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PAR JOSEPH LIOUVILLE

E. R. HEDRICK
LOUIS INGOLD
W. D. A. WESTFALL

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Theory of non-analytic Functions of a complex Variable;

By E. R. HEDRICK, Louis INGOLD
and W. D. A. WESTFALL.

1. *Introduction.* — This paper is concerned with the extension of the classical theory of functions of a complex variable obtained by the removal of the restrictions imposed by the Cauchy-Riemann equations. Only certain very general continuity and differentiability conditions are retained.

The analytic theory is regarded as being equivalent to the theory of a special type of transformations from a pair of variables (x, y) to another pair (u, v). The removal of the above restrictions permits much more general transformations to be considered in the same way.

The paper is written (as is the classical theory) in the language of function-theory rather than in the language of transformations. It is certainly true, as is often insisted [1], that the study of such transformations can be made without the use of complex variables. It is equally true, however, that the study of even an analytic function is only a study of its components, which are real functions, and that the complex variable is, therefore, in no way essentially involved.

The theory of analytic functions, however, is already developed, and it has seemed to the authors that the results of that theory can be much more conveniently utilized in the present paper if the same phraseology is employed; furthermore the use of the complex variable makes it possible to compare the results of this paper with those of the

[1] See PICARD, *Traité d'Analyse*, vol. II, p. 1.

special analytic case, and to better recognize the true position of this special theory in relation to the more general theory.

The word *function* is employed in the general sense defined by Dirichlet. A complex variable w is a function of the complex variable z in a given region R if there exists a value of w corresponding to each value of z in that region. The ordinary technical terms of function-theory, as *multiple-valued*, *single-valued*, etc., are used with their usual meanings.

The two outstanding properties of analytic functions wich lead to the Cauchy-Riemann equations are the conformal property, and the equivalent property of possessing a unique (1) derivative.

In the case of non-analytic functions it is natural to inquire, in connection with the first of these properties, how the angular distortion at a point is related to the directions of the sides of the angle.

It is found that for a fixed angle in the plane of the independent variable there are, in general, two directions for the initial side corresponding to which the angular distortion takes on extreme values.

These directions vary with the angle, but are perfectly determinate even in the limiting case in which the angle under consideration approaches zero. For this special case the two characteristic directions coincide with certain directions discovered by Tissot in connection with mapping problems.

A similar study is made in regard to the difference quotient. It is found that there are two directions at each point in the plane of the independent variable along which the square of the limit of the absolute value of the difference quotient takes on extreme values. It is particularly interesting to note that these directions coincide with the directions of the special case mentioned above.

The lines which these two directions determine (here called *characteristic lines*), are studied by means of the fundamental quantities E, F, G, defined precisely as the fundamental quantities of differential geometry. The lines are somewhat similar to lines of curvature. A

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(1) It is customary to say that the derivative does not exist in case the limit of the difference quotient is not unique. There are, however, occasional departures from this practice,

distinction between analytic and non-analytic functions is that the characteristic lines of the former become indeterminate. This is comparable, in differential geometry, to the case of the sphere, whose lines of curvature become indeterminate. The indicatrix of differential geometry is analogous to Tissot's indicatrix, which becomes a circle in the analytic case.

The notion of the *ellipticity* of a function is introduced. This is the absolute value of the difference between the semi-axes of Tissot's indicatrix. The ellipticity can be used as measure of the divergence of a function at any point from an analytic function.

A second approach to the study of this theory is made through the Beltrami equations. Use is made of the result that any two complex functions on a given surface are analytic functions of each other. By considering the special case in which the given surface is a plane a complete classification of functions is obtained. Functions of the same class in a given region are the functions which are defined in that region, and which are analytic functions of each other. From this point of view analytic functions defined in a given region simply constitute one of these classes.

Many theorems of the usual theory of analytic functions can be generalized to apply to functions of any one class. Thus there exists in each class a theory almost coextensive with the analytic theory. No attempt has been made here to push this generalization to its conclusion, but to illustrate this possibility, the method is applied to one or two of the more important theorems of the ordinary theory.

Finally, it should be mentioned that throughout the paper only those regions are admitted in which the functions under consideration are free from singularities. A future paper is contemplated which will include a study of the properties of non-analytic functions in the neighborhood of their points of discontinuity, together with extensions of the ideas of the present paper.

2. *Directions of maximum and minimum angular distortion.* — One of the most prominent distinctions between analytic and non-analytic functions lies in the fact that in the case of analytic functions, the correspondence between the planes of the dependent and indepen-

dent variables preserves angles. In the case of non-analytic functions, it is of interest to inquire in what manner the amount of distortion of angles at a point varies as the sides of the angle vary in direction. This question will now by investigated.

We denote, as usual, the independent variable by $z = x + iy$ and consider the function

$$(1) \qquad w = u + iv = f(z).$$

Let u and v, the real and imaginary parts of w, and also their first, second, and third partial derivatives (¹) be assumed to be continuous in the regions considered.

We introduce the fundamental quantities E, F and G, defined as follows :

$$(2) \quad E = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2, \quad G = \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2, \quad F = \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y}.$$

These quantities are, of course, not independent. Since they are obtained from a certain representation of the plane, they must satisfy the partial differential equation $K = 0$ where K is the expression for the Gaussian curvature of the differential form $E dx^2 + 2F dx dy + G dy^2$.

Now consider two directions in the z-plane determined by $\tan\theta = h$ and $\tan\varphi = m$, where the angles θ and φ are measured from the x-axis, and suppose that the angle $\alpha = \theta - \varphi$ remains fixed while m and h vary; then we have,

$$(3) \qquad \tan\alpha = k = \frac{h - m}{1 + hm}.$$

Denote by β the angle in the w-plane corresponding to α. An easy computation gives

$$(4) \qquad \tan\beta = \frac{k(1 + m^2)\sqrt{(EG - F^2)}}{E + Fk + [k(G - E) + 2F]\,m + (G - Fk)\,m^2}.$$

In the usual manner the following equation is obtained for the values of m which yield extreme values for $\tan\beta$ when they exist.

$$(5) \quad [k(G - E) + 2F]\,m^2 - 2(G - E - 2kF)\,m - k(G - E) - 2F = 0.$$

If the coefficient of m^2 in this equation does not vanish, it is numeri-

(¹) The curvature K introduced below involves the third derivatives of u and v.

cally equal to the term independent of m but opposite in sign. The solutions of (5) are then real and are the negative reciprocals of each other. The corresponding directions are therefore perpendicular to each other. We shall call them the *principal directions* of the function w *with respect to the angle* α.

If the solutions of equation (5) are distinct, then one of the solutions corresponds to a maximum value of $\tan \beta$ and the other corresponds to a minimum. The general proof of this statement is omitted on account of its length. The proof for the case that is of particular interest to us is given in § 7.

If the coefficient of m^2 vanishes, we shall say that the directions parallel to the axes are the principal directions. These also correspond to maximum and minimum values of $\tan \beta$.

In case the coefficients of both m and m^2 vanish, equation (5) degenerates altogether, and we shall say that any two orthogonal directions are principal directions for this case. Here it may be seen that $F = 0$ and $E = G$, so that $\tan \beta$ is independent of m. We collect the results for the various cases mentioned above, in the following theorem.

THEOREM I. — *For any function* w *of the complex variable* z *there exist, through each point of the* z*-plane, two principal directions with respect to any fixed angle* α. *These directions are perpendicular to each other.*

3. *Special cases.* — There are several interesting special cases of equation (5).

(*a*) If $G = E$, the equation becomes

$$m^2 + 2km - 1 = 0,$$

and is independent of the fundamental quantities, so that for all such functions, the principal directions for a given value of k at a given point are the same.

(*b*) If $F = 0$, the equation becomes

$$km^2 - 2m - k = 0,$$

and is again independent of the fundamental quantities.

(*c*) If α is a right angle, k becomes infinite and the limiting form of the equation is then

$$(G-E)m^2+4Fm-(G-E)=0.$$

The values of $\tan\beta$ corresponding to the values of m which satisfy this equation reduce to

$$\pm\frac{2\sqrt{(EG-F^2)}}{\sqrt{[(G-E)^2+4F^2]}}.$$

(*d*) If both $E=G$ and $F=0$, the equation degenerates. In this case it can be shown that u and v satisfy the Cauchy-Riemann equations, or it can be seen that the expression for $\tan\beta$ reduces to k, and the function in this case will be said to be *analytic at the point* (1).

4. *The limiting case* $k=0$. — If we consider the limiting case in which the angle α reduces to zero we see from formula (4) that $\tan\beta$ also vanishes. In this case, instead of considering the maximum and minimum values of $\tan\beta$, we may consider the maximum and minimum values of the limit of $\frac{\tan\beta}{\tan\alpha}$, as $\alpha=0$. The corresponding form of equation (5) can be obtained by simply putting $k=0$ in that equation, which gives

$$Fm^2-(G-E)m-F=0.$$

The values of m satisfying this equation correspond to maximum and minimum values of $\frac{d\psi}{d\theta}$, where ψ is the angle in the w-plane corresponding to the angle θ in the z-plane.

A detailed discussion of this limiting case will be given in the next

(1) We shall use the expression *analytic at a point* P in this sense, even when the function is not analytic at any other point in the neighborhood of P. This concept seems to be interesting and to be capable of development. Thus, if a function is analytic at each point of a set (E), it follows immediately from the continuity conditions already assumed that the function is analytic at every point of the first derived set (E′). In particular, if a function is analytic at a set of points that are dense in any region (or on any curve) the function is analytic throughout that region (or on that curve) provided the continuity conditions remain true.

few articles, where the equation for m derived above will be obtained in a different connection.

3. *Directional dérivatives of functions.* — Instead of studying the variation in the distortion of angles, the variation in the value of the limit of the difference quotient may investigated. It will be convenient to regard this limit as a derivative associated with a given direction, and to use the usual notation for a derivative. The limit will be called the *directional derivative* of w.

By definition

$$\frac{dw}{dz} = \lim_{\Delta z = 0} \frac{\Delta u + i\Delta v}{\Delta x + i\Delta y}.$$

The assumptions that have been made concerning the functions u and v permit the use of the law of the mean for functions of two variables. Thus we have,

$$\Delta u = \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + R_1,$$

$$\Delta v = \frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y + R_2,$$

where the remainders, R_1 and R_2, are such that

$$\lim_{\Delta x = 0} \frac{R_1}{\Delta x} = 0, \qquad \lim_{\Delta x = 0} \frac{R_2}{\Delta x} = 0.$$

By the use of these formulas we obtain

$$\frac{dw}{dz} = \left[\frac{\partial u}{\partial x} + m\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + m^2\frac{\partial v}{\partial y}\right] : (1 + m^2)$$
$$+ i\left[\frac{\partial v}{\partial x} + m\left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\right) - m^2\frac{\partial u}{\partial y}\right] : (1 + m^2),$$

where Δz is supposed to approach zero in such a way that $\frac{\Delta y}{\Delta x}$ approaches the limit m. The discussion can be easily modified to suit the case in which $\frac{\Delta x}{\Delta y}$ approaches zero.

By squaring and adding the real and imaginary parts of $\frac{dw}{dz}$ we obtain

$$\left|\frac{dw}{dz}\right|^2 = \frac{E + 2mF + m^2(E + G) + 2m^3F + m^4G}{(1 + m^2)^2} = \frac{E + 2mF + Gm^2}{1 + m^2}.$$

This last expression is seen to be equivalent to $\frac{d\sigma^2}{ds^2}$ where ds and $d\sigma$ are the corresponding differentials of arc length in the z-plane, and in the w-plane, respectively.

6. *The principal directions for the derivative.* — We have already considered the distinction between analytic and non-analytic functions based upon the conformal property. A further fundamental distinction between them consists in the fact that the directional derivatives of the latter at a given point are not unique, but depend on the manner in which the increment of the independent variable approaches zero.

It is of interest, then, to know in what way these directional derivatives vary as the method of approach to a point varies, and in particular, to know whether the absolute value of the derivative regarded as a function of m, takes on maximum and minimum values. This question will now be considered.

For convenience we write

$$r = \left|\frac{dw}{dz}\right|^2 = \frac{E + 2Fm + Gm^2}{1 + m^2}. \tag{6}$$

The quadratic equation in m which determines the extreme values of r, when they exist, is

$$F + (G - E)m - Fm^2 = 0. \tag{7}$$

This is the special limiting case of equation (5) which was considered in § 4. That this should be so may be seen by noticing that from equation (4) we have

$$\lim_{k=0} \frac{\tan g\beta}{k} = \frac{\sqrt{(EG - F^2)}}{r}.$$

If F does not vanish equation (7) determines two mutually orthogonal directions. These are the principal directions of the function w *with respect to a zero angle*. We shall call them simply the *principal directions* of the function w.

These directions for any non-conformal transformation of any surface were discovered by Tissot (¹) in connection with mapping problems.

If $F = 0$ and $E \neq G$, we shall say that the directions parallel to the axes are the principal directions. If $F = 0$ and $E = G$ we shall say that any two orthogonal directions are the principal directions. Thus in all cases we have the following theorem.

THEOREM II. — *For any function of the complex variable z there exists, through each point of the z-plane, a pair of principal directions. These directions are perpendicular to each other.*

The pair of principal directions through a point is unique except at points for which $E = G$ and $F = 0$.

The curves determined by the solutions of the differential equation (7) will be called the *characteristic lines* of the function w.

7. *The maximum and minimum values of r.* — The equation (7) regarded as an equation for m may be written in either of the forms

$$(8) \qquad \frac{F + Gm}{m} = \frac{E + 2Fm + Gm^2}{1 + m^2}, \qquad E + Fm = \frac{F + Gm}{m}.$$

Let λ denote either of the values of m satisfying equation (7) and let ρ denote the corresponding value of r. Then from equations (8)

$$(9) \qquad \rho = \frac{F + G\lambda}{\lambda}, \qquad \text{or} \qquad \rho = E + F\lambda.$$

The elimination of λ leads to the equation

$$(10) \qquad \rho^2 - (E + G)\rho - F^2 = 0.$$

The two solutions of this equation, when different, will be called the *principal expansion factors* of the function w.

Equations (9) can be solved for E and G in terms of ρ, λ, and F. When the resulting values are substituted in equation (6), the expres-

(¹) TISSOT, *Sur les cartes géographiques* (*Comptes rendus*, 1849).

sion for r reduces to

$$(11) \qquad r = \rho - \mathrm{F}\frac{(m-\lambda)^2}{\lambda(1+m^2)}.$$

From this it follows directly that if λ is the positive solution of equation (7), then ρ is the maximum of r at points where F is positive, and the minimum where F is negative. This may be seen also by computing $\frac{dr}{dm}$ from equation (11).

If $\mathrm{F} = 0$ while $\mathrm{G} \neq \mathrm{E}$, the equation (6) for r becomes

$$r = \frac{\mathrm{E} + \mathrm{G}m^2}{1+m^2},$$

from which we see that if G is positive and greater than E, $r = \mathrm{E}$ is a minimum corresponding to $m = 0$, and that $r = \mathrm{G}$ is a maximum corresponding to $m = \infty$. If G is negative the maximum and minimum values are interchanged. The argument is similar if $\mathrm{E} > \mathrm{G}$. Thus in every case we have proved the following theorem.

THEOREM III. — *The value of r at each point of the z-plane, corresponding to one of the principal directions is a maximum, and the value corresponding to the other principal direction is a minimum, whenener the solutions of equation* (10) *are distinct.*

8. *Fundamental property of characteristic lines.* — The characteristic lines form a special orthogonal system of lines in the z-plane. The lines of the w-plane corresponding to them also form an orthogonal system. This suggests the problem of determining the conditions which an orthogonal system must satisfy in order that the corresponding system may also be orthogonal. Let $p(x, y) = \text{const.}$, $q = \text{const.}$, be an orthogonal system of curves in the z-plane which corresponds to an orthogonal system in w-plane. Along one of the curves $q = c$, x and y may be regarded as functions of p, and along $p = c$ they may be regarded as functions of q. Similarly, in the w-plane, u and v along one set of the corresponding curves may be regarded as functions of p and along the other set, as functions of q.

The conditions of orthogonality in the two planes are

$$\frac{\partial x}{\partial p}\frac{\partial x}{\partial q}+\frac{\partial y}{\partial p}\frac{\partial y}{\partial q}=0, \qquad \frac{\partial u}{\partial p}\frac{\partial u}{\partial q}+\frac{\partial v}{\partial p}\frac{\partial v}{\partial q}=0.$$

The second of these may be written in the form

$$\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial p}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial p}\right)\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial q}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial q}\right)$$
$$+\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial p}+\frac{\partial v}{\partial y}\frac{\partial y}{\partial p}\right)\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial q}+\frac{\partial v}{\partial y}\frac{\partial y}{\partial q}\right)=0;$$

or

$$E\frac{\partial x}{\partial p}\frac{\partial x}{\partial q}+F\left(\frac{\partial x}{\partial p}\frac{\partial y}{\partial q}+\frac{\partial y}{\partial p}\frac{\partial x}{\partial q}\right)+G\frac{\partial y}{\partial p}\frac{\partial y}{\partial q}=0$$

and by means of the first equation this can be reduced to

$$F\left(\frac{\partial x}{\partial p}\right)^2-(G-E)\frac{\partial x}{\partial p}\frac{\partial y}{\partial p}-F\left(\frac{\partial y}{\partial p}\right)^2=0,$$

or to a similar equation with p replaced by q. Now along one of the curves $q=c$,

$$\frac{\partial q}{\partial x}\frac{\partial x}{\partial p}+\frac{\partial q}{\partial y}\frac{\partial y}{\partial p}=0.$$

By means of this formula the equation above reduces to

$$F\left(\frac{\partial q}{\partial y}\right)^2-(G-E)\frac{\partial q}{\partial y}\frac{\partial q}{\partial x}-F\left(\frac{\partial q}{\partial x}\right)^2=0.$$

The function p satisfies the same equation. These equations show that $m=-\left(\frac{\partial q}{\partial x}\right)\div\left(\frac{\partial q}{\partial y}\right)$ computed from $q=c$ and the corresponding value computed from $p=c$, both satisfy equation (5). These curves are therefore the characteristic lines. Thus we have proved the following theorem.

THEOREM IV. — *The only orthogonal system of curves in the z-plane which corresponds, by means of a given non-analytic function w, to an orthogonal system in the w-plane, is the system of characteristic lines of the function w* (¹).

(¹) For a geometric proof of this theorem see Darboux, *Leçons sur la théorie générale des surfaces*, vol. III, p. 49.

9. *The Beltrami Equations.* — The Cauchy-Riemann equations have been generalized by Beltrami to apply to a complex function on any surface. These generalized equations are valid for complex functions in a plane. We have, therefore, for any function $w = u + iv$ of the complex variable z the following equations :

$$\frac{\partial u}{\partial x} = \pm\left(E\frac{\partial v}{\partial y} - F\frac{\partial v}{\partial x}\right) : J, \qquad \left[J = \sqrt{(EG - F^2)}\right],$$
$$\frac{\partial u}{\partial y} = \pm\left(F\frac{\partial v}{\partial y} - G\frac{\partial v}{\partial x}\right) : J,$$

and also the equations obtained from these by interchanging the functions u and v.

From these equations u may be eliminated by equating the two expressions for the second partial derivative of u with respect to x and y. This is permissible, since the second derivatives of u and v are assumed to be continuous. Similarly, v may be eliminated. When the elimination is performed, it is found that u and v both satisfy the following partial differential equation of the second order :

$$(12) \qquad \frac{G}{J}\frac{\partial^2\theta}{\partial x^2} + \frac{E}{J}\frac{\partial^2\theta}{\partial y^2} - \frac{2F}{J}\frac{\partial^2\theta}{\partial x\,\partial y} + \left[\frac{\partial\left(\frac{E}{J}\right)}{\partial y} - \frac{\partial\left(\frac{F}{J}\right)}{\partial x}\right]\frac{\partial\theta}{\partial y} + \left[\frac{\partial\left(\frac{G}{J}\right)}{\partial x} - \frac{\partial\left(\frac{F}{J}\right)}{\partial y}\right]\frac{\partial\theta}{\partial x} = 0.$$

If, in the equation just obtained, the quantities E, F, and G are replaced by three quantities proportional to them, the equation would obviously remain unaltered, since in each term the factor of proportionality would occur in both numerator and denominator.

If v is a solution of equation (12), the solution u satisfying Beltrami's equations given above is determined except for an additive constant.

10. *Classification of functions.* — The generalized Laplace equation obtained at the close of the previous article may be used as the basis of a very convenient classification of functions. It is well known that all functions on a given surface are analytic functions of each other ([1]). This is true, in particular, for functions in a plane. It follows that all functions whose real and imaginary parts satisfy equation (12) are analytic functions of each other. All such functions which are defined throughout a given region *except at a set of points whose derived set is finite*, will be said to belong to the same class in that region.

Since the ratios of the quantities E, F, G, determine equation (12), these ratios determine the class to which a function belongs; thus if we write $p = \frac{F}{E}$, $q = \frac{G}{E}$, we may say that any function whose fundamental quantities determine the ratios p and q belongs to the class (p, q).

The following statements are easily proved :

Every function (subject to the continuity and differentiability conditions previously specified) *belongs to a definite class, in a given region.*

No function belongs to two different classes in the same region.

The totality of analytic functions in a given region constitute a separate class, the class $(0, 1)$.

There exist functions belonging to every class in a given region.

Equation (12) will be called the *differential equation of the class* determined by the quantities E, F, G, appearing in its coefficients.

11. *The ellipticity of a function.* — In order to picture more vividly the behavior of a function in the neighborhood of a point, an ellipse may be constructed whose axes lie in the direction of the characteristic lines at that point, and whose semi-major and semi-minor axes are equal to the principal expansion factors. The ellipse constructed in this manner has been called *Tissot's indicatrix*.

The semi-axes of Tissot's indicatrix are obtained from equation (10) of § 7. In order to determine under what circumstances the indicatrix reduces to a circle we form the discriminant of this equation, which is, $(E + G)^2 - 4(EG - F^2)$. This can be reduced to $(G - E)^2 + 4F^2$, so that the discriminant vanishes only if $G = E$ and $F = 0$.

([1]) See Picard, *Traité d'Analyse*, vol. II, p. 9. The restriction which we have introduced below, that the functions have a common region of definition, should be included here.

If these equations hold only at certain points, such points will be called *analytic points* of the function. These points may be isolated, or they may fill out a curve in the z-plane, or they may cover an entire region of the plane. In any case, they form a closed set. When they cover an entire region R the function is analytic in R.

Since the two principal expansion factors are equal at analytic points, we have equation

$$\rho_1 - \rho_2 = 0.$$

In the neighborhood of analytic points (assuming continuity) the difference of the principal expansions becomes smaller and the function behaves more and more like an analytic function. The absolute value of the difference between the solutions of equation (10) may be taken as the measure of the divergence of the function from an analytic function.

The expression

$$|\rho_1 - \rho_2| = 0,$$

will be called the *ellipticity* of the function. Given a set of points P_1, P_2, ..., P_n, ... that have a limit point P, if the ellipticity at P_i is ε_i and if $\lim_{i \to \infty} \varepsilon_i = 0$, the function is analytic at P.

12. *The indicatrix for functions of the same class.* — If two functions belong to the same class in a given region, the indicatrix of one of them is related to the indicatrix of the other in a very simple way. The indicatrix, at a given point, is determined by the values of the principal expansion factors. Since the quantities E, F, G, for two functions of the same class are proportional, it follows from the form of equation (10) that the principal expansion factors for one function are proportional to those of another of the same class. It is assumed that the factor of proportionality neither vanishes nor becomes infinite at the point under consideration. Thus we obtain the following theorem.

THEOREM V. — *The indicatrix of any function is similar to the indicatrix of any other function of the same class, at points for which the factor of proportionality beeween the fundamental quantities neither vanishes nor becomes infinite.*

13. *The proportionality factor.* — We have already noted that the fundamental quantities E, F and G, must satisfy the differential equation $K = 0$, where K is the expression for the total curvature of the differential form

$$E\,dx^2 + 2F\,dx\,dy + G\,dy^2.$$

The equation $K = 0$, written out, is

$$\frac{1}{2J}\left[\frac{\partial}{\partial x}\left(\frac{F}{EJ}\frac{\partial E}{\partial y} - \frac{1}{J}\frac{\partial G}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{2}{J}\frac{\partial F}{\partial x} - \frac{1}{J}\frac{\partial E}{\partial y} - \frac{F}{EJ}\frac{\partial E}{\partial x}\right)\right] = 0.$$

If we substitute λE, λF, λG, for E, F, G, in this equation, we obtain equation (12) with $\log\lambda$ in the place of θ. As a result, therefore, we obtain the following theorem.

THEOREM VI. — *The logarithm of the factor of proportionality of the two sets of fundamental quantities that belong to two functions of the same class must satisfy the differential equation of the class.*

14. *The class of a function of a function.* — Let $W_1 = U_1 + iV_1$ and $W_2 = U_2 + iV_2$ be two functions of z of the same class. We wish to consider how they are related when they are regarded as functions of a function $w = u + iv$.

Since the two functions (as functions of z) are of the same class, $E_1 = \lambda E_2$, $F_1 = \lambda F_2$, $G_1 = \lambda G_2$. The variables are now to be changed from the pair (x, y) to the pair (u, v). The formulas for the new values of the fundamental quantities are

$$\frac{e_1}{e_1 g_1 - f_1^2} = \frac{E_1\left(\frac{\partial v}{\partial y}\right)^2 - 2F_1\frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + G_1\left(\frac{\partial v}{\partial x}\right)^2}{E_1 G_1 - F_1^2},$$

$$\frac{e_2}{e_2 g_2 - f_2^2} = \frac{E_2\left(\frac{\partial v}{\partial y}\right)^2 - 2F_2\frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + G_2\left(\frac{\partial v}{\partial x}\right)^2}{E_2 G_2 - F_2^2},$$

with the corresponding formulas for f_1, f_2, and g_1, g_2.

In the first of these formulas we may replace E_1, F_1, G_1, on the right by their values, λE_2, λF_2, λG_2. The result is $\frac{1}{\lambda}$ times the second expression on the right. Thus

$$\frac{\lambda e_1}{f_1^2} = \frac{e_2}{f_2^2} \qquad \text{or} \qquad e_1 = k e_2,$$

where $k = \frac{f_1^2}{(\lambda f_2^2)}$. Similarly

$$f_1 = k f_2, \qquad \text{and} \qquad g_1 = k g_2.$$

Hence we have the following theorem.

THEOREM VII. — *If W_1 and W_2 are two functions of the same class when regarded as functions of some variable z, then they are also of the same class when regarded as functions of any function of z.*

13. *Functions determined by boundary values.* — As mentioned in the introduction, it is possible to generalize many theorems of the analytic function theory so as to apply to functions of any given class in a given region. As an illustration of this application of our theory we give a generalization of the well-known theorem concerning the determination of a function by means of its values on a given boundary.

Let $w = \varphi(z)$ be a non-analytic function of a certain class in a region including the closed region R, and consider the function

$$F(z) = A(w),$$

where A is analytic. $F(z)$ is then determined by its values on the boundary of a closed region R′ in the w-plane. If now, the closed region R′ corresponds uniquely by means of the function $w = \varphi(z)$, to the closed region R in the z-plane, then the function $F(z)$ is determined by its values on the boundary of the region R. We are thus led to following theorem.

THEOREM VIII. — *If by means of a non-analytic function $w = \varphi(z)$, there is a unique correspondence between a closed region R of the z-plane and a closed region R′ of the w-plane, then there is one and only one function $F(z)$ of the class determined by $\varphi(z)$ which takes on given values along the boundary of R.*