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PAR JOSEPH LIOUVILLE

E. T. BELL

General theorems on numerical functions

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General theorems on numerical functions;

BY E. T. BELL.

1. In his paper *Théorèmes généraux concernant des fonctions numériques*, Liouville [1] stated without proof four useful theorems on numerical functions. The first theorem needs a correction, the second is over-conditioned, the third and fourth are exact. We first state Liouville's forms and then point out the required modifications.

THEOREM I. — *Let n be any positive integer prime to the constant integer m, and let the summations refer to all pairs (d, δ) of positive divisors of n such that $n = d\delta$. Let*

$$A(n),\ G(n),\ H(n),\ P(n),\ Q(n)$$

be numerical functions of n, such that, for all n as defined,

$$\Sigma A(d)\, G(\delta) = H(n), \tag{1}$$

$$\Sigma A(d)\, P(\delta) = Q(n). \tag{2}$$

Then (1) *and* (2) *together imply*

$$\Sigma Q(d)\, G(\delta) = \Sigma P(d)\, H(\delta). \tag{3}$$

THEOREM II. — If $B(n)$ is also a numerical function, and $A, B, \ldots, Q$ are such that

$$\Sigma A(d)\, G(\delta) = \Sigma B(d)\, H(\delta), \tag{4}$$

$$\Sigma A(d)\, P(\delta) = \Sigma B(d)\, Q(\delta), \tag{5}$$

[1] *Journal de Mathématiques* (2), 8, 1863, 347-352.

for all n, (d, δ) as above defined, then (4) and (5) together imply

$$\Sigma Q(d) G(\delta) = \Sigma P(d) H(\delta), \tag{6}$$

provided that

$$A(n) = 0, \qquad B(n) = 0, \tag{7}$$

do not hold for all n as defined.

By the usual definition (wich agrees with Liouville's, *ibid.*, vol 2, 1857, 141), $f(x)$ is called a numerical function of x if $f(x)$ is finite and uniform for positive integer values of x. In the first theorem choose $m = q$, $n = p$, $q > p$, where p, q are both prime, and let $A(n)$ be any numerical function of n such that $A(1) = 0$. Then (1), (2) give

$$A(p) G(1) = H(p), \qquad A(p) P(1) = Q(p),$$

and (3) gives

$$Q(1) G(p) + Q(p) G(1) = P(1) H(p) + P(p) H(1);$$

hence, replacing $Q(p)$, $H(p)$ by their values above, we find

$$Q(1) G(p) = H(1) P(p),$$

a condition which is not necessarily satisfied by four numerical functions Q, G, H, P. By the following slight changes theorem I becomes exact and theorem II is generalized.

In order that (1) and (2) shall imoly (3) it is necessary and sufficient that $A(1) \neq 0$, *and in order that (4) and (5) shall imply (6) it is necessary and sufficient that* $A(1) B(1) \neq 0$.

Assuming these changes to have been made we note that the apparently special case $m = 1$ is in fact the general case (the same applies to Liouville's third and fourth theorems). Denote by (m, n) the greatest common divisorof m, n and define $\Phi(m, n)$, for m constant, by

$$\Phi(m, n) = 1 \text{ if } (m, n) = 1,$$
$$\Phi(m, n) = 0 \text{ if } (m, n) > 1.$$

Then $\Phi(m, n)$ is a numerical function of n. To indicate that the functions $A(n), \ldots, Q(n)$ have n prime to m, we may write them $A_m(n), \ldots, Q_m(n)$. Note that $\Phi(m, n) = \Phi(m, d)$, where d is any divisor of n. Hence if the theorems have been proved for $m = 1$, we may choose for $A_1(s), \ldots, Q_1(s)$ the numerical functions $\Phi(m, s) A_1(s), \ldots$ $\Phi(m, s) Q_1(s)$ respectively. In the resulting summations only those terms survive in which d, δ are prime to m, and hence the theorems follow with $A_m, \ldots, Q_m$ in place of $A_1, \ldots, Q_1$. It suffices therefore to prove the revised theorems with the summations referring to *all* pairs (d, δ) of divisors d, of n such that $n = d\delta$.

2. We say that the numerical functions $f(n)$, $g(n)$ are *equal*, and write $f = g$, if, and only if, $f(n) = g(n)$ for all integers $n > 0$. If $f(n)$, $g(n)$, $h(n)$ are any numerical functions of n such that

$$h(n) = \Sigma f(d) g(\delta),$$

for all integers $n > 0$, where the sum refers to all d, δ as defined at the end of § **1**, we write $h = fg$, and call fg the *product* of f, g. This multiplication is commutative, $fg = gf$, and associative, $(fg)k = f(gk)$, $k(n)$ being any numerical function of n. The numerical function $\varepsilon(n)$ defined by

$$\varepsilon(1) = 1, \qquad \varepsilon(s) = 0, \qquad s \neq 1,$$

is called the *unit* function, since $\varepsilon f = f$. A numerical function $f(n)$ is said to be *regular* if, and only if, $f(1) \neq 0$. If a numerical function $f'(n)$ exists such that $ff' = \varepsilon$ we call f' the *reciprocal* (or *inverse*) of f, and denote it by f^{-1}. It was shown in a previous paper (¹) that f^{-1} exists when, and only when, f is regular. Thus with respect to the multiplication above defined the set of all regular $f, g, \ldots$ is an abelian group.

If f is regular and $fg = fh$, then $g = h$, as we see on multiplying by f^{-1}. But if f is not regular, we cannot infer $g = h$. For, taking $n = 1$ in

$$\Sigma f(d) g(\delta) = \Sigma f(d) h(\delta), \qquad n = d\delta,$$

(¹) *Tôhoku Mathematical Journal*, 17, 1920, 221. A much simpler proof will be published in the same journal. A general sketch of the algebra of numerical functions is given, with references, in the *Journal of the Indian Mathematical Society* 17, 1927, 248-260.

we get $f(1)\,g(1) = f(1)\,h(1)$; and if $f(1) = 0$, then $g(1)$, $h(1)$ are arbitrary, contrary to the definition of $f(n)$, $g(n)$ as numerical functions.

Restating (1), (2) of § 1 in terms of this algebra of numerical functions, we may write them (with $m = 1$, as was seen to be sufficient),

$$AG = H, \qquad Q = AP;$$

and therefore, by multiplication in the algebra,

$$AGQ = APH. \tag{8}$$

Let A be regular, and multiply throughout by A^{-1}. Then $\varepsilon GQ = \varepsilon PH$; that is, $GQ = PH$, which is (3). If A is not regular, it cannot be eliminated from (8).

Similarly, if A, B in (4), (5) are regular, the given equations are

$$AG = BH, \qquad AP = BQ,$$

whence

$$AGBQ = BHAP,$$

and we multiply throughout by $A^{-1}B^{-1}$, which (on account of the commutativity and associativity) gives $QG = PH$, namely (6).