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*On the fundamental formulæ of the geometry
of tensor submanifolds;*

By P. DIENES.

General conventions.

I. The summation symbol Σ is suppressed if it applies to terms with two identical suffixes.

II. The first letters of Latin and Greek alphabets — as suffixes — vary from 1 to N, the midletters, $i, j, k, \dots, o; \iota, \kappa, \lambda, \dots, \phi$, from 1 to m , the endletters $p, r, s, \dots; \pi, \rho, \sigma, \dots$ from $m+1$ to N.

III. Greek suffixes are used in initial mathematical axes and Latin suffixes in general geometrical axes.

IV. Structure tensors are denoted by Roman capitals.

1. *The idea of tensor geometry.* — All possible groups $(x^1, x^2, \dots, x^N) \equiv (x^z)$ of N real numbers, also called points, make up the N dimensional mathematical point manifold X_N . When x^z only varies, the other $N-1$ variables $x^1, \dots, x^{z-1}, x^{z+1}, \dots, x^N$ being kept constant, the points so obtained form the z th parametric line of X_N . N different parametric lines pass through every point.

In the construction of every geometry, the directed interval $(x^z, x^z + \Delta x^z)$, *i. e.* all the real numbers between x^z and $x^z + \Delta x^z$ in their natural order of magnitude, is made to correspond to our visual notion of the direction at (x^z) of the line we called the z th parametric line. These N directions at P are symbolised by the N contravariant

base vectors $e_a(P)$, attached to P . This (mathematical) system of local coordinate axes is completed by the N covariant (mathematical) base vectors $e^a(P)$ representing the coordinate hyperplanes.

A tensor at P is, by definition, a polynomial of any finite degree homogeneous both in $e_a(P)$ and in $e^a(P)$. Linear forms in e_a or in e^a are also called contravariant and covariant *vectors* respectively.

In a change of variables $x^z = x^z(y^1, \dots, y^N)$, the increments dx^z are transformed by $dx^z = A_z^{z'} dy^{z'}$ where $A_z^{z'} = \frac{\partial x^{z'}}{\partial x^z}$. Hence $e_a dx^z = e_a A_z^{z'} dy^{z'}$ and thus

$$(1) \quad e_{z'} = A_z^{z'} e_z.$$

Putting $A_z^{z'} = \frac{\partial y^{z'}}{\partial x^z}$ so that

$$A_z^{z'} A_{z'}^{z''} = \delta_z^{z''}, \quad A_z^{z'} A_{z'}^{z''} = \delta_z^{z''},$$

we have for the inverse transformation

$$(2) \quad e_z = A_z^{z'} e_{z'}.$$

We notice that $A_z^{z'}$ and $A_{z'}^{z''}$ denote *different* sets of functions. For instance, A_{z-1}^{z-1} is, in general, different from A_z^{z-1} or from A_{z-1}^{z-1} .

Covariant vectors and tensors might be transformed independently of contravariant tensors. For geometrical purposes, however (See Schouten, 1929, p. 415) it is convenient to put

$$(3) \quad e_{z'} = A_z^{z'} e_z.$$

from which it follows that

$$(4) \quad e_z = A_z^{z'} e_{z'}.$$

If we drop the condition that $dx^z = A_z^{z'} dy^{z'}$ is an exact differential, but still suppose that the rank of the determinant of $A_z^{z'}$ is N , formulae (1) — (4) determine a general *change of base* replacing the original mathematical axes by an arbitrary system of independent axes (base vectors) referred to as (general) *geometrical axes* (Schouten's nonholonomic parameters).

A manifold of tensors becomes a geometrical manifold or space if (a) we define local metric, i. e. length and angle, by means of a quadratic form $g_{\alpha\beta}dx^\alpha dx^\beta$ with arbitrary coefficients $g_{\alpha\beta}(x^\alpha) = g_{\beta\alpha}(x^\alpha)$, (b) we dovetail the local geometrical tensor manifolds so constructed by a definition of equivalence between tensors at neighbouring points. Such a connexion replaces the congruence axioms. For further detail see Schouten 1924 and Dienes 1924, 1926.

2. Tensor submanifolds. — Tensor submanifolds are usually defined in three steps. (I) A point submanifold X_m of X_N is selected, (II) a pencil of tensors is assigned to the points of X_m , (III) metric and connexion are assigned to the selected pencil. In this Art. we shall discuss (I).

The points of X_m may be selected by $N - m$ equations

$$(1) \quad \Phi^r(x^1, \dots, x^N) = 0,$$

the Jacobian of Φ^r having the rank m . To obtain a parametric representation of X_m we complete (1) by m arbitrary functions Φ^i such that the Jacobian of the complete system Φ^z be different from zero and we put

$$(2) \quad u^z = \Phi^z(x^1, \dots, x^N),$$

Solving for x^z we have

$$(3) \quad x^z = \Psi^z(u^1, \dots, u^N),$$

and the points of X_m are characterized by the $N - m$ conditions $u^r = 0$, so that

$$(4) \quad x^z = \Psi^z(u^1, \dots, u^m, 0, \dots, 0) \equiv \dot{\Psi}^z(u^1, \dots, u^m),$$

form a system of parametric equations for X_m .

We put

$$(5) \quad B^i_z = \frac{\partial \Phi^i}{\partial x^z}, \quad C^r_z = \frac{\partial \Phi^r}{\partial x^z}, \quad B^z_i = \frac{\partial \Psi^z}{\partial u^i}, \quad C^z_r = \frac{\partial \Psi^z}{\partial u^r},$$

and notice that

$$(6) \quad \begin{cases} B^i_z B^z_\alpha = \delta^i_\alpha, & B^i_z C^z_r = 0, & B^z_\alpha C^z_r = 0, \\ C^z_r C^z_\alpha = \delta^z_\alpha, & B^i_z B^z_\alpha + C^z_r C^z_\alpha = \delta^i_\alpha. \end{cases}$$

If (x^z) is in X_m , the necessary and sufficient conditions that also $(x^z + dx^z)$ be in X_m are

$$(7) \quad C_x^z dx^z = du^z = 0.$$

Since dx^z is the prototype of contravariant vectors, v^z is considered as a vector of X_m if and only if the N functions v^z satisfy the $N - m$ conditions

$$(8) \quad C_x^z v^z = 0.$$

The rank of the matrix (C_x^z) being $N - m$, if $v_1^z, \dots, v_m^z$ are m independent sets of solutions to (8), every other set of solutions is of the form $c_1 v_1^z + \dots + c_m v_m^z$. It follows from (6)₃ that B_λ^z form m independent sets of solutions to (8), and thus every contravariant vector of X_m is a form in the m independent contravariant vectors

$$(9) \quad e = B_\lambda^z e_\lambda.$$

The corresponding set of covariant vectors $e = x_\lambda^z e_\lambda$ is determined by the equations

$$(10) \quad x_\lambda^z B_\mu^z = \delta_\mu^\lambda,$$

and, from (6)₁, B_λ^z are solutions of (10) so that every covariant vector of X_m is a form in the m independent covariant vectors

$$(11) \quad e = B_\lambda^z e_\lambda.$$

Thus, the tensors of X_m are the forms homogeneous in both systems of base vectors e_λ and e^λ . We add that the vectors

$$(12) \quad e_\sigma = C_\sigma^z e_\lambda, \quad e^\sigma = C_\lambda^z e^\lambda,$$

form a complementary base called the *span* (Spannung) of X_m . The tensors of the span are the forms in e_σ, e^σ . The set of vectors $(e_\lambda, e^\lambda; e_\sigma, e^\sigma)$ is a new mathematical base for X_N , called the *split base*.

The definition of the tensors of X_m hinges on (1). Now, there are

dynamical systems called nonholonomic in which the points i. e. the independent variables, generalised coordinates, are subjected to no restriction, but the changes dx^z of the independent variables are restricted by linear relations, not necessarily exact. This means that in such a case step (1) in the construction of the submanifold is cancelled, and the gradients $C_\alpha^z = \frac{\partial \Phi^z}{\partial x^\alpha}$ in (7) are replaced by arbitrarily assigned functions denoted again by C_α^z . This idea of a nonholonomic manifold has been introduced by Vranceanu (1928), and further developed by Schouten (1929) cf « Papers quoted » in fine), who denotes it by X_x^m .

The admitted contravariant vectors (directions) of X_x^m will again be determined by (8), i. e. any m independent sets of solution to (8), denoted again by B_λ^z , lead to a set of m contravariant base vectors (9). For C_β^z we take a fixed set of independent solutions of

$$(13) \quad C_\alpha^z x_\beta^z = \delta_\beta^\alpha.$$

and putting for a moment $a_{\gamma=\lambda}^z = B_\lambda^z$, $a_{\gamma=\sigma}^z = C_\sigma^z$, the unique set of solutions to the equations

$$(14) \quad a_\gamma^z x_\alpha^z = \delta_\alpha^\gamma.$$

will be denoted by

$$(15) \quad x_\alpha^{\beta=\lambda} = B_\lambda^\beta, \quad x_\alpha^{\beta=\sigma} = C_\sigma^\beta,$$

so that all the five equations (6) will be satisfied. In this case, (11) and (12) determine the remaining base vectors, and together with (9), lead to the initial mathematical axes in X_x^m . These mathematical axes or base vectors, whose determination hinges on the originally given C_α^z , may be replaced by arbitrary « geometrical » base vectors or axes ($e_i, e_i^{\prime}, e_i^{\prime\prime}$) for which Latin suffixes will be used. By definition *forms* in $e_i, e_i^{\prime}, e_i^{\prime\prime}$ or in $e_i, e_i^{\prime}$ are tensors in the (admitted) facet; forms in $e_i, e_i^{\prime}$ or in $e_i, e_i^{\prime\prime}$ those in its span.

The difference between holonomic and non holonomic submanifolds, as manifested in the determination of the tensors of the submanifold, is that, in the nonholonomic case, $B_i^z, B_\alpha^z, C_\sigma^z, C_\alpha^z$ are not necessarily

gradients. The determination of the tensors of the submanifold amounts to splitting the base of X_N into two complementary sets, of dimensions m and $N - m$ respectively. In the holonomic case this process is applied at the points of X_m only, whereas for X_N^m it applies at *every* point of X_N .

The construction of a geometry for such tensor submanifold still requires a definition of connexion and metric in X_N^m (see Arts. 4 and 11). So far [see in particular Levi-Civita (1917) and Schouten (1918, 1924, 1929)] the connexion in X_m or in X_N^m has been defined as the projection of the connexion in X_N . In this paper we shall consider the general case of an arbitrarily assigned connexion in X_N^m . Moreover, we shall establish all the fundamental formulae in general geometrical axes, and shall show that the fundamental formulae of the geometry of tensor submanifolds can all be obtained in a systematic way by the method of resolution of tensors. In this way, we shall obtain, in particular, the extension to X_N^m of the equation of Gauss and Codazzi, and that of Kühne, established for the holonomic case by Schouten (1924, p. 140) and Hlavaty (1926) respectively. Some other equations obtained by the same method seem to be new.

3. Change of base. Resolution of tensors. — In this paper we shall consider the mathematical axes of X_N and X_N^m i. e. the initial mathematical base vectors (e_x, e^x) and $(e_\lambda, e^\lambda; e_\sigma, e^\sigma)$ as fixed once for all. The passage from these mathematical axes to general geometrical axes means to replace e_x, e_λ and e_σ by $N, m, N - m$ arbitrary, but independent, linear combinations of e_x, e_λ and e_σ respectively

$$(1) \quad e_a = A_a^x e_x, \quad e_i = B_i^\lambda e_\lambda, \quad e_\rho = C_\rho^\sigma e_\sigma.$$

the corresponding inverse transformation is obviously given by

$$(2) \quad e_x = A_x^a e_a, \quad e_\lambda = B_\lambda^i e_i, \quad e_\sigma = C_\sigma^\rho e_\rho$$

where

$$(3) \quad A_x^a A_b^x = \delta_b^a, \quad A_x^a A_a^x = \delta_x^x,$$

$$(4) \quad B_\lambda^i B_j^i = \delta_j^\lambda, \quad B_\lambda^i B_i^\lambda = \delta_\lambda^\lambda, \quad C_\sigma^\rho C_\rho^\sigma = \delta_\sigma^\sigma, \quad C_\sigma^\rho C_\rho^\sigma = \delta_\sigma^\sigma.$$

By convention, the transformation of the covariant base is given by

$$(5) \quad \begin{matrix} a \\ e \end{matrix} = A_a^{\alpha} \begin{matrix} \alpha \\ e \end{matrix}, \quad \begin{matrix} i \\ e \end{matrix} = B_i^{\lambda} \begin{matrix} \lambda \\ e \end{matrix}, \quad \begin{matrix} p \\ e \end{matrix} = C_p^{\sigma} \begin{matrix} \sigma \\ e \end{matrix}.$$

$$(6) \quad \begin{matrix} \alpha \\ e \end{matrix} = A_a^{\alpha} \begin{matrix} a \\ e \end{matrix}, \quad \begin{matrix} \lambda \\ e \end{matrix} = B_i^{\lambda} \begin{matrix} i \\ e \end{matrix}, \quad \begin{matrix} \sigma \\ e \end{matrix} = C_p^{\sigma} \begin{matrix} p \\ e \end{matrix}.$$

In a like manner we may pass from one system of geometrical axes

$$(e_a, e_i, e_p)$$

to another

$$(7) \quad \left\{ \begin{array}{ll} e_a = A_a^{\alpha} e_{\alpha}, & e^{\alpha} = A_a^{\alpha} e_a, \\ e_i = B_i^{\lambda} e_{\lambda}, & e^{\lambda} = B_i^{\lambda} e_i, \\ e_p = C_p^{\sigma} e_{\sigma}, & e^{\sigma} = C_p^{\sigma} e_p \end{array} \right.$$

where either A_a^{α} , B_i^{λ} , and C_p^{σ} or $A_a^{\alpha'}$, $B_i^{\lambda'}$, $C_p^{\sigma'}$ can be arbitrarily given and then the other set is determined by either of the two equivalent sets of equations

$$(8) \quad A_a^{\alpha} A_b^{\alpha'} = \delta_b^{\alpha}, \quad B_i^{\lambda} B_j^{\lambda'} = \delta_j^{\lambda}, \quad C_p^{\sigma} C_r^{\sigma'} = \delta_r^{\sigma},$$

$$(9) \quad A_a^{\alpha} A_a^{\alpha'} = \delta_a^{\alpha}, \quad B_i^{\lambda} B_i^{\lambda'} = \delta_i^{\lambda}, \quad C_p^{\sigma} C_p^{\sigma'} = \delta_p^{\sigma}.$$

Now, any set of geometrical base vectors (e_a, e_i, e_p) forms a special kind of base for X_X , a *split base*. If, for a moment, we put

$$\begin{matrix} e \\ a=i \end{matrix} = e_i, \quad \begin{matrix} e \\ a=p \end{matrix} = e_p, \quad \begin{matrix} e \\ a'=i \end{matrix} = e_i, \quad \begin{matrix} e \\ a'=p \end{matrix} = e_p,$$

we have

$$(10) \quad \begin{matrix} e \\ a \end{matrix} = A_a^{\alpha'} e_{\alpha'} = B_a^i e_i + C_a^p e_p, \quad \dots,$$

$$(11) \quad \begin{matrix} e \\ a \end{matrix} = A_a^{\alpha'} e_{\alpha'} = B_i^{\lambda} e_{\lambda} + C_p^{\sigma} e_{\sigma}, \quad \dots,$$

Therefore, from (8) and (9), we have

$$(12) \quad \left\{ \begin{array}{ll} B_i^{\alpha} B_a^{\alpha} = \delta_i^{\alpha}, & C_p^{\alpha} C_a^{\alpha} = \delta_p^{\alpha}, \quad B_i^{\alpha} C_a^{\alpha} = 0, \quad B_a^{\alpha} C_a^{\alpha} = 0, \\ B_i^{\alpha} B_i^{\alpha} + C_p^{\alpha} C_p^{\alpha} = \delta_a^{\alpha}. \end{array} \right.$$

The last two equations in the first row result from the fact that i and p never assume the same value.

We also notice that the result of two consecutive changes of base

$$e_{a'} = A_{a'}^a e_a, \quad e_{i'} = B_{i'}^i e_i, \quad e_{p'} = C_{p'}^p e_p; \quad e_{a''} = A_{a''}^{a'} e_{a'}, \quad e_{i''} = B_{i''}^{i'} e_{i'}, \quad e_{p''} = C_{p''}^{p'} e_{p'}$$

and

$$e_{a'''} = A_{a'''}^{a''} e_{a''}, \quad e_{i'''} = B_{i'''}^{i''} e_{i''}, \quad e_{p'''} = C_{p'''}^{p''} e_{p''}; \quad e_{a''''} = A_{a''''}^{a'''} e_{a'''}, \quad e_{i''''} = B_{i''''}^{i'''} e_{i'''}, \quad e_{p''''} = C_{p''''}^{p'''} e_{p'''}$$

is a change of base

$$e_{a'''} = A_{a'''}^a e_a, \quad e_{i'''} = B_{i'''}^i e_i, \quad e_{p'''} = C_{p'''}^p e_p, \quad e_{a''''} = A_{a''''}^{a'''} e_{a'''}, \quad e_{i''''} = B_{i''''}^{i'''} e_{i'''}, \quad e_{p''''} = C_{p''''}^{p'''} e_{p'''}$$

where

$$(13) \quad \begin{cases} A_{a'''}^a = A_{a'''}^{a''} A_{a''}^a, & B_{i'''}^i = B_{i'''}^{i''} B_{i''}^i, & C_{p'''}^p = C_{p'''}^{p''} C_{p''}^p, \\ A_{a''''}^{a'''} = A_{a''''}^{a''} A_{a''}^{a'''}, & B_{i''''}^{i'''} = B_{i''''}^{i''} B_{i''}^{i'''}, & C_{p''''}^{p'''} = C_{p''''}^{p''} C_{p''}^{p'''} \end{cases}$$

Finally, expressing e_i and e_a in the base $e_{i'}$ and $e_{a'}$, we get

$$e_i = B_i^a e_a = B_i^{a'} e_{a'}, \quad e_p = C_p^a e_a = C_p^{a'} e_{a'}$$

which, from $e_{a'} = A_{a'}^a e_a$, gives

$$(14) \quad B_i^{a'} A_{a'}^a = B_i^a, \quad C_p^{a'} A_{a'}^a = C_p^a.$$

In this way we have established the following rules.

(A) If there is a summation (dummy) suffix in a product of two A's, two B's or two C's, the product is equal to the A, B or C of the remaining suffixes.

(B) If there is a dummy suffix in a product of an A and a B or of an A and a C, the product is equal to the B or the C of the remaining suffixes.

(C) If there is dummy suffix in a product of a B and a C, the product is zero.

$$(D) \quad B_a^{a'} + C_a^{a'} = A_a^{a'}, \quad A_b^a = \delta_b^a, \quad B_j^i = \delta_j^i, \quad C_r^p = \delta_r^p.$$

In particular

$$B_b^a + C_b^a = A_b^a = \delta_b^a.$$

This mechanism of transformation factors is due to Schouten. We

shall also follow his convention of writing one A, B or C only in products of A's, B's or C's. For example

$$B_i^a B_h^j B_k^b \equiv B_{ihk}^{ajb}.$$

For tensor components we introduce the following notation. Every vector $v = v^a e_a$ or $w = w_a e^a$ can be thrown into the form

$$\begin{aligned} v &= v^a B_a^i e_i + v^a C_a^p e_p \equiv \bar{v}^i e_i + \hat{v}^p e_p, \\ w &= w_a B_i^a e^i + w_a C_p^a e^p \equiv \bar{w}_i e^i + \hat{w}_p e^p. \end{aligned}$$

The two resolutes

$$\bar{v}^i = v^a B_a^i, \quad \hat{v}^p = v^a C_a^p,$$

and

$$\bar{w}_i = w_a B_i^a, \quad \hat{w}_p = w_a C_p^a,$$

are called the projections of v^a and w_a on the facet and its span respectively.

Tensors of the rank 3 have 2^3 resolutes, and so on. For example, if $v = v_b^a e^b e^c$, we have

$$\begin{aligned} v &= v_b^a (B_a^i e_i + C_a^p e_p) (B_b^k e^k + C_b^r e^r) \\ &= v_b^a B_{ak}^i e^k e^r + v_b^a B_a^i C_b^r e^r e^k + v_b^a C_a^p B_b^k e^k e^r + v_b^a C_a^p C_b^r e^r e^k. \end{aligned}$$

The first and last resolutes, viz.

$$\bar{v}_k^i = v_b^a B_{ak}^i, \quad \dots, \quad \hat{v}_r^p = v_b^a C_{ar}^p$$

lie in the facet and its span respectively so that we might call them *the facet and span projections* of v . The other two «cross resolutes» having one suffix in the facet, the other in the span show, however, that projection on two complementary facets is not, in general, equivalent to resolution.

4. Connexion and differentiation. — An essentially new idea in the modern conception of geometry is the construction of a finite space by dovetailing infinitesimal local geometries. This connexion is established

by defining equivalence or parallelism between tensors attached to neighbouring points. If $Q(x^z + \Delta x^z)$ is a point in the admitted facet at $P(x^z)$ so that

$$(1) \quad \Delta x^z = B_k^z \Delta u^k$$

with arbitrary Δu^k , the quantities

$$(2) \quad \Delta u^k = B_x^k \Delta x^z$$

are the coordinates of the points in the facet, and we put

$$(3) \quad e_j(Q \parallel P) = e_j + l_{jk}^i e_i \Delta u^k, \quad e^i(Q \parallel P) = e^i - l_{jk}^i e^j \Delta u^k,$$

$$(4) \quad e_r(Q \parallel P) = e_r + \lambda_{rk}^p e_p \Delta u^k, \quad e^r(Q \parallel P) = e^r - \lambda_{rk}^p e^p \Delta u^k.$$

If the point $R(x^z + \Delta x^z)$ is outside the facet so that

$$(5) \quad \Delta x^z = C_s^z \Delta u^s,$$

the quantities

$$(6) \quad \Delta u^s = C_x^s \Delta x^z$$

are its coordinates in the span near P , and we put

$$(7) \quad e_j(R \parallel P) = e_j + s_{js}^i e_i \Delta u^s, \quad e^i(R \parallel P) = e^i - s_{js}^i e^j \Delta u^s$$

$$(8) \quad e_r(R \parallel P) = e_r + \sigma_{rs}^p e_p \Delta u^s, \quad e^r(R \parallel P) = e^r - \sigma_{rs}^p e^p \Delta u^s$$

where (P) has been suppressed. The four sets of connexion parameters l_{jk}^i , λ_{rk}^p , s_{js}^i , σ_{rs}^p are arbitrarily assigned functions of x^z . Equations (3), (4), (7), (8) determine the equivalence also for tensors of any type and rank by the convention that the coefficients in the forms of the e 's are not affected by the equivalence.

For example, if

$$v = v^j e_j \quad \text{and} \quad w = w_i e^i,$$

we have

$$w(R \parallel P) = w_i(R) e^i - w_i(R) s_{is}^j e^j \Delta u^s,$$

$$v(Q \parallel P) = v^j(Q) e_j + v^j(Q) l_{jk}^i e_i \Delta u^k,$$

thus, retaining linear terms only,

$$(9) \quad \begin{cases} \delta w \equiv w(R \parallel P) - w(P) = (\delta w_j - s_{js}^i w_i \Delta u^s)' e, \\ \delta v \equiv v(Q \parallel P) - v(P) = (\delta v^l + l_{jk}^i v^j \Delta u^k) e, \end{cases}$$

where

$$(10) \quad \begin{cases} \delta v^l = v^l(Q) - v^l(P) = \partial_k v^l \Delta u^k, & \partial_k \equiv B_k^\alpha \partial_\alpha, \\ \delta w_j = w_j(R) - w_j(P) = \partial_s w_j \Delta u^s, & \partial_s \equiv C_s^\alpha \partial_\alpha. \end{cases}$$

Therefore we define tensor differentiation by putting

$$(11) \quad \begin{cases} \nabla_k v^l = \partial_k v^l + l_{jk}^i v^j, & \nabla_k v_j = \partial_k v_j - l_{jk}^i v_i, \\ \nabla_k v^p = \partial_k v^p + \lambda_{rk}^p v^r, & \nabla_k v_r = \partial_k v_r - \lambda_{rk}^p v_p, \\ \nabla_s v^l = \partial_s v^l + s_{js}^i v^j, & \nabla_s v_j = \partial_s v_j - s_{js}^i v_i, \\ \nabla_s v^p = \partial_s v^p + \sigma_{rs}^p v^r, & \nabla_s v_r = \partial_s v_r - \sigma_{rs}^p v_p. \end{cases}$$

For tensors of higher rank we extend R. Lagrange's mixed differentiation (R. Lagrange 1922, 1926), used also by Bortolotti (1930) and Schouten and van Kampen (1930, D-symbolik) to the new types of differentiation by putting for example

$$(12) \quad \nabla_k v_r^i = \partial_k v_r^i + l_{rk}^j v_r^j - \lambda_{rk}^p v_p^i, \quad \nabla_s v_r^i = \partial_s v_r^i + s_{js}^j v_r^j - \sigma_{rs}^p v_p^i.$$

We shall also consider partially resolved tensors for which differentiation will be defined by formulae like

$$(13) \quad \nabla_k v_r^a = \partial_k v_r^a + \Lambda_{bc}^a v_r^b B_k^c - \lambda_{rk}^p v_p^a, \quad \nabla_s v_b^i = \partial_s v_b^i + s_{js}^j v_b^j - \Lambda_{bc}^a v_b^i C_s^c.$$

Finally, we put

$$(14) \quad \nabla_c = B_c^k \nabla_k + C_c^s \nabla_s.$$

Λ_{bc}^a are the connexion parameters in X_N (geometrical axes).

In a change of base, the transformation of the connexion parameters is defined by the condition that tensor differential coefficients (or tensors differentials) are transformed as tensors, i. e. by the conditions

$$(15) \quad \nabla_b \Lambda_a' = 0, \quad \nabla_k B_i' = 0, \quad \nabla_s B_i' = 0, \quad \nabla_k C_p' = 0, \quad \nabla_s C_p' = 0.$$

These equations readily lead to the rules of transformation

$$(16) \quad \begin{cases} \Lambda_{b'c'}^{a'} = \Lambda_{ab'c'}^{a'} \Lambda_{b'c}^a + \Lambda_{b'}^{a'} \partial_{c'} \Lambda_{b'}^b \\ l_{j'k'}^{i'} = B_{j'k'}^{i'} l_{jk}^{i'} + B_{j'}^{i'} \partial_{k'} B_{j'}^j, & \lambda_{j'k'}^{i'} = C_{j'k'}^{i'} \lambda_{jk}^{i'} + C_{j'}^{i'} \partial_{k'} C_{j'}^j \\ s_{j's'}^{i'} = B_{j's'}^{i'} s_{js}^{i'} + B_{j'}^{i'} \partial_{s'} B_{j'}^j, & \sigma_{j's'}^{i'} = C_{j's'}^{i'} \sigma_{js}^{i'} + C_{j'}^{i'} \partial_{s'} C_{j'}^j. \end{cases}$$

All these differentiations obey all the ordinary laws of tensor differentiation in X_N . In particular, contraction and any of the differentiations are interchangeable, and

$$(17) \quad \nabla \delta_b^a = 0, \quad \nabla \delta_j^i = 0, \quad \nabla \delta_{b'}^{a'} = 0$$

where ∇ indicates ∇_c , ∇_k or $\nabla_{s'}$.

In the usual theory, $\nabla_a \epsilon^a$, $\nabla_a \epsilon^j$ etc. are not tensors. We have introduced the connexion parameters s and σ in order to make them tensors.

The projected connexions are obtained by requiring that

$$(18) \quad \delta_j^i = {}_i \Delta_j^i, \quad \delta_{r'}^{i'} = {}_i \Delta_{r'}^{i'}, \quad {}' \delta_j^i = {}_n \Delta_j^i, \quad {}' \delta_{r'}^{i'} = {}_n \Delta_{r'}^{i'}$$

where Δ indicates that the following tensor is dealt with as a tensor of X_N , irrespective of the submanifold. These equations when explicit lead to the following specific values for the connexion parameters l , λ , s , σ ,

$$(19) \quad \bar{l}_{jk}^i = B_x^i \nabla_k B_j^x, \quad \bar{\lambda}_{r'k}^{i'} = C_x^{i'} \nabla_k C_{r'}^x, \quad \bar{s}_{j'}^i = B_x^i \nabla_{s'} B_{j'}^x, \quad \bar{\sigma}_{r's}^{i'} = C_x^{i'} \nabla_{s'} C_{r'}^x$$

where x can be replaced by a . A bracketed suffix like (j) after a differentiation symbol indicates that this suffix is disconsidered in the differentiation.

Proof of (19). We see from $e = B_{j'}^a e$ that the components of e_j in (e, e') are $B_{j'}^a$, j being fixed. Thus

$$(20) \quad \Delta_j^i = e_j(Q||P) - e_j = \nabla_{j'} B_{j'}^a \cdot dx^a = \nabla_{j'} B_{j'}^a \cdot dx^a (B_{a'}^i e + C_{a'}^p e).$$

Hence from $dx^a = B_k^a dx^k$, the projection of the difference upon the facet is equal to

$${}_i \Delta_j^i = B_a^i \nabla_k B_{j'}^a \cdot dx^k \cdot e_{j'}$$

On the other hand, by definition, $\partial e = l_{jk}^i dx^k e$. Therefore, if we require that for every i and for every displacement in the facet, i. e. for every value of dx^k , we should have (18)₁, we obtain for l_{jk}^i the value given by (19)₁.

The argument is similar for the other equations. The special values (19) of l , λ , s , σ will be referred to as the connexion parameters of the projected connexion, or, briefly, as *the projected connexion parameters*, while the general arbitrarily assigned values will be called *intrinsic connexion parameters*.

3. *Fundamental tensors of the submanifold.* — Equations (4, 16) show that connexion parameters are not transformed as tensors. On the other hand, the equation

$$\Lambda_{a'b'c'}^{a'bc} \cdot \Lambda_{xbr}^{a\beta\gamma} \Gamma_{\beta\gamma}^x = \Lambda_{xbr}^{a\beta\gamma} \Gamma_{\beta\gamma}^x$$

proves that in any change of base the expression

$$(1) \quad \Lambda_{xbr}^{a\beta\gamma} \Gamma_{\beta\gamma}^x$$

transforms as a tensor. If, however, a suffix of $\Gamma_{\beta\gamma}^x$ is not a dummy suffix, the change of base requires the replacement of $\Gamma_{\beta\gamma}^x$ by $\Lambda_{a'b'}^{x''}$, say, and our equality is destroyed. Similarly, if we try to change dummy suffixes $\alpha\beta\gamma$ into $a'b'c'$. On the other and, the equation

$$(2) \quad B_a^i \nabla_k B_j^x = B_a^i \nabla_k B_j^x$$

resulting from $B_j^x = \Lambda_a^x B_j^a$ and $\nabla_k \Lambda_a^x = 0$, shows that the expression (2) is unaltered by changing the dummy suffix α into a . The definition of (1) requires a specified system of axes, the value of (2) is the same in every system. We shall say that (1) is a *relative tensor*, and in this paper relative tensors will be all defined in the mathematical frame, while (2) will be called an *absolute tensor*. We shall see that torsion and curvature are not absolute tensors.

Another distinction between tensors will be the following. It is known that the Riemann-Christoffel tensor transforms as a tensor in a mathematical change of variables only, while in a general change of base it loses its tensor character. On the other hand, (1) and (2) transform

as tensors in any change of base (axes). We shall say that (1) and (2) are *geometrical tensors*, while the Riemann-Christoffel tensor is a *Ricci tensor* only.

Finally, if a Ricci or geometrical tensor contains metric or connexion (i. e. structure) parameters, it will be called a *structure tensor*. For example (2) is an absolute geometrical structure tensor. The projection factors, B_j^a , say, transform as tensors, for

$$\Lambda_a^{a'} B_j^a \cdot B_j^a = B_j^{a'},$$

but they are not structure tensors. We might call them *formal tensors*.

Table I gives a list of important tensors.

TABLE I.

$D_{jk}^i = B_x^i \nabla_k B_j^x,$	$d_{jk}^i = B_j^x \partial_k B_x^i - B_j^x \partial_k B_x^i,$
$E_{rk}^p = C_x^p \nabla_k C_r^x,$	$e_{rk}^p = C_r^x \partial_k C_x^p - C_r^x \partial_k C_x^p,$
$F_{jk}^p = C_x^p \nabla_k B_j^x = C_x^p \nabla_k B_j^x,$	$f_{jk}^p = B_j^x \partial_k C_x^p,$
$G_{rk}^i = B_x^i \nabla_k C_r^x = B_x^i \nabla_k C_r^x,$	$g_{rk}^i = C_r^x \partial_k B_x^i,$
$H_{js}^i = B_x^i \nabla_s B_j^x,$	$h_{js}^i = B_j^x \partial_s B_x^i - B_j^x \partial_s B_x^i,$
$I_{rs}^p = C_x^p \nabla_s C_r^x,$	$i_{rs}^p = C_r^x \partial_s C_x^p - C_r^x \partial_s C_x^p,$
$J_{js}^i = C_x^i \nabla_s B_j^x = C_x^i \nabla_s B_j^x,$	$j_{js}^i = B_j^x \partial_s C_x^i,$
$K_{rs}^i = B_x^i \nabla_s C_r^x = B_x^i \nabla_s C_r^x,$	$k_{rs}^i = C_r^x \partial_s B_x^i.$

The tensors in the first column are absolute geometrical structure tensors (α may be replaced by a , a' , etc.), those in the second column are relative geometrical formal tensors. F and G are Schouten's — H and — L, the alternated (¹) f is Schouten's Z.

The set of equations

$$(3) \left\{ \begin{array}{ll} l_{jk}^i = \bar{l}_{jk}^i - D_{jk}^i, & \bar{l}_{rk}^p = \bar{l}_{rk}^p - E_{rk}^p, & s_{js}^i = \bar{s}_{js}^i - H_{js}^i, & \sigma_{rs}^p = \bar{\sigma}_{rs}^p - I_{rs}^p, \\ e_j(Q \parallel P) - l_j = (\bar{l}_{jk}^i e_i + F_{jk}^p e_p) dx^k, & e_r(Q \parallel P) - e_r = (G_{rk}^i e_i + \bar{l}_{rk}^p e_p) dx^k, \\ e_j(R \parallel P) - e_j = (\bar{s}_{js}^i e_i + J_{js}^p e_p) dx^s, & e_r(R \parallel P) - e_r = (K_{rs}^i e_i + \bar{\sigma}_{rs}^p e_p) dx^s, \end{array} \right.$$

(¹) *Alternation* is defined by $v_{[ab]} = \frac{1}{2} (v_{ab} - v_{ba})$, *averaging* (Mischung) by $v_{(ab)} = \frac{1}{2} (v_{ab} + v_{ba})$. If there is a suffix between a and b which is not affected by the process, we put it between two bars (Schouten, 1924).

where the point $Q(x^x + dx^x)$ lies in the facet (tangent plane) at $P(x^x)$ and $R(x^x + 'dx^x)$ lies outside the facet so that $dx^x = B_k^x dx^k$, $'dx^x = C_r^x dx^r$, shows that D, E, H, I characterise the difference between projected and intrinsic connexion (*connexion tensors*) while F, G, J, K characterise the parts of the connexion in X_s which is ignored in the definition of the connexions (projected or intrinsic) in X_s^m and in its span (*deviation tensors*).

We notice that in the holonomic case no facet (split base) need be specified at R . Equations (2, 5), however define a split base at R , a natural though highly arbitrary continuation of that at P .

In Table II we give five fundamental tensors whose alternated forms with respect to the low suffixes are the *torsion tensors*.

TABLE II.

$$\begin{aligned} T_{hc}^a &= A_{hc}^a + A_b^x \partial_c A_x^a, \\ T_{jk}^i &= I_{jk}^i + B_j^x \partial_k B_x^i, \\ T_{rk}^p &= I_{rk}^p + C_r^x \partial_k C_x^p, \\ T_{js}^i &= J_{js}^i + B_j^x \partial_s B_x^i, \\ T_{rs}^p &= J_{rs}^p + C_r^x \partial_s C_x^p, \end{aligned}$$

These tensors are relative geometrical structure tensors. It is known that the transform of an infinitesimal square, when we pass from mathematical to geometrical axes, is not closed. The vector closing the transformed square is determined in the usual way by the alternated forms of the T tensors. The four torsions of the submanifold correspond to squares in the facet, span and in both.

We also give the formulae

$$(4) \quad \begin{cases} \nabla_k B_j^a = D_{jk}^a + F_{jk}^a, & \nabla_k C_r^a = E_{rk}^a + G_{rk}^a, \\ \nabla_s B_j^a = H_{js}^a + J_{js}^a, & \nabla_s C_r^a = I_{rs}^a + K_{rs}^a, \\ \nabla_k B_a^i = -D_{ak}^i - G_{ak}^i, & \nabla_k C_a^r = -E_{ak}^r - F_{ak}^r, \\ \nabla_s B_a^i = -H_{as}^i - K_{as}^i, & \nabla_s C_a^r = -I_{as}^r - J_{as}^r, \end{cases}$$

where

$$D_{jk}^a = B_i^a D_{jk}^i, \quad F_{jk}^a = C_p^a F_{jk}^p, \quad \dots$$

Proof of (4)₁ :

$$\nabla_k B_i^a = \delta_b^a \nabla_k B_i^b = (B_b^a + C_b^x) \nabla_k B_i^b = B_j^a \cdot B_b^j \nabla_k B_i^b + C_r^a \cdot C_b^r \nabla_k B_i^b,$$

Proof of (4)₅ :

$$\nabla_k B_a^i = B_a^b \nabla_k B_b^i + C_a^b \nabla_k B_b^i,$$

whose first term is equal to

$$B_a^j \cdot B_j^i \nabla_k B_b^i = - B_a^j \cdot B_b^i \nabla_k B_j^i,$$

and whose second term is equal to $C_a^r \cdot C_r^b \nabla_k B_b^i = - C_a^r \cdot B_b^i \nabla_k C_r^b$.

The proof is similar for the other relations.

Combining $\nabla_b = B^k \nabla_k + C_b^r \nabla_r$ with formulae of (4) we obtain formulae like

$$(5) \quad \nabla_b B_j^a = D_{jb}^a + F_{jb}^a + H_{jb}^a + J_{jb}^a, \quad \nabla_b C_r^a = E_{rb}^a + G_{rb}^a + I_{rb}^a + K_{rb}^a,$$

and

$$(6) \quad \begin{cases} B_a^i \nabla_b B_j^a = D_{jb}^i + H_{jb}^i, \\ C_a^r \nabla_b C_r^a = E_{rb}^a + I_{rb}^a, \\ C_a^r \nabla_b C_j^a = F_{jb}^a + J_{jb}^a, \\ B_a^i \nabla_b C_r^a = G_{rb}^i + K_{rb}^i. \end{cases}$$

6. Curvature tensors. — Another group of fundamental tensors is that of the curvature tensors; they are relative geometrical structure tensors. To simplify writing we put,

$$(1) \quad m_{je}^i = l_{jk}^i B_e^k + s_{js}^i C_e^s, \quad \mu_{re}^p = \lambda_{rk}^p B_e^k + \sigma_{rs}^p C_e^s.$$

TABLE III.

$$\begin{aligned} C_{bcd}^a(\Lambda) &= \partial_{[d} \Lambda_{b]c}^a + \Lambda_{e[d}^a \Lambda_{b]c}^e + \Lambda_{[c}^a \partial_d \Lambda_{b]}^e \cdot \Lambda_{e}^a, \\ C_{jcd}^i(m) &= \partial_{[d} m_{j]c}^i + m_{kd}^i m_{j]c}^k + \Lambda_{[c}^i \partial_d \Lambda_{b]}^e \cdot m_{j]e}^i, \\ C_{rcd}^p(\mu) &= \partial_{[d} \mu_{r]c}^p + \mu_{q[d}^p \mu_{r]c}^q + \Lambda_{[c}^p \partial_d \Lambda_{b]}^e \cdot \mu_{r]e}^p, \\ C_{jkt}^i(l) &= \partial_{[t} l_{j]k}^i + l_{m[t}^i l_{j]k}^m + B_{[k}^i \partial_t B_{j]}^m \cdot l_{jm}^i + f_{[kt]}^i s_{js}^i, \\ C_{jst}^i(s) &= \partial_{[t} s_{j]s}^i + s_{m[t}^i s_{j]s}^m + C_{[s}^i \partial_t C_{j]}^r \cdot s_{jr}^i + k_{[st]}^i l_{jm}^i, \\ 2C_{jkt}^i(l, s) &= \partial_t l_{jk}^i - \partial_k s_{jt}^i + s_{mt}^i l_{jk}^m - l_{mk}^i s_{jt}^m \\ &\quad + B_k^i \partial_t B_{jm}^m \cdot l_{jm}^i - C_t^i \partial_k C_{js}^r \cdot s_{jr}^i + f_{kt}^i s_{js}^i - g_{th}^m l_{jm}^i, \\ C_{jtk}^i &= -C_{jkt}^i, \\ C_{rkt}^p(\lambda) &= \partial_{[t} \lambda_{r]k}^p + \lambda_{q[t}^p \lambda_{r]k}^q + B_{[k}^p \partial_t B_{j]}^m \cdot \lambda_{rm}^p + f_{[kt]}^p \sigma_{rs}^p, \\ C_{rst}^p(\sigma) &= \partial_{[t} \sigma_{r]s}^p + \sigma_{q[t}^p \sigma_{r]s}^q + C_{[s}^p \partial_t C_{j]}^r \cdot \sigma_{rs}^p + k_{[st]}^p \lambda_{jm}^p, \\ 2C_{rkt}^p(\lambda, \sigma) &= \partial_t \lambda_{rk}^p - \partial_k \sigma_{rt}^p + \sigma_{qt}^p \lambda_{rk}^q - \lambda_{qk}^p \sigma_{rt}^q \\ &\quad + B_k^p \partial_t B_{rs}^m \cdot \lambda_{rm}^p + C_t^p \partial_k C_{rs}^q \cdot \sigma_{rs}^p + f_{kt}^p \sigma_{rs}^p - g_{th}^m \lambda_{rm}^p, \\ C_{rth}^p &= -C_{rht}^p. \end{aligned}$$

The geometrical significance of the last eight tensors in X_{Σ}^m and its span is similar to that of the first in X_{Σ} . In Art. 10 we shall express the 16 resolutes of C_{bcd}^a in terms of the last eight and in the absolute tensors D, E, F, G, H, I, J, K. The second and third tensors can be expressed in terms of the last eight :

$$(2) \quad C_{jcd}^i(m) = B_{cd}^{kl} C_{jkl}^i(l) + C_{cd}^{st} C_{jst}^i(s) + B_{lc}^k C_{d1}^l C_{jkl}^i(l, s),$$

$$(3) \quad C_{ced}^p(\rho) = B_{ce}^{kl} C_{rkl}^p(\lambda) + C_{cd}^{st} C_{rst}^p(\sigma) + B_{lc}^k C_{d1}^l C_{rkl}^p(\lambda, \sigma).$$

Proof of (2). $C_{jcd}^i(m)$ is the alternation with respect to c, d of

$$\begin{aligned} (4) \quad & \partial_d l_{jk}^i \cdot B_c^k + l_{jk}^i \partial_d B_c^k + \partial_d s_{js}^i \cdot C_c^s - s_{js}^i \partial_d C_c^s \\ & + (l_{ml}^i B_d^l + s_{ml}^i C_d^l) (l_{jk}^m B_c^k + s_{js}^m C_c^k) - A_b^z \partial_d A_c^z \cdot (l_{jk}^i B_b^k + s_{js}^i C_b^k) \\ & = B_{cd}^{kl} \partial_l l_{jk}^i + B_c^k C_d^l \partial_l l_{jk}^i + B_d^l \partial_l B_c^k \cdot l_{jk}^i + C_d^l \partial_l B_c^k \cdot l_{jk}^i + C_c^l B_d^k \partial_k s_{jt}^i \\ & - C_{cd}^{st} \partial_t s_{js}^i - B_d^l \partial_l C_c^s \cdot s_{js}^i - C_d^l \partial_l C_c^s \cdot s_{js}^i + B_{cd}^{kl} l_{ml}^i l_{jk}^m + C_{cd}^{st} s_{ml}^i s_{js}^m \\ & - B_c^k C_d^l l_{jk}^m s_{ml}^i + C_c^l B_d^k l_{mk}^i s_{jt}^m - B_{cd}^{kl} \partial_l A_c^z \cdot l_{jk}^i - B_c^k C_d^l \partial_l A_c^z \cdot l_{jk}^i \\ & - C_c^l B_d^k \partial_l A_c^z \cdot s_{js}^i - C_{cd}^{st} \partial_t A_c^z \cdot s_{js}^i. \end{aligned}$$

The group 3 + 13 is equal to

$$\begin{aligned} & A_c^z B_d^l \partial_l B_x^k \cdot l_{jk}^i = B_{cd}^{kl} \partial_l B_x^k \cdot l_{jk}^i + B_d^l C_c^z \partial_l B_x^k \cdot l_{jk}^i = \\ (5) \quad & B_{cd}^{kl} \cdot B_c^z \partial_l A_a^m \cdot l_{jm}^i + C_c^z B_d^k \cdot C_l^z \partial_k B_x^m \cdot l_{jm}^i, \dots \quad 3 + 13, \end{aligned}$$

Similarly, 7 + 15 is equal to $A_c^z B_d^l \partial_l C_a^s \cdot s_{js}^i =$

$$(6) \quad B_{cd}^{kl} \cdot B_c^z \partial_l C_x^s \cdot s_{js}^i + C_c^z B_d^k \cdot C_l^z \partial_k C_x^s \cdot s_{js}^i, \dots \quad 7 + 15.$$

Thus, after alternation, 1 + 9 and the first terms of (5) and (6) give $B_{cd}^{kl} C_{jkl}^i(l)$.

Moreover, 8 + 16 is equal to

$$(7) \quad A_c^z C_d^l \partial_l C_a^s \cdot s_{js}^i = B_c^k C_d^l \cdot B_k^z \partial_l C_x^s \cdot s_{js}^i + C_{cd}^{st} \cdot C_l^z \partial_l C_x^s \cdot s_{js}^i, \dots \quad 8 + 16,$$

and 4 + 14 is equal to

$$(8) \quad A_c^z C_x^l \partial_l B_x^k \cdot l_{jk}^i = B_c^k C_d^l \cdot B_k^z \partial_l B_x^m \cdot l_{jm}^i + C_{cd}^{st} \cdot C_x^z \partial_l B_x^m \cdot l_{jk}^i, \dots \quad 4 + 14$$

After alternation, 6 + 10 and the last two terms of (7) and (8) give $C_{cd}^{st} C_{jst}^i(s)$.

The remaining terms are 2, 5, 11, 12, (5)₂, (6)₂, (7)₁, (8)₁. We have to exchange c, d and subtract the terms so obtained (and divide by 2). This difference can be written in the form

$$\begin{aligned} & B_c^k C_d^l [\partial_t l_{jk}^i - \partial_k s_{jt}^i + l_{jk}^m s_{mt}^i - s_{jt}^m l_{mk}^i] \\ & + B_k^z \partial_t B_x^m \cdot l_{jm}^i - C_t^z \partial_k C_x^m \cdot s_{jp}^i - j_{kt}^i s_{js}^i - s_{tk}^m l_{jm}^i] \\ & + C_c^l B_d^k [\partial_k s_{jt}^i - \partial_t l_{jk}^i + s_{jt}^m l_{mk}^i - l_{jk}^m s_{mt}^i] \\ & + C_t^z \partial_k C_x^m \cdot s_{jp}^i - B_k^z \partial_t B_x^m \cdot l_{jm}^i + s_{tk}^m l_{jm}^i - j_{kt}^i s_{js}^i], \end{aligned}$$

where the bar indicates that the corresponding term has been obtained by exchanging c, d and taking the negative sign. This proves (2). A similar argument proves (3).

7. Resolution of T_{bc}^a . — Since the torsion tensor of X_x is the alternated form of T_{bc}^a , we give the resolutes of the latter.

$$\begin{aligned} (1) \quad & B_{ajk}^{ibc} T_{bc}^a = B_{ajk}^{i\beta\gamma} \Gamma_{\beta\gamma}^z = D_{jk}^i + T_{jk}^i(t), \\ (2) \quad & C_{ar}^{pb} B_k^c T_{bc}^a = C_{ar}^{p\beta} B_k^\gamma \Gamma_{\beta\gamma}^z = F_{rk}^p + T_{rk}^p(\tau), \\ (3) \quad & C_a^p B_{jk}^{bc} T_{bc}^a = C_a^p B_{jk}^{i\beta\gamma} \Gamma_{\beta\gamma}^z = F_{jk}^p + f_{jk}^p, \\ (4) \quad & B_a^i C_r^{bc} B_k^c T_{bc}^a = B_a^i C_r^{i\beta} B_k^\gamma \Gamma_{\beta\gamma}^z = G_{rk}^i - g_{rk}^i, \\ (5) \quad & B_{aj}^{ib} C_s^c T_{bc}^a = B_{aj}^{i\beta} C_s^\gamma \Gamma_{\beta\gamma}^z = H_{js}^i + T_{js}^i(s), \\ (6) \quad & C_{ars}^{pb} T_{bc}^a = C_{ars}^{p\beta\gamma} \Gamma_{\beta\gamma}^z = V_{rs}^p - T_{rs}^p(\sigma), \\ (7) \quad & C_a^p B_j^b C_s^c T_{bc}^a = C_a^p B_j^{i\beta} C_s^\gamma \Gamma_{\beta\gamma}^z = J_{js}^p - j_{js}^p, \\ (8) \quad & B_a^i C_{rs}^{bc} T_{bc}^a = B_a^i C_{rs}^{i\beta\gamma} \Gamma_{\beta\gamma}^z = K_{rs}^i - k_{rs}^i. \end{aligned}$$

Proof of (1). From

$$T_{bc}^a = \Lambda_b^a + \Lambda_b^z \partial_c \Lambda_z^a \quad \text{and} \quad \Lambda_{bc}^a = \Lambda_{zbc}^{a\beta\gamma} \Gamma_{\beta\gamma}^z - \Lambda_b^z \partial_c \Lambda_z^a,$$

we obtain the first equality. Moreover

$$\begin{aligned} B_{ajk}^{ibc} (\Lambda_{bc}^a + \Lambda_b^z \partial_c \Lambda_z^a) &= B_{ajk}^{ibc} \Lambda_{bc}^a - B_{aj}^{iz} \partial_k \Lambda_z^a \\ &= B_{ajk}^{ibc} \Lambda_{bc}^a + B_a^i \partial_k B_j^a - B_j^z \partial_k B_a^i - l_{jk}^i + l_{jk}^i, \end{aligned}$$

which proves the second equality of (1) by remarking that, in the definition of D_{jk}^i , the dummy suffix α may be replaced by a . The proof is similar for the other lines.

If we denote the resolutes of T_{br}^a , *i.e.*, the lefthand sides of (1)–(8) by $\bar{T}_{jk}^i$, $\bar{T}_{rk}^p$ etc., and the second column by $\bar{\Gamma}_{jk}^i$ etc., we obtain the following list of formulae for the resolutes of the torsion tensor $T_{[br]}^a$

$$\begin{aligned}
 (9) \quad & T_{[jk]}^i(\Lambda) = \bar{T}_{[jk]}^i = D_{[jk]}^i + T_{[jk]}^i(l), \\
 (10) \quad & {}_2\bar{T}_{[rk]}^p(\Lambda) = {}_2\bar{\Gamma}_{[rk]}^p = F_{rk}^p + T_{rk}^p(\lambda) - J_{kr}^p - f_{kr}^p, \\
 (11) \quad & \bar{T}_{[jk]}^p(\Lambda) = \bar{\Gamma}_{[rk]}^p = F_{rk}^p + f_{[jk]}^p, \\
 (12) \quad & {}_2\bar{T}_{[rk]}^i(\Lambda) = {}_2\bar{\Gamma}_{[rk]}^i = G_{rk}^i + \mathcal{G}_{rk}^i - H_{kr}^i - T_{kr}^i(s), \\
 (13) \quad & \bar{T}_{[kr]}^i = \bar{T}_{rk}^i, \\
 (14) \quad & \bar{T}_{[rs]}^p(\Lambda) = \bar{\Gamma}_{[rs]}^p = l_{rs}^p + T_{[rs]}^p(\sigma), \\
 (15) \quad & \bar{T}_{[kr]}^p = -\bar{T}_{[rk]}^p, \\
 (16) \quad & \bar{T}_{[rs]}^i = \bar{\Gamma}_{[rs]}^i = K_{rs}^i + k_{[rs]}^i.
 \end{aligned}$$

8. Inversion formulae for differentiation of scalars and vectors. — If f is a function of the mathematical variables x^z , we have

$$(1) \quad \partial_a \partial_c f = \Lambda_{\alpha}^b \partial_a \Lambda_{\alpha c}^z \partial_b f,$$

and, from $\partial_l B_k^z = (B_k^z + C_k^z) \partial_l B_k^z = B_k^z \partial_l B_k^z - f_{kl}^z$,

$$(2) \quad \partial_j \partial_k f = (B_k^z \partial_j B_{kl}^z - f_{jkl}^z) \partial_z f.$$

In a like manner,

$$(3) \quad \partial_l \partial_s f = \partial_l C_{s1}^z \partial_z f = (C_s^z \partial_l C_{s1}^z - k_{sl}^z) \partial_z f,$$

$$\begin{aligned}
 (4) \quad & \partial_l \partial_k f - \partial_k \partial_l f = (\partial_l B_k^z - \partial_k C_l^z) \partial_z f \\
 & = (B_k^z \partial_l B_k^z - f_{kl}^z - C_k^z \partial_l C_l^z + \mathcal{G}_{lk}^z) \partial_z f.
 \end{aligned}$$

For every vector or tensor v , where we have suppressed the suffixes to make the statement general, $\nabla_k v = B_k^c \nabla_c v$, $\nabla_c v = C_s^c \nabla_s v$, thus

$$\begin{aligned}
 \nabla_l \nabla_k v &= \nabla_l B_k^c \nabla_c v + B_{kl}^{cd} \nabla_d \nabla_c v, \\
 \nabla_l \nabla_s v &= \nabla_l C_s^c \nabla_c v + C_{sl}^{cd} \nabla_d \nabla_c v.
 \end{aligned}$$

Therefore, from (4, 4),

$$(5) \quad \nabla_l \nabla_k v = (D_{[kl]}^c + F_{[kl]}^c) \nabla_c v + B_{kl}^{cd} \nabla_d \nabla_c v,$$

$$(6) \quad \nabla_l \nabla_s v = (l_{[sl]}^c + K_{[sl]}^c) \nabla_c v + C_{sl}^{cd} \nabla_d \nabla_c v,$$

$$(7) \quad \nabla_l \nabla_k v = \frac{1}{2} (H_{kl}^c + J_{kl}^c - F_{lk}^c - G_{lk}^c) \nabla_c v + B_k^c C_l^d \nabla_d \nabla_c v.$$

We have to calculate $\nabla_d \nabla_c \varphi$ only.

For vectors we have

$$(8) \quad \nabla_d \nabla_c \varphi^i = -T_{[cd]}^b \nabla_b \varphi^i + C_{jcd}^i(m) \varphi^j,$$

$$(9) \quad \nabla_d \nabla_c \varphi^\mu = -T_{[cd]}^b \nabla_b \varphi^\mu + C_{red}^\mu(\mu) \varphi^r,$$

$$(10) \quad \nabla_d \nabla_c \varphi_j = -T_{[cd]}^b \nabla_b \varphi_j - C_{jcd}^i(m) \varphi_i,$$

$$(11) \quad \nabla_d \nabla_c \varphi_r = -T_{[cd]}^b \nabla_b \varphi_r - C_{red}^\mu(\mu) \varphi_\mu.$$

Proof of (8). From $\nabla_c \varphi^i = \partial_c \varphi^i + m_{\mu}^i \varphi^\mu$, we have

$$\begin{aligned} \nabla_d \nabla_c \varphi^i &= \partial_d \nabla_c \varphi^i - \Lambda_{cd}^b \nabla_b \varphi^i + m_{md}^i \nabla_c \varphi^m \\ &= \partial_d \partial_c \varphi^i - \partial_d m_{jc}^i \varphi^j - m_{jc}^i \partial_d \varphi^j - \Lambda_{cd}^b \nabla_b \varphi^i - m_{jd}^i \partial_c \varphi^j + m_{md}^i m_{jc}^m \varphi^j. \end{aligned}$$

From (1)

$$\partial_d \partial_c \varphi^i = \Lambda_{2d}^b \partial_d \Lambda_{c1}^z (\nabla_b \varphi^i - m_{1,2}^i \varphi^j).$$

The group $3+5$ is symmetric in c, d and thus cancels out in the alternation. From $4+1,1$, the coefficient of

$$\nabla_b \varphi^i \text{ is } -\Lambda_{[cd]}^b + B_{2d}^b \partial_d \Lambda_{c1}^z = -T_{[cd]}^b.$$

Finally, in $2+6+1,2$ the coefficient of φ^j is $C_{jcd}^i(m)$, this proves (8). The proof is similar for the other three equations.

Substituting one of the expressions (8), (11) into one of the formulae (5), (7) we obtain the corresponding inversion formula. For instance, substituting (8) into (5), written down for $\varphi = \varphi^i$, and noticing that

$$B_{kl}^{cd} T_{cd}^b = B_{cl}^{cd} T_{cd}^b - C_c^b B_{kl}^{cd} T_{cd}^b = B_m^b (D_{kl}^m + T_{kl}^m) + C_c^b (F_{kl}^c + f_{kl}^c),$$

we obtain

$$(12) \quad \nabla_l \nabla_k \varphi^i = - (T_{[kl]}^b + f_{[kl]}^b) \nabla_b \varphi^i - C_{jkl}^i(l) \varphi^j,$$

since the last two terms of (6,2) cancel out.

If we make use of the equations $\varphi^i = B_a^i \varphi^a$, $\varphi^a = B_j^a \varphi^j$, we have

$$\nabla_b \varphi^i = \nabla_b B_a^i B_j^a \varphi^j = B_a^i \nabla_b \varphi^a,$$

where

$$\begin{aligned} B_j^a \nabla_b B_a^i &= -B_a^i \nabla_b B_j^a = -B_{ba}^m \nabla_m B_j^a - C_b^i B_a^i \nabla_b B_j^a \\ &= -B_{ba}^m D_{jm}^i - C_b^i H_{js}^i. \end{aligned}$$

Thus we obtain

$$(13) \quad \nabla_l \nabla_k \varphi^i = - (T_{[kl]}^b + f_{[kl]}^b) B_a^i \nabla_b \varphi^a + [C_{jkl}^i(l) - T_{[kl]}^m D_{jm}^i + f_{[kl]}^s H_{js}^i] \varphi^j,$$

which, in the case of projected connexion, $D = 0$, $H = 0$, reduces to Schouten's formula (67) (Schouten 1929).

Combining (9) and (6) and noticing that

$$C_{st}^{cd} T_{cd}^b = B_e^b C_{st}^{cd} T_{cd}^e + C_{est}^{bcd} T_{cd}^e = B_m^b (K_{st}^m + L_{st}^m) + C_{st}^b (I_{st}^p + T_{st}^p(\sigma)),$$

we obtain

$$(14) \quad \nabla_d \nabla_s v^b = -[T_{st}^b(\sigma) + L_{st}^b] \nabla_b v^b + C_{st}^b (I_{st}^p + T_{st}^p(\sigma)) v^p.$$

Combining (8) and (6) we get

$$(15) \quad \nabla_d \nabla_s v^i = -[T_{st}^i(\sigma) + L_{st}^i] \nabla_b v^i + C_{st}^i (I_{st}^p + T_{st}^p(\sigma)) v^p$$

and so on.

9. Inversion formulae for tensors. — As typical examples we give the formulae

$$(1) \quad \nabla_d \nabla_c v_j^a = -T_{cd}^b \nabla_b v_j^a + C_{bcd}^a (\lambda) v_j^b + C_{jcd}^a (m) v_i^a,$$

$$(2) \quad \nabla_d \nabla_c v_r^a = -T_{cd}^b \nabla_b v_r^a + C_{bcd}^a (\lambda) v_r^b + C_{rcd}^a (\mu) v_p^a.$$

Proof of (1). From

$$\nabla_c v_j^a = \partial_c v_j^a + \Lambda_{bc}^a v_j^b - m_{jc} v_i^a$$

we have

$$\begin{aligned} \nabla_d \nabla_c v_j^a &= \partial_d \nabla_c v_j^a + \Lambda_{cd}^a \nabla_b v_j^b + m_{jd}^b \nabla_b v_i^a \\ &= \partial_d \partial_c v_j^a + \partial_d \Lambda_{bc}^a v_j^b + \Lambda_{bc}^a \partial_d v_j^b + \partial_d m_{jc}^i v_i^a + m_{jc}^i \partial_d v_i^a \\ &= \Lambda_{cd}^a \partial_b v_j^b + \Lambda_{cd}^a \Lambda_{bc}^e v_j^b + \Lambda_{cd}^a m_{jc}^i v_i^a + \Lambda_{cd}^b \nabla_b v_j^a \\ &= m_{jd}^b \partial_b v_k^a + m_{jd}^b \Lambda_{bc}^a v_k^b + m_{jd}^b m_{kc}^i v_i^a. \end{aligned}$$

The groups 3 + 6, 5 + 10, 8 + 11 are symmetric in c, d and thus cancel out in the alternation. Moreover

$$\partial_d \partial_c v_j^a = \Lambda_{cd}^b \Lambda_{bc}^a (\nabla_b v_j^a + \Lambda_{eb}^a v_j^e + m_{jb}^i v_i^a),$$

so that collecting terms we obtain (1).

The proof is similar for (2) and other analogous formulae.

If we combine (1) and (2) with (8,5) and (8,6) we get

$$(3) \quad \nabla_l \nabla_k v_j^a = -[T_{[kl]}^b(l) + f_{[kl]}^b] \nabla_b v_j^a + B_{kl}^{cd} C_{bcd}^a(\Lambda) v_j^b - C_{jkl}^i(l) v_i^a,$$

$$(4) \quad \nabla_j \nabla_k v_r^a = -[T_{[kl]}^b(l) + f_{[kl]}^b] \nabla_b v_r^a + B_{kl}^{cd} C_{bcd}^a(\Lambda) v_r^b - C_{rkl}^p(\lambda) v_p^a,$$

$$(5) \quad \nabla_l \nabla_s v_j^a = -[T_{[sl]}^b(\sigma) + h_{[sl]}^b] \nabla_b v_j^a + C_{st}^{cd} C_{bcd}^a(\Lambda) v_j^b - C_{jst}^i(s) v_i^a,$$

$$(6) \quad \nabla_l \nabla_s v_r^a = -[T_{[sl]}^b(\sigma) + h_{[sl]}^b] \nabla_b v_r^a + C_{st}^{cd} C_{bcd}^a(\Lambda) v_r^b - C_{rst}^p(\sigma) v_p^a.$$

Combining (1) and (2) with (8,7) and noticing that

$$B_k^c C_l^d T_{cd}^b = B_{ek}^{bc} C_l^d T_{cd}^e + C_e^b B_k^c C_l^d T_{cd}^e = B_m^b [H_{kl}^m + T_{kl}^m(s)] + C_p^b (J_{kl}^p + f_{kl}^p),$$

$$C_l^c B_k^d T_{cd}^b = B_e^b C_l^c B_k^d T_{cd}^e + C_{el}^{bc} B_k^d T_{cd}^e = B_m^b (G_{lk}^m + s_{lk}^m) + C_p^b [E_{lk}^p + T_{lk}^p(\lambda)],$$

we obtain

$$(7) \quad \begin{aligned} \nabla_l \nabla_k v_j^a = & -\frac{1}{2} [T_{kl}^b(s) + f_{kl}^b - T_{lk}^b(\lambda) - s_{lk}^b] \nabla_b v_j^a \\ & - B_k^c C_l^d C_{bcd}^a(\Lambda) v_j^b - C_{jkl}^i(l, s) v_i^a, \end{aligned}$$

$$(8) \quad \begin{aligned} \nabla_l \nabla_k v_r^a = & -\frac{1}{2} [T_{kl}^b(s) + f_{kl}^b - T_{lk}^b(\lambda) - s_{lk}^b] \nabla_b v_r^a \\ & - B_k^c C_l^d C_{bcd}^a(\Lambda) v_r^b - C_{jkl}^i(l, s) v_i^a. \end{aligned}$$

10. Resolution of the curvature tensor. — The inversion formulae (9,3) — (9,8) lead to explicit relations between the resolutes of the curvature tensor $C_{bcd}^a(\Lambda)$ of X_x and the various curvature tensors attached to X_x^m . To obtain these relations we have to replace v_j^a by B_j^a (and i by m , say), v_r^a by C_r^a (and p by q , say) and multiply by B_a^i or C_a^p .

For example, replacing v by B and i by m in (9,3) and multiplying by B_a^i we obtain

$$B_{ajkl}^{bcd} C_{bcd}^a(\Lambda) = C_{jkl}^i(l) + [T_{[kl]}^b(l) + f_{[kl]}^b] B_a^i \nabla_b B_j^a + B_a^i \nabla_l \nabla_k B_j^a.$$

The last two terms can be expressed in the fundamental tensors, by (3,6) and by noticing that the last term in

$$B_a^i \nabla_l \nabla_k B_j^a = D_{jkl}^i - \nabla_l B_a^i \cdot \nabla_k B_j^a$$

is equal to

$$\begin{aligned} \nabla_l B_a^i \cdot (\nabla_b B_j^a - C_b^a) \nabla_k B_j^b &= B_m^a \nabla_l B_a^i \cdot B_b^m \nabla_k B_j^b + C_c^a \nabla_l B_a^i \cdot C_b^c \nabla_k B_j^b \\ &= -D_{ml}^i D_{jk}^m - G_{sl}^i F_{jk}^s. \end{aligned}$$

Thus we obtain

$$(1) \quad B_{ajkl}^{bcd} C_{bcd}^a = C_{jkl}^i + F_{jil}^s G_{ls}^i + T_{[kl]}^m D_{jm}^i + f_{[kl]}^s H_{js}^i + D_{j[kl]}^i + D_{mil}^m D_{jkl}^m$$

which, in the case of projected connexion ($D = 0$, $E = 0$, $H = 0$, $I = 0$) reduces to

$$(2) \quad B_{ajkl}^{ibcd} C_{bcd}^a(\Lambda) = C_{jkl}^i(\Lambda) + F_{j|l}^i G_{l|s}^i.$$

In the holonomic case, $C_{bcd}^a(\Lambda)$ and C_{jkl}^i reduce to the usual (*but halved*) curvature tensors. Since $-F$ and $-G$ are Schouten's H and L , (2) reduces to Schouten's generalisation of the Gauss equation. Therefore, (1) and (2) are the generalised forms of the Gauss equation for non holonomic tensor submanifolds; (1) for general intrinsic connexion, (2) for the special case of projected connexion.

We give the complete list of relevant formulæ for the resolution of $C_{bcd}^a(\Lambda)$:

$$(3) \quad B_{ajkl}^{ibcd} C_{bcd}^a = C_{jkl}^i(\Lambda) + [T_{[kl}^b(\Lambda) + f_{[kl}^b] B_a^i \nabla_b B_f^a + B_a^i \nabla_{[l} \nabla_{k]} B_f^a,$$

$$(4) \quad C_{ar}^{pb} B_{kl}^c C_{bcd}^a = C_{rkl}^p(\Lambda) + [T_{[kl}^b(\Lambda) + f_{[kl}^b] C_a^p \nabla_b C_r^a + C_a^p \nabla_{[l} \nabla_{k]} C_r^a,$$

$$(5) \quad C_a^{pbcd} C_{bcd}^a = [T_{[kl}^b(\Lambda) + f_{[kl}^b] C_a^p \nabla_b B_f^a + C_a^p \nabla_{[l} \nabla_{k]} B_f^a,$$

$$(6) \quad B_a^i C_r^{pb} B_{kl}^c C_{bcd}^a = [T_{[kl}^b(\Lambda) + f_{[kl}^b] B_a^i \nabla_b C_r^a + B_a^i \nabla_{[l} \nabla_{k]} C_r^a,$$

$$(7) \quad B_{aj}^{ib} C_{st}^c C_{bcd}^a = C_{jst}^i(\Lambda) + [T_{[st}^b(\Lambda) + h_{[st}^b] B_a^i \nabla_b B_f^a + B_a^i \nabla_{[l} \nabla_{k]} B_f^a,$$

$$(8) \quad C_{arst}^{pbcd} C_{bcd}^a = C_{rst}^p(\Lambda) + [T_{[st}^b(\Lambda) + h_{[st}^b] C_a^p \nabla_b C_r^a + C_a^p \nabla_{[l} \nabla_{k]} C_r^a,$$

$$(9) \quad C_a^{pb} B_j^c C_{st}^d C_{bcd}^a = [T_{[st}^b(\Lambda) + h_{[st}^b] C_a^p \nabla_b B_f^a + C_a^p \nabla_{[l} \nabla_{k]} B_f^a,$$

$$(10) \quad B_a^i C_{rst}^{bcd} C_{bcd}^a = [T_{[st}^b(\Lambda) + h_{[st}^b] B_a^i \nabla_b C_r^a + B_a^i \nabla_{[l} \nabla_{k]} C_r^a,$$

$$(11) \quad B_{ajk}^{ibc} C_t^d C_{bcd}^a = C_{jkt}^i(\Lambda, s) + \frac{1}{2} [T_{kt}^b(s) + f_{kt}^b - T_{tk}^b(\Lambda) - g_{tk}^b] B_a^i \nabla_b B_f^a \\ + B_a^i \nabla_{[l} \nabla_{k]} B_f^a,$$

$$(12) \quad C_{ar}^{pb} B_k^c C_t^d C_{bcd}^a = C_{rkt}^p(\Lambda, \sigma) + \frac{1}{2} [T_{kt}^b(s) + f_{kt}^b - T_{tk}^b(\Lambda) - g_{tk}^b] C_a^p \nabla_b C_r^a \\ + C_a^p \nabla_{[l} \nabla_{k]} C_r^a,$$

$$(13) \quad C_a^{pb} B_{jk}^{hc} C_t^d C_{bcd}^a = \frac{1}{2} [T_{kt}^b(s) + f_{kt}^b - T_{tk}^b(\Lambda) - g_{tk}^b] C_a^p \nabla_b B_f^a \\ + C_a^p \nabla_{[l} \nabla_{k]} B_f^a,$$

$$(14) \quad B_a^i C_r^{pb} B_k^c C_t^d C_{bcd}^a = \frac{1}{2} [T_{kt}^b(s) + f_{kt}^b - T_{tk}^b(\Lambda) - g_{tk}^b] B_a^i \nabla_b C_r^a \\ + B_a^i \nabla_{[l} \nabla_{k]} C_r^a.$$

The terms containing derivatives of projection factors can be

expressed in the fundamental tensors by (3.6) and by the formulae

$$(15) \quad B_a^i \nabla_l \nabla_k B_j^a = -\nabla_l D_{jk}^i - D_{jk}^m D_{ml}^i + F_{jk}^q G_{ql}^i,$$

$$(16) \quad C_a^p \nabla_l \nabla_k C_r^a = -\nabla_l E_{rk}^p - E_{rk}^q E_{ql}^p + G_{rk}^m E_{ml}^p,$$

$$(17) \quad C_a^p \nabla_l \nabla_k B_j^a = -\nabla_l F_{jk}^p - D_{jk}^m F_{ml}^p + F_{jk}^q E_{ql}^p,$$

$$(18) \quad B_a^i \nabla_l \nabla_k C_r^a = -\nabla_l G_{rk}^i - E_{rk}^q G_{ql}^i + G_{rk}^m D_{ml}^i,$$

$$(19) \quad B_a^i \nabla_l \nabla_s B_j^a = -\nabla_l H_{js}^i - H_{js}^m H_{ml}^i + J_{js}^q K_{ql}^i,$$

$$(20) \quad C_a^p \nabla_l \nabla_s C_r^a = -\nabla_l I_{rs}^p - I_{rs}^q I_{ql}^p + K_{rs}^m J_{ml}^p,$$

$$(21) \quad C_a^p \nabla_l \nabla_s B_j^a = -\nabla_l J_{js}^p - H_{js}^m J_{ml}^p + J_{js}^q I_{ql}^p,$$

$$(22) \quad B_a^i \nabla_l \nabla_s C_r^a = -\nabla_l K_{rs}^i + I_{rs}^q K_{ql}^i + K_{rs}^m H_{ml}^i,$$

$$(23) \quad B_a^i \nabla_l \nabla_k B_j^a = -\nabla_l D_{jk}^i - D_{jk}^m H_{ml}^i + F_{jk}^q K_{ql}^i,$$

$$(24) \quad C_a^p \nabla_l \nabla_k C_r^a = -\nabla_l E_{rk}^p - E_{rk}^q I_{ql}^p + G_{rk}^m J_{ml}^p,$$

$$(25) \quad C_a^p \nabla_l \nabla_k B_j^a = -\nabla_l F_{jk}^p - D_{jk}^m J_{ml}^p + F_{jk}^q I_{ql}^p,$$

$$(26) \quad B_a^i \nabla_l \nabla_k C_r^a = -\nabla_l G_{rk}^i - E_{rk}^q K_{ql}^i + G_{rk}^m H_{ml}^i,$$

$$(27) \quad B_a^i \nabla_k \nabla_l B_j^a = -\nabla_k H_{jl}^i - H_{jl}^m D_{mk}^i + J_{jl}^q G_{qk}^i,$$

$$(28) \quad C_a^p \nabla_k \nabla_l C_r^a = -\nabla_k I_{rl}^p - I_{rl}^q E_{ql}^p + K_{rl}^m F_{mk}^p,$$

$$(29) \quad C_a^p \nabla_k \nabla_l B_j^a = -\nabla_k J_{jl}^p - H_{jl}^m F_{mk}^p + J_{jl}^q E_{qk}^p,$$

$$(30) \quad B_a^i \nabla_k \nabla_l C_r^a = -\nabla_k K_{rl}^i + I_{rl}^q G_{ql}^i + K_{rl}^m D_{mk}^i.$$

Substituting (3) — (14) we obtain the final forms of the resolutes indicated by a bar,

$$(31) \quad \bar{C}_{jkl}^i(\Lambda) = C_{jkl}^i(\Lambda) + F_{jk}^q G_{ql}^i - T_{kl}^m D_{jm}^i - f_{kl}^q H_{jq}^i - D_{j(k,l)}^i - D_{jk}^m D_{ml}^i,$$

$$(32) \quad \bar{C}_{rkl}^p(\Lambda) = C_{rkl}^p(\Lambda) + G_{rk}^m E_{ml}^p - T_{kl}^m E_{rm}^p - f_{kl}^q I_{rq}^p - E_{r(k,l)}^p - E_{rk}^q E_{ql}^p,$$

$$(33) \quad \bar{C}_{jkl}^i(\Lambda) = -F_{jk}^q E_{ql}^i - T_{kl}^m F_{jm}^i - f_{kl}^q J_{jq}^i - F_{j(k,l)}^i - D_{jk}^m F_{ml}^i,$$

$$(34) \quad \bar{C}_{rkl}^p(\Lambda) = -G_{rk}^m D_{ml}^p - T_{kl}^m G_{rm}^p - f_{kl}^q K_{rq}^p + G_{r(k,l)}^p - E_{rk}^q G_{ql}^p,$$

$$(35) \quad \bar{C}_{jst}^i(\Lambda) = C_{jst}^i(s) + J_{js}^q K_{qt}^i - T_{st}^m H_{jq}^i - k_{st}^m D_{jm}^i - H_{j(s,t)}^i - H_{js}^m H_{mt}^i,$$

$$(36) \quad \bar{C}_{rst}^p(\Lambda) = C_{rst}^p(\sigma) + K_{rs}^m J_{mt}^p - T_{st}^m I_{rq}^p - k_{st}^m E_{rm}^p - I_{r(s,t)}^p - I_{rs}^q I_{qt}^p,$$

$$(37) \quad \bar{C}_{jst}^i(\Lambda) = J_{js}^q I_{qt}^i - T_{st}^m J_{jq}^i - k_{st}^m F_{jm}^i - J_{j(s,t)}^i - H_{js}^m J_{mt}^i,$$

$$(38) \quad \bar{C}_{rst}^p(\Lambda) = K_{rs}^m H_{mt}^p - T_{st}^m K_{rq}^p - k_{st}^m G_{rm}^p - K_{r(s,t)}^p - I_{rs}^q K_{qt}^p,$$

$$(39) \quad {}_2\bar{C}_{jkl}^i(\Lambda) = {}_2C_{jkl}^i(\Lambda, s) + F_{jk}^q K_{ql}^i - J_{jl}^q G_{qk}^i + [T_{kl}^m(s) - g_{lk}^m] D_{jm}^i \\ + [f_{kl}^q - T_{lk}^q(\lambda)] H_{jq}^i - D_{j(k,l)}^i - H_{jk}^m H_{ml}^i - D_{jk}^m D_{ml}^i,$$

$$(40) \quad {}_2\bar{C}_{rkl}^p(\Lambda) = {}_2C_{rkl}^p(\Lambda, \sigma) + G_{rk}^m J_{mt}^p - K_{rt}^m E_{mk}^p + [T_{kl}^m(s) - g_{lk}^m] E_{rm}^p \\ + [f_{kl}^q - T_{lk}^q(\lambda)] I_{rq}^p + E_{r(k,l)}^p - I_{rk}^q I_{qt}^p + E_{rk}^q E_{ql}^p - I_{rt}^q E_{qk}^p,$$

$$(41) \quad {}_2\bar{C}_{jkl}^i(\Lambda) = F_{jk}^q I_{ql}^i - J_{jl}^q E_{qr}^i + [T_{kl}^m(s) - g_{lk}^m] F_{jm}^i \\ - [f_{kl}^q - T_{lk}^q(\lambda)] J_{jq}^i - F_{j(k,l)}^i - J_{jk}^m J_{ml}^i - D_{jk}^m J_{ml}^i - H_{ji}^m F_{mt}^i,$$

$$(42) \quad {}_2\bar{C}_{rst}^p(\Lambda) = G_{rk}^m H_{mt}^p - K_{rt}^m D_{mk}^p + [T_{kl}^m(s) - g_{lk}^m] G_{rm}^p \\ + [f_{kl}^q - T_{lk}^q(\lambda)] K_{rq}^p + G_{r(k,l)}^p - K_{rtk}^p - E_{rk}^q K_{ql}^p - I_{rt}^q G_{qk}^p.$$

From $C_{bde}^a = -C_{bed}^a$, the remaining four resolutes are given by the negative values of (39) — (42).

Equation (31) is the extension of Schouten's form of Gauss' equation, (32) that of Ilavaty's and Schouten's form (Ilavaty 1926, Schouten and van Kampen 1930) of the Kühne-Ricci equation, (33) and (34) are the extension of Schouten's form of Codazzi's equations, to non holonomic tensor submanifolds with independent (intrinsic) fourfold connexion, in general geometrical axes. The other relations seem to be new.

In the case of projected connexion, *i. e.* when $D = 0$, $E = 0$, $H = 0$, $I = 0$, the formulae become quite simple even for nonholonomic submanifolds. This simplicity is due to the introduction of fourfold connexion with the corresponding fundamental tensors.

11. Resolution of metric tensors. — From $v' = \bar{v}^i B_i^a + \hat{v}^p C_p^a$, the squared length of v' is resolved into four parts :

$$(1) \quad g_{ab} v'^a v'^b = g_{ab} B_i^a \bar{v}^i \bar{v}^j + g_{ab} B_i^a C_p^b \bar{v}^i \hat{v}^p + g_{ab} B_p^a B_j^b \hat{v}^p \bar{v}^j + g_{ab} C_{pr}^a \hat{v}^p \hat{v}^r.$$

If v' is in the admitted facet, *i. e.* if $\hat{v}^p = 0$, we put

$$(2) \quad [v']^2 = \bar{g}_{ij} \bar{v}^i \bar{v}^j,$$

where

$$(3) \quad \bar{g}_{ij} = g_{ab} B_{ij}^{ab}.$$

In this way, the length of v measured in the facet by $\bar{g}_{ij}$ will be equal to its original length in X_N .

In a like manner, if we put

$$(4) \quad g_{pr} = g_{ab} C_{pr}^{ab},$$

the length of a vector v' in the span ($\bar{v}^i = 0$), measured in the span by $\hat{g}_{pr}$ will be equal to its original length in X_N .

We also put

$$(5) \quad g_{ip} = g_{ab} B_i^a C_p^b,$$

and notice that $g_{pi} = g_{ip}$ follows from the symmetry of g_{ab} . Thus we

can write (1) in the form

$$(6) \quad g_{ab} v^a v^b = \bar{g}_{ij} \bar{v}^i \bar{v}^j + 2 g_{ip} \bar{v}^i \hat{v}^p + \hat{g}_{pr} \hat{v}^p \hat{v}^r.$$

The angle (v, w) between v^a and w^a is defined by putting

$$(7) \quad |v| |w| \cos(v, w) = g_{ab} v^a w^b.$$

Since the components v^a of e and w^a are B_i^a and C_p^a respectively, we have

$$(8) \quad |e_i| |e_p| \cos(e_i, e_p) = g_{ab} B_i^a C_p^b = g_{ip}.$$

Therefore, $g_{ip} = 0$ if (a) e_i or e_p has a zero length, (b) e_i and e_p are orthogonal. If the latter is true for every i and p , we obtain from (6) the pythagorean relation

$$(9) \quad |v|^2 = |\bar{v}|^2 + |\hat{v}|^2.$$

Similar considerations lead to the corresponding formulae for covariant vectors with arbitrary g^{ab} . We obtain, however, more significant relations if we suppose as usual that g^{ab} are determined by the equations

$$(10) \quad g^{ab} g_{ac} = \delta_c^b.$$

We also put

$$(11) \quad g^{ij} = B_{ab}^i g^{ab}, \quad g^{ir} = B_a^i C_b^r g^{ab}, \quad g^{pr} = C_{ab}^p g^{ab}.$$

All these metric parameters are absolute geometrical structure tensors. To keep, however, to general usage, we do not denote them by capitals.

We give a list of fundamental tensors derived from a given g_{ab} .

TABLE IV.

$Q_{abc} = \nabla_c g_{ab},$	$Q_c^{ab} = \nabla_c g^{ab},$		
$Q_{ijk} = \nabla_k g_{ij},$	$W_{ijs} = \nabla_s g_{ij},$	$Q_k^{ij} = \nabla_k g^{ij},$	$W_s^{ij} = \nabla_s g^{ij},$
$P_{irk} = \nabla_k g_{ir},$	$V_{irs} = \nabla_s g_{ir},$	$P_k^{ir} = \nabla_k g^{ir},$	$V_s^{ir} = \nabla_s g^{ir},$
$O_{prk} = \nabla_k g_{pr},$	$U_{prs} = \nabla_s g_{pr},$	$O_k^{pr} = \nabla_k g^{pr},$	$U_s^{pr} = \nabla_s g^{pr}.$

In the resolution of metric tensors we restrict ourselves to Q_{abc} , since, from (10),

$$(12) \quad g_{ac}Q_d^{ab} + g^{ab}Q_{acd} = 0.$$

The relations between the resolutes of Q_{abc} and the metric tensors of the submanifold are given by the following formulae, where, again, resolutes are indicated by a bar,

$$(13) \quad \bar{Q}_{ijk} = Q_{ijk} - 2g_{m'j}D_{i,k}^m - 2g_{q,j}F_{i,k}^q,$$

$$(14) \quad \bar{Q}_{irk} = P_{irk} - g_{mr}D_{ik}^m - g_{qr}F_{ik}^q - g_{ql}E_{rk}^q - g_{ml}G_{rk}^m,$$

$$(15) \quad \bar{Q}_{prk} = O_{prk} - 2g_{q,r}E_{q,k}^q - 2g_{m,r}G_{p,k}^m,$$

$$(16) \quad \bar{Q}_{ijs} = W_{ijs} - 2g_{m,j}H_{i,s}^m - 2g_{q,j}J_{i,s}^q,$$

$$(17) \quad \bar{Q}_{irs} = V_{irs} - g_{mr}H_{is}^m - g_{qr}J_{is}^q - g_{ql}I_{rs}^q - g_{ml}K_{rs}^m,$$

$$(18) \quad \bar{Q}_{prs} = U_{prs} - 2g_{q,r}I_{p,s}^q - 2g_{m,r}K_{p,s}^m.$$

The remaining two resolutes are obtained from $Q_{buc} = Q_{abc}$.

Proof of (13). From

$$\begin{aligned} Q_{ijk} &= \nabla_k g_{ij} = \nabla_k (B_{ij}^{ab}) \cdot g_{ab} + B_{ij}^{ab} \nabla_k g_{ab} \\ &= (B_j^b \nabla_k B_i^a + B_i^a \nabla_k B_j^b) g_{ab} + B_{ij}^{ab} Q_{abc}. \end{aligned}$$

(3,4) readily leads to (13). The proof is similar in the other cases.

We notice that, for an orthogonal span, P and V vanish, but not the other four tensors. In the special case of projected connexion and orthogonal space, formulae (13) — (18) become

$$(19) \quad \bar{Q}_{ijk} = Q_{ijk},$$

$$(20) \quad \bar{Q}_{irk} = -g_{qr}F_{ik}^q - g_{ml}G_{rk}^m,$$

$$(21) \quad \bar{Q}_{prk} = O_{prk},$$

$$(22) \quad \bar{Q}_{ijs} = W_{ijs},$$

$$(23) \quad \bar{Q}_{irs} = -g_{qk}J_{is}^q - g_{ml}K_{rs}^m,$$

$$(24) \quad \bar{Q}_{prs} = U_{prs}.$$

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