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**Some expansions in vector space**

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# *Some expansions in vector space;*

By A. D. MICHAL and R. S. MARTIN [1].

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1. Introduction. — The general theory of linear vector spaces has been studied by several authors [2]. The present paper is mainly concerned with the study of the properties of two expansions B(T) and D(T) in a special linear vector space S, closed under multiplication by numbers of a real or complex system A, and having additional properties abstracted from those of a space of linear transformations. The functions B(T) and D(T) are on S to S and S to A respectively. D(T) is analogous to the Fredholm determinant and B(T) to the first Fredholm minor. A simple derivation of some results on generalized rotations is incidentally given in Section 5.

2. Postulates and Definitions. — Let A denote the system of real or the system of complex numbers.

We consider a space S consisting of a set $\{T, U, \ldots\}$ of elements T, U, ... and five opérations

$$(\oplus), \quad (\odot), \quad (\otimes), \quad \|\ldots\|, \quad |\ldots|$$

satisfying the following postulates.

1. The set $\{T, U, \ldots\}$ forms a complete linear vector space with

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(2) *See*, for example, S. Banach, *Fundamenta Mathematicae*, vol. 3, 1922, p. 133-181, and M. Fréchet, *Espaces abstraits*, Paris, 1928. *See* also *Fréchet's fundamental papers on abstract spaces in*, *Rend. Circ. Mat. Palermo*, vol. 22, 1906, p. 1-74, and in *Journal de Mathématiques*, 1929, p. 71-92.

respect to addition, $\oplus$, multiplication, $\odot$, by numbers $a, b, \ldots$ of A, and the norm $\| \ldots \|$.

II. The set $\{T, U, \ldots\}$ forms a ring with respect to $\oplus$ and composition, $\otimes$, having a unique two-sided unit I, that is

$$I \otimes T = T \otimes I = T \qquad (\text{all } T).$$

III. $\|I\| = 1$.

IV. The product $T \otimes U$ is a bilinear operation of modulus unity on the space $S^2$ to S.

V. $[\ldots]$ is a homogeneous linear continuous operation on S to the number system A, having the property that

$$[T \otimes U] = [U \otimes T] \qquad (\text{all } T, U).$$

That the five postulates are consistent is shown by taking S to be A, the operations

$$(\oplus), \quad (\odot), \quad (\otimes), \quad \| \ldots \|,$$

to be ordinary addition, multiplication, and modulus respectively, and $[a]$ to be $a$.

With A the real number system and with the following interpretation of the elements and operations, the set of ordered pairs of functions $\big(T(x, y), T(x)\big)$ form a non-trivial instance of a space S. Let

$$T = \big(T(x, y), T(x)\big), \qquad U = \big(U(x, y), U(x)\big),$$
$$I = (0, 1),$$

where $T(x, y)$ and $U(x, y)$ are continuous on the square $a \leq x, y \leq b$, and $T(x)$ and $U(x)$ are continuous on the interval $(a, b)$. Take

$$T \oplus U = \big(T(x, y) + U(x, y), T(x) + U(x)\big),$$

$$a \odot T = \big(a\,T(x, y), a\,T(x)\big),$$

$$T \otimes U = \left(\int_a^b T(x, z)\,U(z, y)\,dz + T(x)\,U(x, y) + U(y)\,T(x, y), T(x)\,U(x)\right),$$

$$\|T\| = (b - a) \max_{x, y} |T(x, y)| + \max_x |T(x)|,$$

$$[T] = \int_a^b T(x, x)\,dx.$$

We may also take

$$\|T\| = c\,T(x_1),$$

where $c$ is a constant and $x_1$ is a fixed number in $(a, b)$.

Unless ambiguity arises we shall write $\oplus$ as simply $+$ and omit to write $\odot$ and $\otimes$. As immediate consequences of postulate (4) we have

$$(2.1) \qquad (a \odot T) \otimes U = T \otimes (a \odot U) = a \odot (T \otimes U)$$

and

$$(2.2) \qquad \|T \otimes U\| \leq \|T\| \cdot \|U\|.$$

For purposes of exposition it is convenient to make several definitions, some of whose further implications have been studied elsewhere (1).

*Definition I.* — By a polynomial $p(x)$ of degree $n$ on a vector space $V_1(A)$ to a vector space $V_2(A)$ we shall mean a continuous function $p(x)$ such that $p(x + \lambda y)$ is a polynomial of degree $n$ in $\lambda$ of the number system A, with coefficients in $V_2(A)$. If further

$$p(\lambda x) = \lambda^n p(x),$$

$p(x)$ is said to be homogeneous of degree $n$. It is convenient to consider a null polynomial as having any degree whatever.

*Definition II.* — By the modulus of a homogeneous polynomial of degree $n$ we shall mean

$$\max_x \frac{\|p(x)\|}{\|x\|^n}$$

and shall denote it by $mp$.

*Definition III.* — We shall say that a function $f(x)$ on $V_1(A)$ to $V_2(A)$ is analytic at a point $x_0$ if there exists a sequence $\{h_n(x)\}$ of homogeneous polynomials $\big(h_n(x)$ of degree $n\big)$ such that $\Sigma \lambda^n m h_n$ has a positive radius of convergence $r$, and such that for $\|x - x_0\| < r$, $\Sigma h_n(x - x_0)$ converges (in the norm) to $f(x)$. We shall call $r$ the

(1) R. S. Martin, *California Institute thesis*, 1932.

radius of analyticity of $f(x)$ at $x_0$ and shall often refer to such an $f(x)$ as analytic $(r)$ at $x = x_0$.

3. The Polynomials $B_n(T)$ and $a_n(T)$. — Let T be in S. Let $a_0 = 1$ and $B_0 = 0$ where 0 is the zero element of S. Define a sequence of numbers $\{a_n(T)\}$ and a sequence of elements $\{B_n(T)\}$ of S by means of the recurrence relations

$$B_n(T) = a_{n-1}(T)T - B_{n-1}(T)T, \tag{3.1}$$

$$a_n(T) = \frac{1}{n}[B_n(T)]. \tag{3.2}$$

We have

$$B_n(T) = \sum_{r=1}^{n} (-1)^{r-1} a_{n-r}(T) T^r. \tag{3.3}$$

$$n\, a_n(T) = \sum_{r=1}^{n} (-1)^{r-1} a_{n-r}(T) [T^r]. \tag{3.4}$$

Solving (3.4) (whose determinant is $n\,!$) we have

$$a_n(T) = \frac{1}{n\,!} \begin{vmatrix} [T] & [T^2] & \dots & . & [T^n] \\ n-1 & [T] & \dots & . & [T^{n-1}] \\ 0 & n-2 & \dots & . & [T^{n-2}] \\ . & \dots\dots & \dots & . & \dots\dots \\ 0 & 0 & \dots & 1 & [T] \end{vmatrix} \qquad (n > 0). \tag{3.5}$$

Furthermore

$$B_n(T) = \frac{1}{(n-1)\,!} \begin{vmatrix} T & T^2 & \dots & T^{n-1} & T^n \\ n-1 & [T] & \dots & [T^{n-2}] & [T^{n-1}] \\ 0 & n-2 & \dots & [T^{n-3}] & [T^{n-2}] \\ . & \dots\dots & \dots & \dots\dots & \dots\dots \\ 0 & 0 & \dots & 1 & [T] \end{vmatrix} \qquad (n > 0). \tag{3.6}$$

as can be seen by expanding (3.6) in its first row and using (3.3) and (3.5).

Theorem 3.I. — $B_n(T)$ *is a homogeneous polynomial of degree $n$ on the space* S *to the space* S. *The modulus of* $B_n(T)$ *satisfies the inequality*

$$m\, B_n \leq \frac{(\gamma+1)(\gamma+2)\dots(\gamma+n-1)}{(n-1)\,!} \qquad (n > 0). \tag{3.7}$$

*where $\gamma$ is the modulus of the operation $[\ldots]$. Similarly $a_n(\mathrm{T})$ is a homogeneous polynomial of degree $n$ on $\mathrm{S}$ to $\mathrm{A}$ and*

$$m\, a_n \leqq \frac{\gamma(\gamma+1)(\gamma+2)\ldots(\gamma+n-1)}{n!} \qquad (n>0). \tag{3.8}$$

***Proof*** : The polynomial property of $a_n(\mathrm{T})$ and $\mathrm{B}_n(\mathrm{T})$ is evident for $n=1$, for then $a_1(\mathrm{T})=[\mathrm{T}]$ and $\mathrm{B}_1(\mathrm{T})=\mathrm{T}$. Assuming the theorem true for $n-1$, the continuity of $\mathrm{B}_n(\mathrm{T})$ follows from the inequality

$$\begin{aligned}
\|\mathrm{B}_n(\mathrm{T})-\mathrm{B}_n(\mathrm{U})\| \leqq{} & \|a_{n-1}(\mathrm{T})\mathrm{T}-a_{n-1}(\mathrm{U})\mathrm{T}\|+\|a_{n-1}(\mathrm{U})\mathrm{T}-a_{n-1}(\mathrm{U})\mathrm{U}\| \\
& +\|\mathrm{B}_{n-1}(\mathrm{U})\mathrm{T}-\mathrm{B}_{n-1}(\mathrm{T})\mathrm{T}\|+\|\mathrm{B}_{n-1}(\mathrm{U})\mathrm{U}-\mathrm{B}_{n-1}(\mathrm{U})\mathrm{T}\| \\
\leqq{} & \big(|a_{n-1}(\mathrm{T})-a_{n-1}(\mathrm{U})|+\|\mathrm{B}_{n-1}(\mathrm{T})-\mathrm{B}_{n-1}(\mathrm{U})\|\big)\|\mathrm{T}\| \\
& +\big(|a_{n-1}(\mathrm{U})|+\|\mathrm{B}_{n-1}(\mathrm{U})\|\big)\|\mathrm{T}-\mathrm{U}\|
\end{aligned}$$

and that of $a_n(\mathrm{T})$ from the fact that $[\ldots]$ is a linear continuous operation.

Observing that $a_{n-1}(\mathrm{T}+\lambda\mathrm{U})$ and $\mathrm{B}_{n-1}(\mathrm{T}+\lambda\mathrm{U})$ are polynomials in $\lambda$ on $\mathrm{A}$ to $\mathrm{A}$ and $\mathrm{A}$ to $\mathrm{S}$ respectively we see at once from (3.1) that $\mathrm{B}_n(\mathrm{T}+\lambda\mathrm{U})$ is a polynomial in $\lambda$ on $\mathrm{A}$ to $\mathrm{S}$, say $\Sigma\lambda^r\mathrm{K}_r$. Then

$$a_n(\mathrm{T}+\lambda\mathrm{U})=\frac{1}{n}\Big[\sum\lambda^r\mathrm{K}_r\Big]=\sum\lambda^r\Big[\frac{\mathrm{K}_r}{n}\Big]$$

is a polynomial in $\lambda$ on $\mathrm{A}$ to $\mathrm{A}$. The homogeneity is obvious from (3.1) and (3.2).

The proof of the inequalities (3.7) and (3.8) is by induction and the use of the postulates.

## 4. The Expansions B(T) and D(T). — Theorem 4.1 :

$$\mathrm{B}(\mathrm{T})=\sum_1^\infty \mathrm{B}_n(\mathrm{T}) \qquad \textit{and} \qquad \mathrm{D}(\mathrm{T})=\sum_0^\infty a_n(\mathrm{T})$$

*are functions on $\mathrm{S}$ to $\mathrm{S}$ and $\mathrm{S}$ to $\mathrm{A}$ respectively analytic at $\mathrm{T}=0$. Their radii of analyticity are not less than unity.*

***Proof*** :

$$\Sigma\lambda^n m\,\mathrm{B}_n \quad \text{and} \quad \Sigma\lambda^n m\,a_n$$

are dominated respectively by

$$\sum_{1}^{\infty} \lambda^n \frac{(\gamma+1)\ldots(\gamma+n-1)}{(n-1)!} = \lambda(1-\lambda)^{-(\gamma+1)} \qquad (|\lambda|<1)$$

and

$$\sum_{0}^{\infty} \lambda^n \frac{\gamma(\gamma+1)\ldots(\gamma+n-1)}{n!} = (1-\lambda)^{-\gamma} \qquad (|\lambda|<1).$$

THEOREM 4.II. — *If* $|\lambda|<1$ *then*

$$\text{(4.1)} \qquad D(\lambda I) = (1+\lambda)^l$$

*and*

$$\text{(4.2)} \qquad B(\lambda I) = \lambda(1+\lambda)^{l-1} I$$

*where*

$$l = |I|.$$

*Proof*: From the homogeneity of $a_n(T)$ we have

$$a_n(\lambda I) = \lambda^n a_n(I)$$

and from (3.4)

$$\text{(4.3)} \qquad n\, a_n(I) = l \sum_{r=1}^{n} (-1)^{r-1} a_{n-r}(I).$$

The recursion formula (4.3) together with the condition $a_0 = 1$ determine the successive coefficients $b_n$ in the expansion of $(1+\lambda)^l$, for differentiating

$$\text{(4.4)} \qquad (1+\lambda)^l = \Sigma b_n \lambda^n$$

we obtain

$$\sum n b_n \lambda^{n-1} = l(1+\lambda)^{l-1} = l\left(\sum b_n \lambda^n\right)\frac{1}{1+\lambda} = l\left(\sum b_n \lambda^n\right)\sum(-1)^r \lambda^r.$$

Equating coefficients of $\lambda^n$ we see that $b_0 = 1$ and that $b_n$ satisfies the recursion formulæ (4.3) for $a_n(I)$. Hence

$$b_n = a_n(I).$$

Furthermore from (3.3) we see that

$$B_n(I) = \beta_n I$$

where

$$\beta_n = \sum_{r=1}^{n} (-1)^{r-1} a_{n-r}(\mathrm{I}) \qquad (\beta_0 = 0).$$

Since $\beta_0 = 0$ and since

$$\lambda(1+\lambda)^{l-1} = \left(\sum_0^\infty a_n(\mathrm{I})\lambda^n\right)\left(\sum_0^\infty (-1)^r \lambda^{r+1}\right)$$

it follows that $\beta_n$ is precisely the coefficient of $\lambda^n$ in the expansion of

$$\lambda(1+\lambda)^{l-1}.$$

COROLLARY 1. — *There exist spaces* S *in which the radii of analyticity about* $\mathrm{T} = 0$ *of* $\mathrm{B}(\mathrm{T})$ *and* $\mathrm{D}(\mathrm{T})$ *are unity.*

By a proper choice of the constant $c$ in the instance of a space S given in section (2) we may secure that $l$ is not a positive integer or zero.

Since $\|\mathrm{I}\| = 1$ we have that

$$m\, a_n \geqq |a_n(\mathrm{I})|$$

and

$$m\, \mathrm{B}_n \geqq |\beta_n|.$$

Hence the radii of convergence of the power series $\Sigma m a_n \lambda^n$ and $\Sigma m \mathrm{B}_n \lambda^n$ are respectively at most equal to the radii of convergence of $\Sigma |a_n(\mathrm{I})| \lambda^n$ and $\Sigma |\beta_n| \lambda^n$. The latter two series, however, have the same radius of convergence as $\Sigma a_n(\mathrm{I})\lambda^n$ and $\Sigma \beta_n \lambda^n$, namely unity. Combining this result with Theorem 4.I. we have the corollary.

COROLLARY 2. — *If the real part of* $[\mathrm{I}]$ *is positive then* $\mathrm{D}(-\mathrm{I}) = 0$.

We have

$$a_n(-\mathrm{I}) = (-1)^n b_n$$

where $b_n$ is given by (4.4).

If the real part of $l$ is positive then the series $\Sigma(-1)^n b_n$ is known to converge. Hence using Abel's theorem on convergence up to the unit circle we have

$$\Sigma a_n(-\mathrm{I}) = \lim_{\lambda \to -1-0} \Sigma b_n \lambda^n = \lim_{\lambda \to -1-0} (1+\lambda)^l = 0.$$

COROLLARY 3. — *If the real part of* $[I]$ *is negative then the series* $\Sigma a_n(-I)$ *representing* $D(-I)$ *is divergent.*

COROLLARY 4. — *If* $[I] \equiv 0$, *then* $D(-I) = 1$.

We observe that for fixed T, $B(\lambda T)$ and $D(\lambda T)$ are analytic functions of $\lambda$ near $\lambda = 0$ on A to S and A to A respectively and that their radii of analyticity are not less than $\frac{1}{\|T\|}$.

THEOREM 4.III. — *If* $\{E_n\}$ *is a sequence of elements of* S *such that* $E \equiv \Sigma E_n$ *converges in the norm, then* $\Sigma TE_n$ *and* $\Sigma E_n T$ *converge in the norm and are equal to* TE *and* ET *respectively.*

*Proof:* Let

$$C_m = \sum^{m} E_n,$$

then

$$\left\| \sum^{m} TE_n - TE \right\| = \| T(C_m - E) \| \leq \| T \| . \| C_m - E \|.$$

Since by hypothesis $\| C_m - E \|$ approaches zero with $\frac{1}{m}$, the first part of the theorem is proved. The proof of the second part is similar.

THEOREM 4.IV. — *If* $E(T) = \Sigma E_n(T)$, *where* $E_n(T)$ *is a homogeneous polynomial of degree n, is a function on* S *to* S *analytic at* $T = 0$, *then* $TE(T)$ *and* $E(T)T$ *are analytic on* S *to* S *and their radii of analyticity are at least as large as that of* $E(T)$.

*Proof:* That $TE_n(T)$ is a homogeneous polynomial of degree $n + 1$ is shown by a similar argument to that employed in the proof of Theorem 3.I From the inequality

$$\frac{\| TE_n(T) \|}{\| T \|^{n-1}} \leq \frac{\| T \|}{\| T \|} \frac{\| E_n(T) \|}{\| T \|^n} \leq m E_n$$

and Theorem 4.III the rest of Theorem 4.IV follows The proof is similar for $E(T)T$.

5. INVERSES AND ROTATIONS. — In this section we shall not make use

of postulate 5 for the space S. Hence throughout this section S will denote a space that satisfies the first four postulates.

*Definition.* — By a left-hand (right hand) inverse to an element T of S we shall understand an element $T_1$ such that

$$T_1 \otimes T = I \qquad \left(T \otimes T_1 = I\right).$$

A two-sided inverse $T^{-1}$ we shall call an inverse of T.

THEOREM 5.I. — *If* T *of* S *has at least one right hand inverse and at least one left hand inverse then it has unique right and left hand inverses and they are identical.*

*Proof :* Let $T_1$ be any left hand inverse and $T_2$ any right hand inverse of T. From

$$T \otimes T_2 = I,$$

we have

$$T_2 = I \otimes T_2 = T_1 \otimes T \otimes T_2 = T_1 \otimes I = T_1.$$

COROLLARY 1. — *If* T *has an inverse, then it is unique.*

COROLLARY 2. — *If* $T_1, T_2, \ldots, T_n$ *of* S *have inverses, then*

$$T_1 \otimes T_2 \otimes \ldots \otimes T_n.$$

*has an inverse, namely*

$$T_n^{-1} \otimes T_{n-1}^{-1} \otimes \ldots \otimes T_1^{-1}.$$

COROLLARY 3. — *If* T *has an inverse* $T^{-1}$ *then* $T^{-1}$ *has an inverse, namely* T.

THEOREM 5.II. — *If* $\|T\| < 1$, *then* $I + T$ *has an (unique) inverse, namely*

$$I + \sum_{n=1}^{\infty} (-1)^n T^n.$$

*Proof :* By successive applications of (2.2) we have

$$\|(-1)^n T^n\| \leqq \|T\|^n.$$

It follows from the completeness of S that $\sum_{n=1}^{\infty}(-1)^n T^n$ is in S. By an evident calculation and the use of Theorem 4.III, it is seen that

$$I + \sum_{n=1}^{\infty}(-1)^n T^n$$

is an inverse of $I + T$, which by Corollary 1 of the preceding theorem is unique.

Suppose now there exist an operation /, wich we shall call transposition, with the following three properties :

$$(a) \qquad 'T \text{ is a linear operation on S to S.}$$

$$(b) \qquad 'T'U = '(UT).$$

$$(c) \qquad '('T) = T.$$

It follows immediately from the properties $(b)$ and $(c)$ that $'I = I$. From $(b)$ and this result it follows that if T has an inverse then the transposed of T has an inverse given by the transposed of the inverse of T.

*Definition.* — If $I - \lambda T$ has an inverse $I + \lambda\Gamma(\lambda)$, then $\Gamma(\lambda)$ will be called the resolvent of $\lambda T$.

THEOREM 5.III. — *Suppose $I - \lambda T$ and $I - (\lambda + \mu)T$ have inverses, then if $\Gamma(\lambda)$ is the resolvent of $\lambda T$ it follows that $\Gamma(\lambda + \mu)$ is the resolvent of $\mu\Gamma(\lambda)$.*

*Proof :* By hypothesis

$$(I - \lambda T)\big(I + \lambda\Gamma(\lambda)\big) = I.$$

From this relation we obtain

$$(5.1) \qquad (I - \lambda T)\Gamma(\lambda) = T.$$

Consider an element J defined by

$$(5.2) \qquad J = \big(I - \mu\Gamma(\lambda)\big)\big(I + \mu\Gamma(\lambda + \mu)\big).$$

On applying (5.1) twice successively to $(I - \lambda T)J$ we obtain

$$(I - \lambda T)J = \big(I - (\lambda + \mu)T\big)\big(I + \mu\Gamma(\lambda + \mu)\big) = I - \lambda T.$$

Since $I - \lambda T$ has an inverse it follows that $J = I$ and hence the theorem is proved.

*Definition of a rotation.* — If $I + T$ has an inverse given by $I + {}'T$ then $T$ will be called a rotation.

THEOREM 5.IV. — *If* $T$ *is a rotation and if* $(\lambda - 1)T$ *has a resolvent* $\Gamma(\lambda - 1)$ *then* $-\lambda T$ *has a resolvent and*

$$\Gamma(\lambda - 1) = -{}'\Gamma(-\lambda). \tag{5.3}$$

*Proof:* Since $T$ is a rotation

$$'T = -\Gamma(-1). \tag{5.4}$$

By hypothesis and the previous theorem we have

$$\left(I - \lambda\Gamma(\lambda - 1)\right)(I + \lambda\,'T) = I. \tag{5.5}$$

Applying the properties of transposition, the transposed of (5.5) becomes

$$(I + \lambda T)\left(I + \lambda\,'\Gamma(\lambda - 1)\right) = I. \tag{5.6}$$

Hence the inverse of $I - (-\lambda)T$ exists and we have

$$\left(I - (-\lambda)T\right)\left(I - \lambda\Gamma(-\lambda)\right) = I. \tag{5.7}$$

By Corollary I to Theorem 5.I

$$'\Gamma(\lambda - 1) = -\Gamma(-\lambda).$$

From this result (5.3) follows immediately.

*Corollary.* — *If* $T$ *is a rotation and* $I + \frac{1}{2}T$ *has an inverse then*

$$H = \Gamma\left(-\tfrac{1}{2}\right)$$

*is defined and*

$$H = -{}'H.$$

THEOREM 5.V. — *If* $T$ *is a rotation such that* $I + \frac{1}{2}T$ *has an inverse, then* $T$ *is the résolvent of* $\frac{1}{2}H$ *where* $H$ *satisfies the relation*

$$'H = -H.$$

*Proof :* Take $\lambda = \frac{1}{2}$ in (5.7) and apply the previous corollary.

THEOREM 5.VI. — *If* $I - \frac{1}{2}T$ *has an inverse and if*

$$(5.8) \qquad {}'T = -T$$

*then the resolvent* $\Lambda\left(\frac{1}{2}\right)$ *of* $\frac{1}{2}T$ *is a rotation.*

*Proof :* By hypothesis we have

$$(5.9) \qquad \left(I + \frac{1}{2}\Lambda\left(\frac{1}{2}\right)\right)\left(I - \frac{1}{2}T\right) = I.$$

Transposing and making use of (5.8) we obtain

$$(5.10) \qquad \left(I + \frac{1}{2}T\right)\left(I + \frac{1}{2}{}'\Lambda\left(\frac{1}{2}\right)\right) = I.$$

Since by hypothesis $I - \frac{1}{2}T$ has an inverse and since $I + \frac{1}{2}T$ is the transposed of $I - \frac{1}{2}T$ it follows that $I + \frac{1}{2}T$ has an inverse, namely,

$$I - \frac{1}{2}\Lambda\left(-\frac{1}{2}\right).$$

Hence from (5.10) we have

$$(5.11) \qquad {}'\Lambda\left(\frac{1}{2}\right) = -\Lambda\left(-\frac{1}{2}\right).$$

Since $\Lambda\left(\frac{1}{2}\right)$ and $\Lambda\left(-\frac{1}{2}\right)$ exist we can apply Theorem 5.III and obtain

$$\left(I - \Lambda\left(\frac{1}{2}\right)\right)\left(I - \Lambda\left(-\frac{1}{2}\right)\right) = I.$$

With the aid of (5.11) we can write this relation in the form

$$\left(I + \Lambda\left(\frac{1}{2}\right)\right)\left(I - {}'\Lambda\left(\frac{1}{2}\right)\right) = I.$$

This proves the theorem.

**6. A Lemma on Sequences of Real Numbers Defined by Recurrent Inequalities.** — The results of the following lemma are necessary for the development of the succeeding sections.

Lemma. — *Let $\{r_n\}$ be a sequence of non-negative real numbers such that $\Sigma r_n$ converges. Let two sequences of non-negative real numbers $\{e_n\}$ and $\{f_n\}$ satisfy the recurrent inequalities*

$$(6.1)\qquad ne_n \geqq \mu\{\lambda(e_{n-1}+f_{n-1})+r_{n-1}\},$$

$$(6.2)\qquad f_n \geqq \lambda(e_{n-1}+f_{n-1})+r_{n-1},$$

$$e_0 = f_0 = 0,$$

*where $\mu \geqq 0$. Then if $0 \leqq \lambda \leqq 1$, $\Sigma e_n x^n$ and $\Sigma f_n x^n$ have radii of convergence at least unity. If furthermore $0 \leqq \lambda < 1$, $\Sigma e_n$ and $\Sigma f_n$ converge and their sums tend to zero with $\Sigma r_n$.*

*Proof*: Since $e_n$, $f_n$, $r_n$ are non-negative it is clearly sufficient to prove the lemma for the case where the inequalities (6.1) and (6.2) are replaced by the corresponding equalities.

Assume first that $\mu > 0$, $0 \leqq \lambda \leqq 1$. From the hypotheses on $\{r_n\}$ it is clear that $\Sigma r_n x^n$ converge uniformly and absolutely for $|x| \leqq 1$ and hence definies an analytic function $R(x)$ of $x$ in the interior of the unit circle.

Consider the differential equation

$$(6.3)\qquad u'(x) = \frac{\mu}{1-\lambda x}\left(\lambda u(x) + R(x)\right).$$

Let $E(x) \equiv \Sigma E_n x^n$ be the unique analytic solution satisfying the initial condition $E(0) = 0$. Let

$$(6.4)\qquad F(x) = \sum F_n x^n = x\frac{E'(x)}{\mu}.$$

We have at once

$$(6.5)\qquad \frac{E'(x)}{\mu} = \lambda\left(E(x) + F(x)\right) + R(x),$$

$$(6.6)\qquad \frac{F(x)}{x} = \lambda\left(E(x) + F(x)\right) + R(x).$$

Equating the coefficients of $x^{n-1}$ and observing that $F(0) = E(0) = 0$

we have

$$
(6.7) \qquad \begin{cases} nE_n = \mu[\lambda(E_{n-1} + F_{n-1}) - r_{n-1}], \\ F_n = \lambda(E_{n-1} - F_{n-1}) - r_{n-1}, \\ E_0 = F_0 = 0. \end{cases}
$$

Since the coefficients of differential equation (6.3) are analytic within the unit circle it follows that the solution $E(x)$, and hence $F(x)$, is analytic within the unit circle. For $|x| < 1$ we may write the solution $E(x)$ in the form

$$
(6.8) \qquad E(x) = (1 - \lambda x)^{-\mu} \int_0^x (1 - \lambda t)^{\mu - 1} R(t)\, dt.
$$

Furthermore for $0 \geq \lambda < 1$ the sums of the coefficients of $x^n$ in the expansions of $(1 - \lambda x)^{-\mu}$ and $(1 - \lambda x)^{\mu - 1}$ are absolutely convergent. Since the result of Cauchy multiplication of two power series the sums of whose coefficients are absolutely convergent is a power series having the same property, and since the property is preserved under term by term integration, it follows from (6.8) that $\Sigma E_n$ is convergent.

On differentiating (6.8) and applying a similar argument we see that $\Sigma F_n$ is convergent.

Applying Abel's theorem on continuity on the circle of convergence we have that $R(x)$ and $E(x)$ are continuous on the left at $x = 1$.

Hence

$$
\sum E_n = (1 - \lambda)^{-\mu} \int_0^1 (1 - \lambda t)^{\mu - 1} R(t)\, dt.
$$

Since

$$
R(t) \leq R(1) \qquad (0 \leq t \leq 1),
$$
$$
\Sigma E_n \leq M(\lambda, \mu) R(1) = M(\lambda, \mu) \Sigma r_n.
$$

By a similar argument we prove that

$$
\Sigma F_n \leq N(\lambda, \mu) \Sigma r_n.
$$

Assume now that $\mu = 0$. The reccurence relations (6.7) become

$$
(6.9) \qquad \begin{cases} E_n = 0, \\ F_n = \lambda F_{n-1} + r_{n-1}. \end{cases}
$$

We simply observe that the coefficients in the power series expansion of

$$F(x) = \frac{x\,B(x)}{1-\lambda x}$$

are given by (6.9). The rest of the argument is similar to that for $\mu > 0$.

**7. Fréchet differentiability of B(T) and D(T).** — If the number system A of a vector space $V_1(A)$ is the *complex number system* it can be shown by a general argument that a function $f(x) = \sum_{0}^{\infty} h_n(x)$ analytic $(r)$ at $x = 0$ on $V_1(A)$ to a complete $V_2(A)$ has for $\|x\| < r$ a Fréchet differential [1] $\delta f(x)$ given by $\sum_{0}^{\infty} \delta h_n(x)$ which, as a function of $x$, is also analytic $(r)$ at $x = 0$. For the special analytic functions B(T) and D(T) it is possible to give a direct proof of their term by term differentiability for an *unrestricted number system* A.

THEOREM 7.1. — *The functions* $D(T) \equiv \Sigma a_n(T)$ *and* $B(T) \equiv \Sigma B_n(T)$ *have Fréchet differentials* $\delta D$ *and* $\delta B$ *given for* $\|T\| < 1$ respectively by $\Sigma \delta a_n(T)$ *and* $\Sigma \delta B_n(T)$. *These last two series define analytic functions of* T *whose radii of analyticity are at least unity.*

*Proof* : By Theorem 3.1 and by a theorem on the differentials of polynomials proved elsewhere [2], it follows that $a_n T$ and $B_n(T)$ possess differentials for all T in S.

From the definition of a Fréchet differential we have that $\varepsilon_n(T, \delta T)$ and $\varepsilon'_n(T, \delta T)$ defined by

$$(7.1)\qquad \begin{cases} \|\delta T\|\,\varepsilon_n = \Delta a_n(T) - \delta a_n(T) & (\delta T \neq 0), \\ \varepsilon_n = 0 & (\delta T = 0); \end{cases}$$

$$(7.2)\qquad \begin{cases} \|\delta T\|\,\varepsilon'_n = \Delta B_n(T) - \delta B_n(T) & (\delta T \neq 0), \\ \varepsilon'_n = 0 & (\delta T = 0) \end{cases}$$

where

$$\Delta f(T) = f(T + \delta T) - f(T).$$

---

[1] FRÉCHET, *La notion de différentielle dans l'Analyse générale* (*Ann. Éc. Norm. sup.*, t. XLII, 1925, p. 293-323).

[2] MARTIN, *loc. cit.*

are continuous functions of $\delta T$ at $\delta T = 0$. Incorder to show Fréchet differentiability for $\|T\| < 1$ it is clearly sufficient to prove that for $\|T\| < 1$ $\Sigma \delta a_n(T)$ and $\Sigma \delta B_n(T)$ converge and that

$$\varepsilon(T, \delta T) = \Sigma \varepsilon_n(T, \delta T)$$

and

$$\varepsilon'(T, \delta T) = \Sigma \varepsilon'_n(T, \delta T)$$

converge for

$$\|T\| + \|\delta T\| < 1$$

and represent continuous functions of $\delta T$ at $\delta T = 0$.

From the recurrence formulæ (3.1) and (3.2), and from the evident identity

$$\Delta\big(U(T)\,V(T)\big) = \big(\Delta U(T)\big)\,V(T) + U(T + \delta T)\,\Delta V(T),$$

we have

$$(7.3) \qquad n\,\Delta a_n(T) = \big(\Delta a_{n-1}(T)\big)[T] + a_{n-1}(T + \delta T)[\delta T] + \Big[\big(\Delta B_{n-1}(T)\big)T + B_{n-1}(T + \delta T)\,\delta T\Big].$$

$$(7.4) \qquad \Delta B_n(T) = \big(\Delta a_{n-1}(T)\big)T + a_{n-1}(T + \delta T)\,\delta T + \big(\Delta B_{n-1}(T)\big)T + B_{n-1}(T + \delta T)\,\delta T.$$

Taking norms of (7.3) and (7.4) we obtain

$$n|\Delta a_n(T)| \leqq \gamma\Big\{\big(|\Delta a_{n-1}(T)| + \|\Delta B_{n-1}(T)\|\big)\|T\| + \big(|a_{n-1}(T + \delta T)| + \|B_{n-1}(T + \delta T)\|\big)\|\delta T\|\Big\},$$

$$\|\Delta B_n(T)\| \leqq \big(|\Delta a_{n-1}(T)| + \|\Delta B_{n-1}(T)\|\big)\|T\| + \big(|a_{n-1}(T + \delta T)| + \|B_{n-1}(T + \delta T)\|\big)\|\delta T\|,$$

where $\gamma$ is the modulus of the operation $[\ldots]$. By Theorem 3.1 the quantities

$$r_n = \big(|a_n(T + \delta T)| + \|B_n(T + \delta T)\|\big)\|\delta T\|$$

are for $\|T\| + \|\delta T\| < 1$ the terms of an absolutely convergent series.

Hence from the preceding Lemma taking

$$e_n = |\Delta a_n(T)|, \qquad f_n = \|\Delta B_n(T)\|, \qquad \lambda = \|T\|, \qquad \mu = \gamma$$

and $r_n$ as defined above we have that $\Sigma|\Delta a_n(T)|$ and $\Sigma\|\Delta B_n(T)\|$ converge and tend to zero with $\Sigma r_n$ and hence with $\|\delta T\|$.

Making use of the formulae, valid when $a$, $U$, $V$ are differentiable

$$\partial\left(a \,(\widehat{\cdot})\, U\right) = (\partial a)\,(\widehat{\cdot})\, U,$$
$$\partial\left(U \,(\widehat{\times})\, V\right) = (\partial U)\,(\widehat{\times})\, V + U \,(\widehat{\times})\, \partial V,$$
$$\partial[U] = [\partial U]$$

we have again from (3.1) and (3.2)

$$(7.5) \qquad n\,\partial a_n(T) = \left(\partial a_{n-1}(T)\right)[T] + a_{n-1}(T)[\partial T] + \left[\left(\partial B_{n-1}(T)\right)T + B_{n-1}(T)\,\partial T\right].$$
$$(7.6) \qquad \partial B_n(T) = \left(\partial a_{n-1}(T)\right)T + a_{n-1}(T)\,\partial T + \left(\partial B_{n-1}(T)\right)T + B_{n-1}(T)\,\partial T.$$

Substracting (7.5) from (7.3) and (7.6) from (7.4) we obtain, using (7.1) and (7.2)

$$(7.7) \qquad n\|\partial T\|\varepsilon_n = \|\partial T\|\varepsilon_{n-1}[T] + \left(\Delta a_{n-1}(T)\right)[\partial T] + \left[\|\partial T\|\varepsilon'_{n-1}T + \left(\Delta B_{n-1}(T)\right)\partial T\right]$$

and

$$(7.8) \qquad \|\partial T\|\varepsilon'_n = \|\partial T\|\varepsilon_{n-1}T + \left(\Delta a_{n-1}(T)\right)\partial T + \|\partial T\|\varepsilon'_{n-1}T + \left(\Delta B_{n-1}(T)\right)\partial T.$$

Taking norms of (7.7) and (7.8) we obtain the inequalities

$$(7.9) \qquad n|\varepsilon_n| \leqq \gamma\left(|\varepsilon_{n-1}|\,\|T\| + |\Delta a_{n-1}| + \|\varepsilon'_{n-1}\|\,\|T\| + \|\Delta B_{n-1}\|\right).$$
$$(7.10) \qquad \|\varepsilon'_n\| \leqq |\varepsilon_{n-1}|\,\|T\| + |\Delta a_{n-1}| + \|\varepsilon'_{n-1}\|\,\|T\| + \|\Delta B_{n-1}\|.$$

Again Applying the Lemma to (7.9) and (7.10) and using what we have just proved for $\Sigma|\Delta a_n|$ and $\Sigma\|\Delta B_n\|$ we obtain the result that $\Sigma|\varepsilon_n|$ and $\Sigma\|\varepsilon'_n\|$ converge and tend to zero with $\partial\|T\|$.

Taking norms of (7.5) and (7.6), dividing by $\|T\|^{n-1}$ and taking the maximum of both sides quà T we have

$$(7.11) \qquad n(m\,\partial a_n) \leqq \gamma\{(m\,\partial a_{n-1}) + (m\,\partial B_{n-1}) + (m a_{n-1} + m B_{n-1})\|\partial T\|\}.$$
$$(7.12) \qquad (m\,\partial B_n) \leqq (m\,\partial a_{n-1}) + (m\,\partial B_{n-1}) + (m a_{n-1} + m B_{n-1})\|\partial T\|.$$

Let $\gamma_1$ be a positive number less than unity. Multiply both sides of (7.11), (7.12) by $\gamma_1^{n-1}$ and apply the Lemma taking

$$e_n = \gamma_1^{n-1} m\,\partial a_n, \qquad f_n = \gamma_1^{n-1} m\,\partial B_n, \qquad \lambda = \gamma_1, \qquad \mu = \gamma$$

and

$$r_n = \eta^{n-1}(m a_{n-1} + m B_{n-1}) \| \delta T \|.$$

Since by Theorem 4.I, $\Sigma r_n$ converges it follows at once that

$$\Sigma \eta^{n-1}(m\, \delta a_n)$$

and

$$\Sigma \eta^{n-1}(m\, \delta B_n)$$

have radii of convergence not less than unity. Thus $\Sigma\, \delta a_n$ and $\Sigma\, \delta B_n$ represent functions analytic $(r \geqq 1)$ at $T = 0$.

Finally from the convergence of $\Sigma \| \varepsilon_n \|$ and $\Sigma \| \varepsilon'_n \|$, and the completeness of the space S we have that $\Sigma \varepsilon_n$ and $\Sigma \varepsilon'_n$ converge in the norm, and that $\| \Sigma \varepsilon_n \|$ and $\| \Sigma \varepsilon'_n \|$ tend to zero with $\| \delta T \|$. Summing (7.1) and (7.2) we have for $\| T \| + \| \delta T \| < 1$

$$\Delta D(T) - \Sigma\, \delta a_n(T) = \| \delta T \|\, \varepsilon(T, \delta T),$$
$$\Delta B(T) - \Sigma\, \delta B_n(T) = \| \delta T \|\, \varepsilon'(T, \delta T).$$

This completes the proof of the theorem.

8. The inverse of $I + T$ for $\| T \| < 1$. Theorem 8.I. — *If* $\| T \| < 1$ *then*

$$I - \frac{B(T)}{D(T)}$$

*is an inverse to* $I + T$.

*Proof*: We have for $\| T \| < 1$

$$(8.1) \qquad D(T)T - B(T) - B(T)T = 0,$$

for from the definitions of $D(T)$ and $B(T)$, the recurrence formulæ (3.1) and (3.2), Theorem 4.III and the vanishing of $B_0(T)$ we have

$$D(T)T - B(T)T = \sum_{0}^{\infty} \big( a_n(T)T - B_n(T)T \big) = \sum_{0}^{\infty} B_{n+1}(T) = B(T).$$

By a simple calculation we see from (8.1) that the theorem is proved for those values of T for which $D(T) \neq 0$. For all T for which $\| T \| < 1$ and $D(T) \neq 0$ we have by Theorem 5.II

$$(8.2) \qquad \frac{B(T)}{D(T)} = \sum_{1}^{\infty} (-1)^{n-1} T^n.$$

Let $T_1$ be any chosen T such that $\|T_1\| < 1$. From the remark preceding Theorem 4.III we have that

$$D(\lambda T_1) = \sum_0^\infty \lambda^n a_n(T_1)$$

is an analytic function on A to A for

$$|\lambda| < \frac{1}{\|T_1\|}.$$

From the recursion formula (3.2) we have

$$(8.3) \qquad \lambda \frac{d}{d\lambda} D(\lambda T_1) = \sum_0^\infty n \lambda^n a_n(T_1) = \sum_0^\infty \lambda^n | B_n(T_1) |$$
$$= \sum_0^\infty | B_n(\lambda T_1) | = | B(\lambda T_1) |.$$

For all $\lambda$ in A such that $|\lambda| < \frac{1}{\|T_1\|}$ and $D(\lambda T_1) \neq 0$ we have using (8.3) and (8.2) with $T = \lambda T_1$

$$(8.4) \qquad \frac{\sum_0^\infty n \lambda^{n-1} a_n(T_1)}{\sum_0^\infty \lambda^n a_n(T_1)} - \sum_0^\infty (-1)^n \lambda^n | T_1^{n+1} | = 0.$$

Since we clearly have

$$|(-1)^n | T_1^{n+1} || \leqq \gamma \|T_1\|^{n+1}$$

it follows that

$$\sum_0^\infty (-1)^n \lambda_n | T_1^{n+1} |$$

definies an analytic function of $\lambda$ for $|\lambda| < \frac{1}{\|T_1\|}$.

The left hand side of (8.4) regarded as a function of a general complex variable $\lambda$ is analytic in the region $|\lambda| < \frac{1}{\|T_1\|}$ except possibly for poles, and vanishes for $\lambda$ in A and in the above region except at the zeros of $D(\lambda T_1)$, that is, it has zeros which are not isolated. It follows [1] that (8.4) is an identity for complex $\lambda$ in the region $|\lambda| < \frac{1}{\|T_1\|}$. Let C be a circle of radius $\rho\left(1 < \rho < \frac{1}{\|T_1\|}\right)$ whose center is at the origin and whose circumference passes trough none of the zeros (if any) of $\sum_0^\infty \lambda^n a_n(T_1)$. Integrating (8.4) around C we have

$$\int_C \frac{\frac{d}{d\lambda}\left(\sum_0^\infty \lambda^n a_n(T_1)\right)}{\sum_0^\infty \lambda^n a_n(T_1)} d\lambda = 0.$$

It follows immediately from a well known theorem that $\sum_0^\infty \lambda^n a_n(T_1)$ has no zeros inside C. Placing $\lambda = 1$ we see that $D(T_1) \neq 0$. But $T_1$ is any T in the region $\|T\| < 1$. The argument of the first part of the proof is therefore valid for all T in $\|T\| < 1$.

*Corollary* 1. — *For any chosen* $T_1$, *if* $|\lambda| < \frac{1}{\|T_1\|}$, *then* $D(\lambda T_1)$ *is not zero.*

*Corollary* 2. — *For* $\|T\| < 1$

$$\log D(T) = \sum_1^\infty (-1)^{n-1} \frac{|T^n|}{n}. \tag{8.5}$$

*Proof:* Integrate (8.4) from 0 to 1.

**9. The Product Theorem for D(T). Theorem 9.1.** — *The differential of the polynomial* $a_n(T)$ *is given by*

$$\delta a_n(T) = a_{n-1}(T)[\delta T] - [B_{n-1}(T)\delta T] \qquad (n > 0). \tag{9.1}$$

*Proof:* We first establish by induction the formula, évident for $n = 1$,

$$\delta a_n(T) = \sum_0^{n-1} (-1)^p a_{n-p-1}(T)[T^p \delta T]. \tag{9.2}$$

---

[1] E. Goursat, *Cours d'Analyse mathématique*, 4e édition, t. II, p. 98.

Differentiation of (3.4) gives

$$(9.3)\qquad n\,\delta a_n(T) = \sum_{j=0}^{n-1} (-1)^j (\delta\alpha_{n-j-1})[T^{j+1}] + \sum_{j=0}^{n-1} (-1)^j (j+1) a_{n-j-1}[T^j \delta T].$$

Assuming (9.2) for (1, 2, ..., *n*-1), substituting the result in the first sum on the right hand side of (9.3), reversing the summations and making use of (3.4), we have

$$n\,\delta a_n(T) = \sum_{r=0}^{n-1} (-1)^r (n-r-1) a_{n-r-1}[T^r \delta T] + \sum_{j=0}^{n-1} (-1)^j (j+1) a_{n-j-1}[T^j \delta T]$$

which gives (9.2)

From (3.3) we have

$$(9.4)\qquad [B_{n-1}(T)\,\delta T] = \sum_{r=1}^{n-1} (-1)^{r+1} a_{n-r-1}[T^r \delta T].$$

The theorem follows by adding (9.2) and (9.4).

With the aid of Theorems 7.I and 9.I we obtain the corollary.

COROLLARY. — *The differential* $\delta D(T)$ *is given for* $\|T\| < 1$ *by*

$$\delta D(T) = D(T)[\delta T] - [B(T)\,\delta T].$$

THEOREM 9.II. — *If* $\|T\| < 1$, $\|U\| < 1$ *and if for* $0 \leqq \lambda \leqq 1$

$$\|\lambda(T+U) + \lambda^2 TU\| < 1$$

*then*

$$(9.5)\qquad D(T+U+TU) = D(T)\,D(U).$$

*Proof* : By successive use of a theorem (¹) (proved elsewhere) on the differentials of a function of a function in vector spaces, by the

(¹) M. FRÉCHET, *Annales sc. de l'École Normale supérieure*, vol. 42, 1925. *See* also R. S. MARTIN, *loc. cit.*

corollary to Theorem 9.1 and by Corollary 1 to Theorem 8.1 we have for $|\lambda| \leqq 1$

$$\frac{d}{d\lambda}\log D(\lambda T) = [T] - \left[\frac{B(\lambda T)}{D(\lambda T)}T\right]$$

which may be written in the convenient form

$$(9.6) \qquad \frac{d}{d\lambda}\log D(\lambda T) = [\overline{T}(\lambda T)T],$$

where

$$\overline{T}(\lambda T) = I - \frac{B(\lambda T)}{D(\lambda T)},$$

is the inverse of $I + \lambda T$. Similarly

$$(9.7) \qquad \frac{d}{d\lambda}\log D(\lambda U) = [\overline{U}(\lambda U)U].$$

Let $W(\lambda)$ be defined by means of

$$(9.8) \qquad I + W(\lambda) = (I + \lambda T)(I + \lambda U).$$

Then

$$(9.9) \qquad \frac{d}{d\lambda}\log D(W(\lambda)) = \left[\frac{dW}{d\lambda}\right] - \left[\frac{B(W)}{D(W)}\frac{dW}{d\lambda}\right] = \left[\overline{W}\frac{dW}{d\lambda}\right].$$

where $\overline{W}$ is the inverse of $I + W(\lambda)$. By Corollary 2 to Theorem 5.1

$$\overline{W} = \overline{U}\,\overline{T}.$$

Differentiating (9.8) we have

$$\frac{dW}{d\lambda} = (I + \lambda T)U + T(I + \lambda U).$$

Placing the last two results in (9.9) and making spécial use of postulate 5 for the space S, we have

$$\frac{d}{d\lambda}\log D(W(\lambda)) = [\overline{T}T] + [\overline{U}U].$$

Hence by (9.6) and (9.7)

$$(9.10) \qquad \frac{d}{d\lambda}\log D(W(\lambda)) = \frac{d}{d\lambda}\log D(\lambda T) + \frac{d}{d\lambda}\log D(\lambda U).$$

Since the derivatives in (9.10) are in the ordinary sense we may integrate from O to 1 and take exponentials. This gives (9.5).

COROLLARY. — *If* $|T| < \sqrt{2} - 1$ *and* $|U| < \sqrt{2} - 1$ *then the conclusions* (9.5) *hold.*

**10.** *Concluding Remarks.* — If the elements T of the space S are matrices $(t_j^i)$ of a finite order $s$ and if

$$|T| = \sum_{i,j}^{s} t_j^i$$

then $a_r$ is precisely the coefficient of $\lambda^r$ in the determinant

$$D = |\delta_j^i + \lambda t_j^i| \quad [1]$$

and $B_r$ is given by $a_r I - A_r$ where $A_r$ is the coefficient of $\lambda^r$ in the adjoint of D so that for $r > s$, $B_r = 0$. The equation (3.3) for $n = s + 1$ with the condition $B_{s+1} = 0$ is equivalent to the theorem that T satisfies its characteristic equation. In the case of a general space S, if T is such a point that D(T) and B(T) converge, then $|B_r| \to 0$ with $\frac{1}{r}$ and one may, if he so chooses, regard the limiting form of (3.3) as the generalization of the algebraic theorem.

[1] Where $\delta_j^i = \begin{cases} 1 & \textit{if } i = j, \\ 0 & \textit{if } i \neq j. \end{cases}$