

JOURNAL
DE
MATHÉMATIQUES

PURES ET APPLIQUÉES

FONDÉ EN 1836 ET PUBLIÉ JUSQU'EN 1874

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Journal de mathématiques pures et appliquées 8^e série, tome 4 (1921), p. 327-341.

[<http://www.numdam.org/item?id=JMPA_1921_8_4_327_0>](http://www.numdam.org/item?id=JMPA_1921_8_4_327_0)



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*Integral solutions of the equation $\xi^2 + \eta^2 = \zeta^2$
in the quadratic realms of rationality;*

By HARRIS HANCOCK.

One of the simplest Diophantine equations is $x^2 + y^2 = z^2$.

The solution of this equation gives the so called Pythagorean numbers $x = 2pqt$, $y = (p^2 - q^2)t$, $z = (p^2 + q^2)t$, where p, q, t are rational integers such that $p > q > 0$, $t > 0$ with the further condition that p and q are relatively prime and both must not be odd.

For example,

$$\begin{array}{llllll} p = 2, & q = 1, & x = 4, & y = 3, & z = 5, & 5^2 = 4^2 + 3^2, \\ p = 5, & q = 4, & x = 40, & y = 9, & z = 41, & 41^2 = 40^2 + 9^2, \text{ etc.} \end{array}$$

A treatment of such problems in the realm of natural integrals is found in the second volume of Dickson's admirable *History of the Theory of Numbers*. As every problem, whose history is given by Dickson, admits a generalized treatment in the algebraic realms of the second, third and higher degrees, it is seen that a vast field of further investigation is open to Mathematicians. And by developing this algebraic number-theory new light may be thrown upon the theory of algebraic equations.

By definition a quadratic algebraic integer is the root of the equation $t^2 + At + B = 0$, where A and B are rational integers and the coefficient of t^2 is unity.

Let

$$\xi = a_1 + \sqrt{a_1^2 + a_2}, \quad \eta = b_1 + \sqrt{b_1^2 + b_2}, \quad \zeta = c_1 + \sqrt{c_1^2 + c_2}$$

be three quadratic algebraic integers, being respectively the roots of

$$x^2 - 2a_1x - a_2 = 0, \quad y^2 - 2b_1y - b_2 = 0, \quad z^2 - 2c_1z - c_2 = 0,$$

where $a_1, a_2, b_1, b_2, c_1, c_2$ are arbitrary rational integers.

If then $\xi^2 + \eta^2 = \zeta^2$, it follows that

$$(I) \quad 2a_1\xi + a_2 + 2b_1\eta + b_2 = 2c_1\zeta + c_2,$$

or

$$(I^a) \quad \begin{aligned} 2a_1^2 + a_2 + 2a_1\sqrt{a_1^2 + a_2} + 2b_1^2 + b_2 + 2b_1\sqrt{b_1^2 + b_2} \\ = 2c_1^2 + c_2 + 2c_1\sqrt{c_1^2 + c_2}. \end{aligned}$$

Our problem is to solve this latter equation in integral values of $a_1, a_2, b_1, b_2, c_1, c_2$.

First let $a_1 = 0$. It follows that

$$(II) \quad a_2 + 2b_1^2 + b_2 = 2c_1^2 + c_2,$$

$$(III) \quad b_1\sqrt{b_1^2 + b_2} = c_1\sqrt{c_1^2 + c_2}.$$

Writing (III) in the form

$$(IV) \quad b_1^3 - c_1^3 = c_1^2c_2 - b_1^2b_2,$$

it is seen that b_1, c_1 , may be taken at pleasure and c_2, b_2 so chosen as to satisfy (IV). Then from (II) a_2 is determined.

For example let $b_1 = 4, c_1 = 3$, then is $b_2 = -700, c_2 = -1225$ and $a_2 = -539$. And that is, if η is a root of $y^2 - 8y + 700 = 0$ and ξ a root of $z^2 - 6z + 1225 = 0$, then $\eta^2 = \xi^2 + 539$. See *Diophantus* II, 11; *Dickson* II, p. 402.

Next observe that from (I^a) follow the two equations

$$(1) \quad 2a_1^2 + a_2 + 2b_1^2 + b_2 = 2c_1^2 + c_2,$$

$$(2) \quad a_1\sqrt{a_1^2 + a_2} + b_1\sqrt{b_1^2 + b_2} = c_1\sqrt{c_1^2 + c_2},$$

That these equations admit solution is seen at once if we put

$$a_1 = b_1 = c_1; \quad c_2 = 2a_1^2 + a_2 + b_2. \quad a_2 = b_2 = -\frac{a_1^2}{2}.$$

For it is evident that if ξ, η, ζ are respectively the roots of

$$x^2 - 2a_1x + \frac{a_1^2}{2} = 0, \quad y^2 - 2a_1y + \frac{a_1^2}{2} = 0, \quad z^2 - 2a_1z - a_1^2 = 0,$$

that $\xi^2 + \eta^2 = \zeta^2$.

Note that a_1 must be an even integer in order that ξ and η be algebraic integers.

For example,

$$x^2 - 4x + 2 = 0, \quad y^2 - 4y + 2 = 0, \quad z^2 - 4z - 4 = 0,$$

have roots

$$\xi = \eta = 2 + \sqrt{2}, \quad \zeta = 2 + 2\sqrt{2}$$

such that

$$\xi^2 + \eta^2 = \zeta^2.$$

A solution of (1) and (2) is had as follows :

Put

$$(3) \quad \sqrt{b_1^2 + b_2} = m \sqrt{a_1^2 + a_2},$$

where m is a rational integer to be determined.

We have at once

$$(4) \quad c_1 = a_1 + mb_1 \quad \text{and} \quad \sqrt{a_1^2 + a_2} = \sqrt{c_1^2 + c_2}.$$

In the latter expression put for c_1 its value so that

$$\sqrt{a_1^2 + a_2} = \sqrt{(a_1 + mb_1)^2 + c_2} = \sqrt{a_1^2 + 2a_1mb_1 + m^2b_1^2 + c_2}$$

and regarding a_2 fixed, make c_2 satisfy the relation

$$(5) \quad a_2 = 2a_1mb_1 + m^2b_1^2 + c_2.$$

This expression may be simplified by observing from (1) that

$$(6) \quad a_2 + 2b_1^2 + b_2 = 4a_1mb_1 + 2m^2b_1^2 + c_2.$$

From (5) and (6) it is seen that

$$(7) \quad 2b_1^2 + b_2 = a_2 - c_2,$$

and also that

$$(8) \quad m^2b_1^2 + 2a_1b_1m = 2b_1^2 + b_2.$$

The latter equation in m may be satisfied in an infinite number of ways. For let $2a_1 = 3mb_1$, thus making a perfect square on the left hand side and put respectively $b_2 = 2b_1^2, = 14b_1^2, = 34b_1^2, = 62b_1^2$, etc.

It is seen that m may take the values 1, 2, 3, 4,

For example let $m = 2$, so that $a_1 = 3b_1, b_2 = 14b_1^2$. From (3) it follows that $b_1^2 + b_2 = m^2(a_1^2 + a_2)$, or $-21b_1^2 = 4a_2$, an expression which offers an infinite number of values of a_2 and b_1 . Write $b_1 = 2, a_2 = -21, a_1 = 6, b_2 = 56$. From (4) and (5) it is seen that $c_1 = 10$ and $c_2 = -85$.

It is thus shown that the roots of

$$\text{arc} \quad \begin{aligned} x^2 - 12x + 21 = 0, \quad y^2 - 4y - 56 = 0, \quad z^2 - 20z + 85 = 0, \\ \xi = 6 + \sqrt{15}, \quad \eta = 2 + 2\sqrt{15}, \quad \zeta = 10 + \sqrt{15} \end{aligned}$$

and that

$$\xi^2 + \eta^2 = \zeta^2.$$

A second method. — Write as above

$$x^2 - 2a_1x + a_2 = 0, \quad y^2 - 2b_1y - b_2 = 0, \quad z^2 - 2c_1z - c_2 = 0$$

with roots, the algebraic integers,

$$\xi = a_1 + \sqrt{a_1^2 + a_2}, \quad \eta = b_1 + \sqrt{b_1^2 + b_2}, \quad \zeta = c_1 + \sqrt{c_1^2 + c_2}.$$

Put

$$\eta = m\xi + n,$$

so that

$$\eta^2 = m^2\xi^2 + 2mn\xi + n^2.$$

It follows that

$$2b_1\eta + b_2 = m^2(2a_1\xi + a_2) + 2mn\xi + n^2,$$

or

$$2b_1m\xi + 2b_1n + b_2 = 2a_1m^2\xi + a_2m^2 + 2mn\xi + n^2.$$

It is clear from this last expression that

$$(1) \quad b_1 = ma_1 + n,$$

$$(2) \quad 2b_1n + b_2 = m^2a_2 + n^2.$$

If n is eliminated from these equation we have

$$(i) \quad b_1^2 + b_2 = m^2(a_1^2 + a_2).$$

From equation (1), it follows that

$$2a_1\xi + a_2 + 2b_1m\xi + 2b_1n + b_2 = 2c_1\xi + c_2.$$

If the rational and irrational part of this expression are equated, it is seen that

$$(3) \quad a_1 + mb_1 = c_1,$$

$$(4) \quad 2a_1^2 + a_2 + 2b_1ma_1 + 2b_1n + b_2 = 2c_1^2 + c_2.$$

Eliminate n from (4) by means of (1), we have equation (ii) above

$$(ii) \quad a_2 + 2b_1^2 + b_2 = 4a_1mb_1 + 2m^2b_1^2 + c_2.$$

Noting that the relation

$$\sqrt{a_1^2 + a_2} = \sqrt{c_1^2 + c_2} = \sqrt{(a_1 + mb_1)^2 + c_2}$$

must be satisfied, it is seen that

$$(5) \quad a_2 = 2a_1mb_1 + m^2b_1^2 + c_2.$$

From (ii) and (5) the quantities $a_2 - c_2$ and m may be respectively eliminated, and the following equations take their places

$$(4^a) \quad 2b_1^2 + b_2 = m^2b_1^2 + 2a_1b_1m,$$

$$(5^a) \quad 2b_1^2 + b_2 = a_2 - c_2.$$

If as in the previous solution, we put in (4^a) the values $2a_1 = 3b_1m$, $b_2 = 14b_1^2$, we have $16b_1^2 = 4m^2b_1^2$, or $m = 2$, $2a_1 = 6b_1$. From (1) $n = -5b_1$ and from (2) $-21b_1^2 = 4a_2$. Put $b_1 = 2$, $a_2 = -21$, $a_1 = 6$, $b_2 = 56$. From (3) and (5^a) it is seen that $c_1 = 10$ and $c_2 = -85$. Observe, however, that equations (1), (2) et (4) may by eliminating n be replaced by

$$b_1^2 + b_2 = m^2(a_1^2 + a_2)$$

and

$$a_2 + 2b_1^2 + b_2 = 4a_1mb_1 + 2m^2b_1^2 + c_2;$$

and it is seen that this solution is only a different form of the preceding one.

A more general solution. — Writing as in the preceding method

$$\eta = m\xi + n,$$

we have

$$(1) \quad b_1 = m c_1 + n$$

and

$$(2) \quad 2 b_1 n + b_2 = m^2 c_2 + n^2.$$

Again note if n is eliminated from (1) and (2), that

$$b_1^2 + b_2 = m^2 (c_1^2 + c_2).$$

Further write $\xi = p\zeta + q$ and it follows that

$$(3) \quad a_1 = p c_1 + q$$

and

$$(4) \quad 2 a_1 q + a_2 = p^2 c_2 + q^2.$$

If q is eliminated from these latter equations, we have

$$a_1^2 + a_2 = p^2 (c_1^2 + c_2).$$

The above values substituted in (1) cause that equation to become

$$2 a_1 (p\zeta + q) + a_2 + 2 b_1 (m\zeta + n) + b_2 = 2 c_1 \zeta + c_2.$$

In this expression equate the real parts and also the imaginary parts and it is seen that

$$(5) \quad \begin{aligned} c_1 &= p a_1 + m b_1, \\ 2 p a_1 c_1 + 2 m b_1 c_1 + 2 a_1 q + 2 b_1 n + a_2 + b_2 &= c_2 + 2 c_1^2. \end{aligned}$$

Due to (5) the last equation becomes

$$(6) \quad 2 a_1 q + 2 b_1 n + a_2 + b_2 = c_2.$$

From (1), (3) and (5) it is seen that

$$(7) \quad c_1 = \frac{mn + pq}{1 - m^2 - p^2};$$

and from (2), (4) and (6)

$$(8) \quad c_2 = \frac{n^2 + q^2}{1 - m^2 - p^2}.$$

A solution of (7) and (8) is had, if we put

$$u = k(1 - m^2 - p^2) \quad \text{and} \quad q = l(1 - m^2 - p^2),$$

where k and l are arbitrary integers.

It follows that

$$c_1 = mk + pl, \\ c_2 = (k^2 + l^2)H, \quad \text{where} \quad H = 1 - m^2 - p^2.$$

From (1) and (3) we have

$$b_1 = k + p(ml - pk), \\ a_1 = l + m(pk - ml);$$

and from (2) and (4)

$$b_2 = [(ml - pk)^2 - k^2]H, \\ a_2 = [(pk - ml)^2 - l^2]H.$$

Further note that

$$c_1^2 + c_2^2 = l^2 + k^2 - (kp - lm)^2, \\ a_1^2 + a_2^2 = p^2(c_1^2 + c_2^2), \quad b_1^2 + b_2^2 = m^2(c_1^2 + c_2^2).$$

For example, let $l = 2$, $k = 3$, $p = 5$, $m = 7$.

It is seen that

$$c_1^2 + c_2^2 = 12, \quad a_1^2 + a_2^2 = 5^2 \cdot 12, \quad b_1^2 + b_2^2 = 7^2 \cdot 12, \\ c_1 = 3i, \quad a_1 = 9, \quad b_1 = -2, \\ \xi = 3 + 2\sqrt{3}, \quad \zeta = 9 + 10\sqrt{3}, \quad \eta = -2 + 14\sqrt{3},$$

where $\xi^2 + \eta^2 = \zeta^2$.

Again put

$$m = 3, \quad l = 1, \quad p = 2, \quad k = 5,$$

and we have

$$\xi = 22 + 3i\sqrt{23}, \quad \eta = 9 + 3i\sqrt{23}, \quad \zeta = 17 + i\sqrt{23}, \\ \xi^2 + \eta^2 = \zeta^2.$$

Finally make $c_1^2 + c_2^2$ a perfect square, and we have the Pythagorean numbers. For example, let

$$m = 5, \quad l = 2, \quad p = 3, \quad k = 4,$$

and we have

$$\begin{aligned} c_1^2 + c_2 &= 16, & c_1 &= 26, & \zeta &= 26 + 4 = 30, \\ a_1^2 + a_2 &= 3^2 \cdot 16, & a_1 &= 12, & \xi &= 12 + 12 = 24, \\ b_1^2 + b_2 &= 5^2 \cdot 16, & b_1 &= -2, & \eta &= -2 + 20 = 18, \\ \zeta^2 + \eta^2 &= 900 = 30^2. \end{aligned}$$

The quadratic realms of rationality.

We are now in a position to express the above results in a more definite form, which is done through proof of the following theorem:

In every quadratic realm there is an infinite number of solutions of the equation

$$\xi^2 + \eta^2 = \zeta^2$$

through algebraic integers.

It is well known that any quadratic integer may be expressed in the form $a \cdot 1 + b \cdot \omega$, where a and b are any two rational integers and where $1, \omega$ form the *basis* of the realm, say $R(\sqrt{t})$, t any integer. If $t \equiv 2 \pmod{4}$ or $t \equiv 3 \pmod{4}$, then is $\omega = \sqrt{t}$; if, however, $t \equiv 1 \pmod{4}$, then is $\omega = \frac{1 + \sqrt{t}}{2}$. The case where t contains a squared factor may be reduced to one or the other of the above cases, so that it is unnecessary to consider the case $m \equiv 0 \pmod{4}$. As the present paper is concerned particularly with the *existence* of such solutions as have been indicated, it is seen that the solutions of the second case necessarily imply those of the first case; for if

$$\xi^2 + \eta^2 = \zeta^2,$$

then necessarily

$$(2\xi)^2 + (2\eta)^2 = (2\zeta)^2.$$

We may therefore consider here only the first case: If

$$\xi = m_1 + m\sqrt{t},$$

where m_1 and m are any rational integers and if ξ satisfies the equation

$$x^2 - 2a_1x - a_2 = 0,$$

where a_1, a_2 are arbitrary rational integers, we have

$$m_1^2 + 2mm_1\sqrt{t} + m^2t = 2a_1m_1 + 2a_1m\sqrt{t} + a_2.$$

It follows that

$$m_1 = a_1 \quad \text{and} \quad m^2t = a_1^2 + a_2.$$

Similarly if η and ζ are roots of the two equations

$$y^2 - 2b_1y - b_2 = 0 \quad \text{and} \quad z^2 - 2c_1y - c_2 = 0,$$

we have

$$\begin{aligned} \xi &= a_1 + m\sqrt{t}, \\ \eta &= b_1 + n\sqrt{t}, \end{aligned}$$

where n like m is an arbitrary integer,

$$\zeta = c_1 + p\sqrt{t},$$

p being likewise any rational integer; with the conditions

$$m^2t = a_1^2 + a_2, \quad n^2t = b_1^2 + b_2, \quad p^2t = c_1^2 + c_2.$$

Writing their values in equation (1) namely

$$2a_1\xi + a_2 + 2b_1\eta + b_2 = 2c_1\zeta + c_2,$$

it is seen that

$$2a_1^2 + a_2 + 2a_1m\sqrt{t} + 2b_1^2 + b_2 + 2b_1n\sqrt{t} = 2c_1^2 + c_2 + 2c_1p\sqrt{t}.$$

Equating the rational and irrational terms in this expression, we have the following equations to solve

- (1) $2a_1^2 + a_2 + 2b_1^2 + b_2 = 2c_1^2 + c_2,$
- (2) $a_1m + b_1n = c_1p,$
- (3) $m^2t = a_1^2 + a_2,$
- (4) $n^2t = b_1^2 + b_2,$
- (5) $p^2t = c_1^2 + c_2.$

From (1), (3), (4) and (5) it follows that

$$2(m^2 + n^2 - p^2)t = a_2 + b_2 - c_2;$$

or, what is the same thing,

$$(6) \quad (m^2 + n^2 - p^2)t = c_1^2 - a_1^2 - b_1^2,$$

Eliminate respectively a_1, b_1, c_1 , between (1) and (2) after (1) has been put in the form

$$a_1^2 + b_1^2 + (m^2 + n^2)t = c_1^2 + p^2 t,$$

and we have

$$(7) \quad p^2 (m^2 + n^2 - p^2)t = (a_1 m + b_1 n)^2 - p^2 (a_1^2 + b_1^2).$$

$$(8) \quad n^2 (m^2 + n^2 - p^2)t = n^2 (c_1^2 - a_1^2) - (c_1 p - a_1 m)^2.$$

$$(9) \quad m^2 (m^2 + n^2 - p^2)t = m^2 (c_1^2 - b_1^2) - (c_1 p - b_1 n)^2.$$

Note that if we put

$$m^2 + n^2 = p^2,$$

in these equations, we have

$$\frac{a_1}{m} = \frac{b_1}{n} = \frac{c_1}{p}.$$

If then Pythagorean numbers are chosen for m, n, p , the same are had for a_1, b_1, c_1 , and from (6) the number t is indeterminate.

Observe that equations (2) are satisfied if we put

$$m = k a_1, \quad n = k b_1, \quad p = k c_1 \quad (k \text{ any rational integer})$$

and choose for a_1, b_1, c_1 any Pythagorean numbers π_1, π_2, π_3 , $\pi_1^2 + \pi_2^2 = \pi_3^2$. For example, in any realm of nationality $R(\sqrt{t})$, it is evident that the algebraic integers

$$\xi = \pi_1 + k \pi_1 \sqrt{t}, \quad \eta = \pi_2 + k \pi_2 \sqrt{t}, \quad \zeta = \pi_3 + k \pi_3 \sqrt{t}$$

satisfy the equation

$$\xi^2 + \eta^2 = \zeta^2.$$

The theorem has resolved itself in the solution of the equation

$$(6) \quad (m^2 + n^2 - p^2)t = c_1^2 - a_1^2 - b_1^2,$$

subject to the condition

$$(2) \quad a_1 m + b_1 n = c_1 p.$$

One method of procedure is the following :

The integer t being fixed, give to m, n, p fixed values and put

$$p^2 - m^2 - n^2 = k,$$

where k is constant.

From (6), it is seen that

$$a_1^2 + b_1^2 - c_1^2 = t.k.$$

If c_1 is eliminated from this latter equation and (2), we have

$$(II) \quad (p^2 - m^2)a_1^2 - 2mna_1b_1 + (p^2 - n^2)b_1^2 = p^2tk,$$

which equation is a quadratic form in the two unknown quantities a_1, b_1 , whose discriminant D is $-p^2k$.

The problem has now resolved itself into the solution of the quadratic form (II) with negative, zero or positive, discriminant.

1° If k is positive the discriminant is negative. In this case multiply (II) by $4(p^2 - m^2)$ and put

$$(i) \quad 2(p^2 - m^2)a_1 - 2mnb_1 = s.$$

It is seen that (II) becomes

$$s^2 + 4p^2kb_1^2 = 4(p^2 - m^2)tkp^2,$$

or

$$(ii) \quad s^2 = 4kp^2[(p^2 - m^2)t - b_1^2].$$

Hence through a finite number of trials, we may determine whither or not there is a value for b_1 which makes the right hand side of (ii) a perfect square -- a condition which is evidently necessary for the solution of the problem.

For example write $p = 3, m = 2, n = 1$, so that $k = 4$. Take $t = 5$. From (ii) it is seen that, when

(a)	$b_1 = 0,$	$s = 4.3.5;$
(b)	$b_1 = 3,$	$s = 4.3.4;$
(c)	$b_1 = 4,$	$s = 4.3.3;$
(d)	$b_1 = 5,$	$s = 0.$

Hence from (i) and (2), we have

- (a) $a_1 = 6, \quad b_1 = 0, \quad c_1 = 4;$
 (b) $a_1 = 6, \quad b_1 = 3, \quad c_1 = 5;$
 (c) *no solution;*
 (d) $a_1 = 3, \quad b_1 = 5, \quad c_1 = 3.$

It is also evident from (ii) that negative values may be given to b_1 and s so that, for example, $b_1 = -3, s = -4.3.4.$

In this case $a_1 = -b_1, b_1 = -3, c_1 = -4.$

Hence in the realm $R(\sqrt{5})$ if values 2, 1, 3 are taken respectively for m, n, p , it is seen that :

$$(a) \quad \begin{cases} \xi = \pm 6 + 2\sqrt{5}, \\ \eta = 0 + \sqrt{5}, \\ \zeta = \pm 4 + 3\sqrt{5}; \end{cases}$$

$$(b) \quad \begin{cases} \xi = \pm 6 + 2\sqrt{5}, \\ \eta = \pm 3 + \sqrt{5}, \\ \zeta = \pm 5 + 3\sqrt{5}; \end{cases}$$

$$(c) \quad \begin{cases} \xi = \pm 3 + 2\sqrt{5}, \\ \eta = \pm 5 + \sqrt{5}, \\ \zeta = \pm 3 + 3\sqrt{5}; \end{cases}$$

arc solutions of the equation

$$\xi^2 + \eta^2 = \zeta^2.$$

It is evident that if $-2, -1, -3$ had been chosen respectively at values of m, n, p , the same values of a_1, b_1, c_1 as those above would have been derived.

If we put $m = 1, n = 3, p = 2$, it is seen that $k = -6$ and $D = +24$. Equation (II) becomes, if t is taken = 3,

$$5b_1^2 + 6b_1a_1 - 3a_1^2 = 72.$$

A solution is $a_1 = 3, b_1 = 3, c_1 = \frac{a_1m + b_1n}{p} = 6$. The corresponding values of ξ, η, ζ are

$$\xi = 3 + \sqrt{3}, \quad \eta = 3 + 3\sqrt{3}, \quad \zeta = 6 + 2\sqrt{3}.$$

If $t = 10$, the equation

$$5b_1^2 + 6b_1a_1 - 3a_1^2 = 240,$$

admits solution $a_1 = 2$, $b_1 = 6$, $c_1 = 10$.

However if $t = 5$, the equation

$$5b_1^2 + 6b_1a_1 - 3a_1^2 = 120$$

does *not* admit solution, as is evident from elementary consideration.

Having *one* solution it is possible by means of the Pell's equation to derive an infinity of others.

In the first case, for example, where $D = 24$, the equation of Pell, namely $T^2 - 24U^2 = 1$, admits the solution $T = 5$, $U = 1$.

Note that the equation

$$5b_1^2 + 6b_1a_1 - 3a_1^2 = 72,$$

may be written in the form

$$(5b_1 + 3a_1)^2 - 24a_1^2 = 5 \cdot 72.$$

Hence writing

$$5b_1 + 3a_1 + \sqrt{24}a_1 = (24 + 3\sqrt{24})(5 + \sqrt{24})^k,$$

we have the solutions corresponding to values of $k = 1, 2, 3, \dots$

If $k = 1$, it is seen that

$$a_1 = 39, \quad b_1 = 15, \quad c_1 = 42,$$

and

$$\xi = 39 + \sqrt{3}, \quad \eta = 15 + 3\sqrt{3}, \quad \zeta = 42 + 2\sqrt{3}.$$

In the two examples given above it is seen that a_1 , b_1 , c_1 have a common factor other than unity.

Take, however,

$$m = 3, \quad n = 2, \quad p = 1$$

and let

$$t = 22.$$

It is seen that $k = -12$, and the discriminant $D = +12$. Equation (II) becomes

$$8a_1^2 + 12a_1b_1 + 3b_1^2 = 12 \cdot 22.$$

Note that 12 is a quadratic residue of 12.22, the congruence

$$n^2 \equiv 12 \pmod{12.22}$$

admitting the solution $n = 54$; and observe that

$$a_1 = 3, b_1 = 4$$

is a *proper* solution of the quadratic form, that is, one in which a_1 and b_1 are relatively prime.

The corresponding Pell's equation

$$T^2 - 12U^2 = 1$$

admits the solution

$$T = 7, \quad U = 2.$$

Write

$$8a_1^2 + 12a_1b_1 + 3b_1^2 = 12.22$$

in the form

$$(8a_1 + 6b_1)^2 - 12b_1^2 = 8.12.22,$$

and it is seen that the general solution is

$$8a_1 + 6b_1 + \sqrt{12}b_1 = (48 + 4\sqrt{12})(7 + 2\sqrt{12})^k \quad (k = 0, 1, 2, 3, \dots).$$

When

$k = 1,$	$a_1 = -39,$	$b_1 = 127,$	$c_1 = 131,$
$k = 2,$	$a_1 = -549,$	$b_1 = 1732,$	$c_1 = 1817,$
$k = 3,$	$a_1 = -7647,$	$b_1 = 24124,$	$c_1 = 25307,$
.....,	,	,	

The corresponding values of ξ, η, ζ are

$k = 0,$	$k = 1,$
$\xi = 3 + 3\sqrt{22},$	$\xi = -39 + 3\sqrt{22},$
$\eta = 4 + 2\sqrt{22},$	$\eta = 124 + 2\sqrt{22},$
$\zeta = 17 + \sqrt{22};$	$\zeta = 131 + \sqrt{22};$
$k = 2,$	$k = 3,$
$\xi = -549 + 3\sqrt{22},$	$\xi = -7647 + 3\sqrt{22},$
$\eta = 1732 + 2\sqrt{22},$	$\eta = 24124 + 2\sqrt{22},$
$\zeta = 1817 + \sqrt{22};$	$\zeta = 25307 + \sqrt{22};$
.....,	

Note that on the transition from a negative to a positive discriminant we must have

$$D = 0 = -p^2 k.$$

If $k = 0$, we have the Pythagorean numbers considered above. And if $p = 0$, the quadratic form (II) becomes

$$a_1 m + b_1 n = 0$$

and the further solution is without difficulty.

