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An integral equation satisfied by the Lame's functions;

By J. L. SHARMA.

1. All the Lame's functions of the first kind are given by the scheme

$$\left\{\begin{matrix} & \sqrt{p(u)-e_1}, & \sqrt{[p(u)-e_2][p(u)-e_3]}, & \\ 1, & \sqrt{p(u)-e_2}, & \sqrt{[p(u)-e_1][p(u)-e_3]}, & p'(u) \\ & \sqrt{p(u)-e_3}, & \sqrt{[p(u)-e_2][p(u)-e_1]}, & \end{matrix}\right\} \prod_r [p(u)-a_r].$$

For each of these types of functions Whittaker [1] has investigated intégral equations, involving Jacobian elliptic functions. The object of the present note is to show that the integral equation

$$E_n^m(u)=\lambda\int_\alpha^{\alpha+4\omega_1}\left[p'\left(\frac{u}{2}\right)p'\left(\frac{v}{2}\right)\right]^{-n} \times\left\{1-\frac{\left[p\left(\frac{u}{2}\right)-e_2\right]\left[p\left(\frac{v}{2}\right)-e_2\right]}{(e_1-e_2)(e_3-e_2)}\right\}^{2n} E_n^m(v)\,dv,$$

is satisfied by all the Lame's functions of the first kind.

2. Before we establish this result we shall prove the following two lemmas.

Lemma I. — All the Lame's functions of the first kind can be

(1) Whittaker, *Proc. Lond. Math. Soc.*, 2e série, t. XIV, 1915, p. 260-268.

expressed in the form

$$(2.1)\qquad \left[p'\left(\frac{u}{2}\right)\right]^{-n}\left\{a_{2n}\left[p\left(\frac{u}{2}\right)\right]^{2n}+a_{2n-1}\left[p\left(\frac{u}{2}\right)\right]^{2n-1}+\ldots+a_0\right\}.$$

Consider functions of the first species. They are polynomials of degree $\frac{n}{2}$ in $p(u)$. But

$$p(u)=\frac{1}{4\left[p'\left(\frac{u}{2}\right)\right]^2}\left\{4\left[p\left(\frac{u}{2}\right)\right]^4+2\left[p\left(\frac{u}{2}\right)\right]^2+8g_3p\left(\frac{u}{2}\right)+\frac{g_2^2}{4}\right\}.$$

Hence by substituting for $p(u)$ we obtain an expression, of which the numerator is a function of degree $4\times\frac{n}{2}$, *i.e.*, $2n$ in $p\left(\frac{u}{2}\right)$ and the denominator is $\left[p'\left(\frac{u}{2}\right)\right]^n$. Thus Lame's function of the first species can be put in the form (2.1).

Since

$$\sqrt{p(u)-e_1}=\frac{1}{p'\left(\frac{u}{2}\right)}\left[p\left(\frac{u}{2}\right)-p\left(\frac{\omega_1}{2}\right)\right]\left[p\left(\frac{u}{2}\right)-p\left(\omega_2+\frac{\omega_1}{2}\right)\right]$$

therefore, the first function of the second species, viz.

$$\sqrt{p(u)-e_1}\prod_r^{\frac{n-1}{2}}[p(u)-a_r],$$

will have for its denominator

$$\left[p'\left(\frac{u}{2}\right)\right]\left[p'\left(\frac{u}{2}\right)\right]^{2\times\frac{n-1}{2}}\ i.e.\left[p'\left(\frac{u}{2}\right)\right]^n$$

and its numerator will be of degree $2+4\times\frac{n-1}{2}$, *i.e.*, $2n$ in $p\left(\frac{u}{2}\right)$. Thus it is also of the form (2.1). Similarly we can show that other functions are of the same type.

It can also be deduced directly from the differential equation. Putting (1)

$$u=2v\qquad\text{and}\qquad z=y\,[p'(v)]^n.$$

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(1) Halphen, *Traités des Fonctions elliptiques*, vol. II, p. 471.

Lame's equation takes the form

$$\frac{d^2z}{dv^2} - 2n\frac{\wp''(v)}{\wp'(v)}\frac{dz}{dv} + 4[n(2n-1)\wp(v) - B] = 0.$$

If this equation is solved in a series of powers of $\wp(v)$, then it will be found that one integral is of degree $2n$ in $\wp\left(\frac{u}{2}\right)$ for values of B given by an equation of degree $2n+1$, *i.e.*, all the four equations satisfied by B, viz.,

$$P_n(B) = 0, \qquad Q^n_{e_1}(B) = 0, \qquad Q^n_{e_2}(B) = 0, \qquad Q^n_{e_3}(B) = 0,$$

are included in one equation only. Hence we find that all the Lame's functions can be put in the form (2.1).

LEMMA II.

$$(2.2) \qquad F(u, v) \equiv \left[\wp'\left(\frac{u}{2}\right)\wp'\left(\frac{v}{2}\right)\right]^{-n}\left[1 - \frac{\left[\wp\left(\frac{u}{2}\right) - e_2\right]\left[\wp\left(\frac{v}{2}\right) - e_2\right]}{[e_1 - e_2][e_3 - e_2]}\right]^{2n}$$

is annihilated by the operator

$$D^2_u - D^2_v \equiv \frac{\partial^2}{\partial u^2} - n(n+1)\wp(u) - \frac{\partial^2}{\partial v^2} + n(n+1)\wp(v).$$

Differentiating (2.2) twice, we get,

$$(2.3) \quad D^2_u F(u, v) = [2a(U+V)(a+UV) + 2e_2UV(UV+4a) + 2a^2e_2]K,$$

where

$$K = n\left(n - \frac{1}{2}\right)\left[\wp'\left(\frac{u}{2}\right)\wp'\left(\frac{v}{2}\right)\right]^{-n}\left[1 - \frac{\left[\wp\left(\frac{u}{2}\right) - e_2\right]\left[\wp\left(\frac{v}{2}\right) - e_2\right]}{(e_1 - e_2)(e_3 - e_2)}\right]^{2n-2}$$

and

$$U = \wp\left(\frac{u}{2}\right) - e_2, \qquad V = \wp\left(\frac{v}{2}\right) - e_2, \qquad 4a = 12e_2^2 - g_2,$$

by making use of

$$\wp(u) = \frac{1}{4}\left[\frac{\wp''\left(\frac{u}{2}\right)}{\wp'\left(\frac{u}{2}\right)}\right]^2 - 2\wp\left(\frac{u}{2}\right),$$

$$\wp'(u) = 4U^3 + 12e_2U^2 + 4aU,$$

$$\wp''(u) = 6U^2 + 12e_2U + 2a.$$

As (2.3) is symmetrical in u and v, therefore it is clear that

$$[D_u^2 - D_v^2]\,F(u, v) = 0.$$

3. Apply the operator

$$\frac{\partial^2}{\partial u^2} - n(n+1)\,p(u) - B_n^m$$

to the integral

$$\int_\alpha^{\alpha+i\omega_1} F(u, v)\,E_n^m(v)\,dv,$$

we get,

$$\int_\alpha^{\alpha+i\omega_1} \left\{ \frac{\partial^2}{\partial u^2} - n(n+1)\,p(u) - B_n^m \right\} F(u, v)\,E_n^m(v)\,dv$$

$$= \int_\alpha^{\alpha+i\omega_1} \left[\left\{ \frac{\partial^2}{\partial v^2} - n(n+1)\,\wp(v) - B_n^m \right\} F(u, v) \right] E_n^m(v)\,dv,$$

$$= \int_\alpha^{\alpha+i\omega_1} E_n^m(v)\,\frac{\partial^2}{\partial v^2}\,F(u, v)\,dv$$

$$- \int_\alpha^{\alpha+i\omega_1} [n(n+1)\,p(v) + B_n^m]\,F(u, v)\,E_n^m(v)\,dv.$$

Integrate twice by parts the first of the two integrals on the right hand side, we get,

$$\left[\frac{\partial F}{\partial v}\,E_n^m(v) - F\,\frac{dE_n^m(v)}{dv} \right]_\alpha^{\alpha+i\omega_1} + \int_\alpha^{\alpha+i\omega_1} F(u, v)\,D_v^2\,E_n^m(v)\,dv.$$

It is clear from the form of $E_n^m(u)$ that it is doubly periodic function. Moreover $E_n^m(u)$ and $F(u, v)$ are both uniform.

Hence the expression within the brackets vanishes. Further

$$D_v^2\,E_n^m(v) = 0,$$

therefore, we find that the integral

$$\int_\alpha^{\alpha+i\omega_1} F(u, v)\,E_n^m(v)\,dv$$

is annihilated by the operator

$$\frac{d^2}{du^2} - n(n+1)\,p(u) - B_n^m$$

and it is evidently a polynomial in $\wp\left(\frac{u}{2}\right)$ of degree $2n$ multiplied by $\left[\wp'\left(\frac{u}{2}\right)\right]^{-n}$. Since Lame's equation possesses only one integral of this form, it follows that the integral is a constant multiple of $E_n^m(u)$. Therefore,

$$E_n^m(u) = \lambda \int_\alpha^{\alpha+4\omega_1} \left[\wp'\left(\frac{u}{2}\right)\wp'\left(\frac{v}{2}\right)\right]^{-n} \times \left\{ 1 - \frac{\left[\wp\left(\frac{u}{2}\right) - e_2\right]\left[\wp\left(\frac{u}{2}\right) - e_2\right]}{(e_3 - e_2)(e_1 - e_2)} \right\}^{2n} E_n^m(v)\, dv,$$

where λ is one of the *characteristic numbers* and α is any point other than zero or its congruent points.