K_2(u, v) = \left[p'\left(\frac{u}{2}\right)p'\left(\frac{v}{2}\right)\right]^n \sum_{1}^{n+\frac{1}{2}} \frac{F_n^m(v)\, F_n^m(u)}{\lambda_m \theta_m}.$$