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A note on abstract Polynomials in complex Spaces

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A note on abstract Polynomials in complex Spaces;

By I. E. HIGHBERG [1].

Fréchet [2], in his 1929 paper, gave a definition of polynomials in a very general sort of a space — an « espace algébrophile » — with a real multiplier domain. His definition is essentially as follows. A function $f(x)$ defined on an « espace algébrophile » E, to a like space E′, will be called a polynomial, if $f(x)$ is continuous and for some integer n, $\Delta^n f(x) \equiv 0$, where

$$\Delta^n f(x) = \Delta_n[\Delta^{n-1} f(x)], \qquad \Delta^0 f(x) = f(x), \qquad \Delta_i f(x) = f(x + \Delta_i x) - f(x)$$

and the $\Delta_i x$ are arbitrary increments.

Gateaux [3] has defined a polynomial in a different manner and Michal [4] and Martin [5] have considered similar definitions in Banach spaces. Let E and E′ be Banach spaces and A the associated number system, where A is either R, the real number system, or C, the complex number system. If $f(\mu)$ is a function on A to E, Martin

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(1) I wish to thank Professor A. D. Michal for many helpful criticisms and suggestions in the preparation of this paper.

(2) *Les polynomes abstraits* (*Journal de Mathématiques pures et appliquées*, 9e série, t. **8**, 1929, p. 71).

(3) *Sur diverses questions du Calcul fonctionnel* (*Bull. Soc. de France*, vol. **50**, 1922).

(4) A. D. Michal and R. S. Martin, *Some Expansions in Vector Space* (*Journal de Mathématiques pures et appliquées*, 9e série, t. **13**, 1934, p. 69).

(5) R. S. Martin, *Contributions to the Theory of Functionals* (*Thesis*, California Institute of Technology, 1932).

defines it to be a polynomial if it is expressible in the form

$$f(\mu) = a_0 + \mu . a_1 + \ldots + \mu^n . a_n$$

where the a_i are fixed elements in E. Let $p(x)$ be a function on E to E'. Martin calls it a polynomial if, 1° $p(x)$ is continuous, 2° for each pair x, y, $p(x + \mu . y)$ is a polynomial in μ with coefficients in E'. When A is R, Martin showed that his definition and Fréchet's were equivalent. (Incidentally, Fréchet proved half of the equivalence in his paper). Martin conjectured that if A is C, we would have to add to Fréchet's conditions the further condition of Fréchet differentiability of $p(x)$ at $x = 0$ in order that the two definitions be equivalent. That this is not enough I will show later.

In this paper will be considered what additional restrictions must be imposed in a complex « espace algébrophile » in order that the definition of a polynomial given by Fréchet be equivalent to the definition considered by Martin and Michal.

I.

Let E be a complex « espace algébrophile. » In Fréchet's postulates we can replace the real number system R by C, and all the theorems on continuity remain valid. I shall assume them in the remainder of this paper.

Definition 1. — If $f(x)$ is a function on a space E to a space E' of like nature, it will be said to possess a Gateaux differential at the point x_0, if for any z in E

$$\lim_{\mu \to 0} \frac{f(x_0 + \mu . z) - f(x_0)}{\mu} \qquad (\mu \text{ in C})$$

exists, independent of the way in which $\mu \to 0$.

We do not require this limit to be linear in z.

Lemma 1. — *Let $\chi(\mu) = f(\mu) . a$, where a is in* E *and $f(\mu)$ is a function on* C *to* C *having a derivative everywhere. Then $\chi(\mu)$ is Gateaux differentiable everywhere.*

Proof

$$\frac{\chi(\mu+t\lambda)-\chi(\mu)}{t}=\lambda\,\frac{f(\mu+t\lambda)-f(\mu)}{t\lambda}\cdot a.$$

Since $\lim\limits_{t\to 0}\frac{f(\mu+t\lambda)-f(\mu)}{t\lambda}=f'(\mu)$, and since $g(\mu).a$ is a continuous function of μ, we conclude that the Gateaux differential exists and equals $\lambda f'(\mu).a$.

That $f(\mu).a+g(\mu).b$ has a Gateaux differential everywhere if $f(\mu)$ and $g(\mu)$ have derivatives everywhere follows from the continuity of the operation $x+y$. The extension to any finite number of terms is obvious.

Definition 2. — If $\Phi(\mu)$ is a function on C to E, then it will be called a C polynomial if it can be expressed in the form

$$\Phi(\mu)=a_0+\mu.a_1+\ldots+\mu^n.a_n \tag{1}$$

where $a_0, \ldots, a_n$ are fixed elements in E. If $a_n\neq 0$ it will be said to be of degree n.

Definition 2'. — Let $\Phi(\mu)$ be a function on C to E. Then $\Phi(\mu)$ will be said to be a C polynomial if :

1° $\Phi(\mu)$ is continuous,

2° for some integer n, $\Delta^{n+1}\Phi(\mu)\equiv 0$,

3° $\Phi(\mu)$ possesses a Gateaux differential everywhere. It will be said to be of degree n, if $\Delta^n\Phi(\mu)\not\equiv 0$.

I shall now prove the equivalence of the two definitions. First I shall show that if $\Phi(\mu)$ is a polynomial of degree n according to definition 2, then it is a polynomial of degree n according to definition 2'.

The proof that $\Phi(\mu)$, where $\Phi(\mu)$ has the form (1), satisfies condition 1° and 2° in definition 2' is the same as in Fréchet's paper. That it satisfies 3° is a consequence of lemma 1 and the remarks following the lemma. That $\Delta^n\Phi(\mu)\not\equiv 0$ is obvious.

To prove the converse, that a polynomial of degree n according to definition 2' is a polynomial of degree n according to definition 2, we have.

Case I : $n = 0$. Then $\Delta\Phi(\lambda) \equiv 0$, or $\Phi(\lambda + \mu) - \Phi(\lambda) \equiv 0$. Hence $\Phi(\lambda) = a_0$, which is of the form (1).

Case II : $n = 1$, $\Delta^2\Phi(\lambda) \equiv 0$. Then

$$\Phi(\lambda + \mu + \nu) - \Phi(\lambda + \mu) - \Phi(\lambda + \nu) + \Phi(\lambda) \equiv 0.$$

Setting $\lambda = 0$, we get

$$\Phi(\mu + \nu) - \Phi(\mu) - \Phi(\nu) + \Phi(0) \equiv 0. \tag{2}$$

Set $\chi(\lambda) \equiv \Phi(\lambda) - \Phi(0)$. Then $\chi(\lambda)$ is continuous since $\Phi(\lambda)$ is continuous, and moreover is Gateaux differentiable for the same reason. Using equation (2) we get

$$\chi(\lambda + \mu) = \chi(\lambda) + \chi(\mu). \tag{3}$$

Then by familiar methods we have

$$\chi(a.\mu) = a.\chi(\mu)$$

where a is a real multiplier. Hence if $\lambda = \lambda_1 + i\lambda_2$

$$\chi(\lambda) = \lambda_1.\chi(1) + \lambda_2.\chi(i) = \frac{\lambda + \bar{\lambda}}{2}\chi(1) + \frac{\lambda - \bar{\lambda}}{2i}\chi(i)$$

where $\bar{\lambda}$ is the complex conjugate of λ. Hence

$$\Phi(\lambda) = a_0 + \lambda.a_1 + \bar{\lambda}.b_1.$$

Since it was assumed that $\Phi(\lambda)$ was Gateaux differentiable we see that $\bar{\lambda}.b_1$ must also be. This is a contradiction and hence $b_1 = 0$. Then $\Phi(\lambda)$ is of the form (1).

It is to be noted that in this case we do not require the full condition on $\Phi(\lambda)$ of Gateaux differentiability everywhere, differentiability at one point is sufficient to make the two definitions equivalent. When $n = 1$, condition 3° of definition 2′ may be replaced by the algebraic condition,

$$\frac{\Phi(1) - \Phi(0)}{1} = \frac{\Phi(i) - \Phi(0)}{i}. \tag{3°′}$$

I shall now prove the general case by induction.

Case III : $n = n$, $\Delta^{n+1}\Phi(\lambda) \equiv 0$. Then $\Delta^n[\Phi(\lambda + \mu) - \Phi(\lambda)] \equiv 0$.

Since $\Phi(\lambda)$ is continuous, $\Phi(\lambda+\mu)-\Phi(\lambda)$ considered as a function of λ is continuous. Since $\Phi(\lambda)$ possesses a Gateaux differential everywhere, $\Phi(\lambda+\mu)-\Phi(\lambda)$ is also Gateaux differentiable everywhere [1]. Hence under the induction hypothesis we will assume that $\Phi(\lambda+\mu)-\Phi(\lambda)$ is a C polynomial in λ of the form (1), and of degree $n-1$ at most. Let us set

$$\psi(\lambda, \mu)=\Phi(\lambda+\mu)-\Phi(\lambda)-\Phi(\mu). \tag{4}$$

Evidently, $\psi(\lambda, \mu)$ is also a C polynomial of degree at most $n-1$ in λ and since it is symmetric in λ, μ it is also a C polynomial in μ of degree at most $n-1$.

In exactly the same manner as in Fréchet's paper we prove that

$$\psi(\lambda, \mu)=g(\lambda+\mu)-g(\lambda)-g(\mu) \tag{5}$$

where

$$g(\lambda)=-\psi_0+\sum_1^r{}_s \lambda^s . B_s,$$

and where ψ_0 and B_s are constant elements in E. We set

$$H(\lambda)=\Phi(\lambda)-g(\lambda)$$

and it follows that

$$H(\lambda+\mu)=H(\lambda)+H(\mu).$$

Now $\Phi(\lambda)$ is continuous and Gateaux differentiable, and $g(\lambda)$ is continuous and is Gateaux differentiable by lemma 1. Hence $H(\lambda)$ is continuous and Gateaux differentiable and we may conclude that $H(\lambda)=\lambda . H(1)$. Hence

$$\Phi(\lambda)=-\psi_0+\lambda . H(1)+\sum_1^r{}_s \lambda^s . B_s .$$

Now $\psi(\lambda, \mu)$ is of degree $n-1$ at most in λ, but the right hand side of equation (5) is of degree $r-1$ at most and hence $r \leqq n$.

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(1) It is essential that $\Phi(\lambda)$ be differentiable everywhere. For if it is differentiable at only one point we cannot assert that $\Phi(\lambda+\mu)-\Phi(\lambda)$ is differentiable at all. For example $\Phi(\lambda)=\bar{\lambda}^2 . a$ is differentiable at $\lambda=0$ but nowhere else.

If $\Delta^n\Phi(\lambda) \not\equiv 0$, $r = n$. Thus the equivalence of the two definitions is established.

II.

In this section we will complete the equivalence proofs by discussing polynomials on a complex « espace algébrophile » E to a space E′ of like nature.

Definition 3. — Let $p(x)$ be a function on E to E′. Then $p(x)$ will be said to be an E polynomial if :

1° $p(x)$ is continuous,
2° for every pair x, y, $p(x + \lambda.y)$ is a C polynominal in λ.

It will be said to be of degree n, if for some x, y $p(x + \lambda.y)$ is a C polynomial of degree n and for all x, y is a C polynomial of degree $\leqq n$.

Definition 3′. — Let $p(x)$ be a function on E to E′. Then $p(x)$ will be said to be an E polynomial if :

1° $p(x)$ is continuous,
2° for some integer n, $\Delta^{n+1}p(x) \equiv 0$,
3° $p(x)$ possesses a Gateaux differential everywhere.

It will be said to be of degree n, if $\Delta^n p(x) \not\equiv 0$.

I shall first prove that *a polynomial of degree n according to definition* 3′ *is a polynomial of degree n by definition* 3.

Let $\Phi(\mu) = p(x + \mu.y)$. Then $\Phi(\mu)$ is a function on C to E′ and is continuous. Furthermore $\Delta^{n+1}\Phi(\mu) \equiv 0$. It may also be readily shown that $\Phi(\mu)$ is Gateaux differentiable everywhere. Hence, using the results of section I, we conclude that $p(x + \mu.y)$ is a C polynomial of degree $\leqq n$. That its degree is exactly n, or that for some x, y $\Delta^n\Phi(\mu) \not\equiv 0$ will be shown later.

In order to prove that *if* $p(x)$ *is an* E *polynomial of degree n by definition* 3, *it is also an* E *polynomial of degree n by definition* 3′, I shall state some results without proof from Martin's thesis. These results can be readily proved.

Let $p(x+\mu.y)$ be represented in the form

$$p(x+\mu.y)=k_0(x,y)+\mu.k_1(x,y)+\ldots+\mu^n.k_n(x,y). \tag{6}$$

The following lemmas all assume that $p(x)$ is a polynomial according to definition 3.

LEMMA 2. — *If $p(x)$ is an E polynomial, then $k_r(x,y)$ is homogeneous in y of degree r.*

LEMMA 3. — *For fixed x, $k_r(x,y)$ is a polynomial in y of degree $\leqq n$, and for fixed y, $k_r(x,y)$ is a polynomial of degree $\leqq n$ in x.*

LEMMA 4. — *If $p(x)$ is a homogeneous polynomial, then $k_r(x,y)$ is homogeneous of degree r in y and homogeneous of degree $n-r$ in x.*

LEMMA 5. — *If $p(x)$ is a polynomial of degree n and is homogeneous of degree m, then $m=n$.*

LEMMA 6. — *If $p(x)$ is a homogeneous polynomial of degree m, then for some Δx, $\Delta p(x)$ is a polynomial of degree $m-1$.*

We can also express $p(x)$ as a sum of homogeneous polynomials,

$$p(y)=h_0(y)+h_1(y)+\ldots+h_n(y), \tag{7}$$

by setting $h_r(y)=k_r(0,y)$. By taking $p(x)$ as the sum of homogeneous polynomials and using lemma 6 successively, we prove :

LEMMA 7. — *If $p(x)$ is an E polynomial of degree n, then $\Delta^{n+1}p(x)\equiv 0$ and for some choice of the increments $\Delta_i x$, $\Delta^n p(x)\not\equiv 0$.*

We must now prove that $p(x)$ is Gateaux differentiable.

$$p(x+\mu.\Delta x)=k_0(x,\Delta x)+\mu.k_1(x,\Delta x)+\ldots+\mu^n.k_n(x,\Delta x)$$

$$p(x+\mu.\Delta x)-p(x)=\mu.k_1(x,\Delta x)+\ldots+\mu^n.k_n(x,\Delta x)$$

since $k_0(x,\Delta x)=k_0(x,0)$. Dividing by μ we see that the limit as $\mu\to 0$ exists and equals $k_1(x,\Delta x)$. Using this result and lemma 7 we conclude that if $p(x)$ is an E polynomial of degree n by definition 3 then it is also an E polynomial of degree n by definition 3′.

In the proof of the converse which preceded this we did not show that if $p(x)$ is a polynomial of degree n by definition 3′ then it is exactly of degree n by definition 3. This now follows from lemma 7, for if it were of degree $< n$ by definition 3 then $\Delta^n p(x) \equiv 0$, and it could not be of degree n by definition 3′. Hence we have proved *the complete equivalence of the two definitions* 3 *and* 3′.

Note. — It seems to be true that if in definition 2′ of a C polynomial we leave out the condition of Gateaux differentiability, or in other words, if we do not add the requirements of Gateaux differentiability to Fréchet's definition of a polynomial, then $\Phi(\mu)$ will have the form

$$\Phi(\mu) = a_{00} + \mu . a_{10} + \bar{\mu} . a_{11} + \mu^2 . a_{20} + \mu\bar{\mu} . a_{21} + \bar{\mu}^2 . a_{22} + \dots \\ + \mu^n . a_{n0} + \mu^{n-1}\bar{\mu} . a_{n1} + \dots + \mu\bar{\mu}^{n-1} . a_{n,n-1} + \bar{\mu}^n . a_{nn}.$$

This has been verified for $n = 0, 1, 2$, but the general case has not yet been proved. It is my intention to discuss the properties of functions like the above — and which we might call $\overline{\mathrm{C}}$ polynomials — in a later paper.