Journées Équations aux dérivées partielles

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Journées Équations aux dérivées partielles (2000), p. 1-8

<http://www.numdam.org/item?id=JEDP_2000____A16_0>

Journées *Équations aux dérivées partielles*
Nantes, 5–9 juin 2000
GDR 1151 (CNRS)

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Résumé

Il est bien connu que les fréquences propres associées à un d'Alembertien amorti sont confinées dans une bande parallèle à l'axe réel. Nous rappelons l'asymptotique de Weyl pour la distribution des parties réelles des fréquences propres, nous montrons que "presque toutes" les fréquences propres appartiennent à une bande déterminée par la limite de Birkhoff du coefficient d'amortissement. Nous montrons aussi que certaines moyennes des parties imaginaires convergent vers la moyenne du coefficient d'amortissement.

Abstract

The eigenfrequencies associated to a damped wave equation, are known to belong to a band parallel to the real axis. We review Weyl asymptotics for the distribution of the real parts of the eigenfrequencies, we show that up to a set of density 0, the eigenfrequencies are confined to a band determined by the Birkhoff limits of the damping coefficient. We also show that certain averages of the imaginary parts converge to the average of the damping coefficient.

1. Introduction and results

Let M be a smooth compact Riemannian manifold of dimension n and let Δ be the corresponding Laplace Beltrami operator. In control theory one is interested in the long time behaviour ($t \to +\infty$) of solutions to

$$(\partial_t^2 - \Delta + 2a(x)\partial_t)v(t,x) = 0 \text{ on } \mathbf{R} \times M. \tag{1.1}$$

Here we let $a \in C^\infty(M; \mathbf{R})$ be the "damping" coefficient (where true damping corresponds to taking $a \geq 0$). Much more general problems can and have been considered : M could have a boundary, a could have discontinuities, we could replace the scalar equation (1.1) by a system and so on. The reason for us to

Acknowledgements: We are greatful to G. Lebeau, P. Freitas and A. Laptev for stimulating discussions, and to M. Solomjak for drawing our attention to the work of Markus–Matsaev
MSC 2000 : 35P15, 35P20
Keywords : non selfadjoint, eigenvalue

look at this eqution is that it leads to simplified model problems for resonances for strictly convex obstacles.

In this talk we will only discuss the stationary problem obtained by putting $v(t,x) = e^{it\tau}u(x)$:

$$(-\Delta - \tau^2 + 2ia(x)\tau)u(x) = 0 \tag{1.2}$$

If there exists a non-trivial solution to (1.2), we call $\tau \in \mathbf{C}$ an eigenfrequency. Equivalently τ is an eigenvalue of

$$\mathcal{P} = \begin{pmatrix} 0 & 1 \\ -\Delta & 2ia(x) \end{pmatrix} : H^1 \times H^0 \to H^1 \times H^0,$$

with domain $H^2 \times H^1$, where $H^s = H^s(M)$ is the standard Sobolev space on M. The spectrum (i.e. the set of eigenfrequencies) is discrete and symmetric under reflection in the imaginary axis. There are several equivalent definitions of the multiplicity $m(\tau) \in \{1, 2, ..\}$ of an eigenfrequency τ (see [7] for more details) and we shall always count the eigenfrequencies with their multiplicity. If τ is an eigenfrequency, then it is easy to see that

$$\begin{cases} \inf a \leq \operatorname{Im} \tau \leq \sup a, \ \operatorname{Re} \tau \neq 0, \\ 2\min(\inf a, 0) \leq \operatorname{Im} \tau \leq 2\max(\sup a, 0), \ \operatorname{Re} \tau = 0. \end{cases} \tag{1.3}$$

In the case $a \geq 0$, the energy of solutions to (1.1) is non-increasing when $t \to +\infty$ and Lebeau [4] has obtained a lower bound on the rate of decay in terms of $\inf \operatorname{Im} \tau$ and A_- introduced below, and an earlier result in the same direction was obtained by J. Rauch and M. Taylor [6]. Lebeau also obtained a sharpening of (1.3) for large eigenfrequencies: Put

$$\langle a \rangle_T = \frac{1}{2T} \int_{-T}^{T} a \circ \exp(tH_p) dt, \ T > 0,$$

where $p(x, \xi) = \xi^2$ denotes the principal symbol of $-\Delta$, and H_p is the corresponding Hamilton field. Let

$$A_+ := \inf_{T>0} \sup_{p^{-1}(1)} \langle a \rangle_T = \lim_{T \to \infty} \sup_{p^{-1}(1)} \langle a \rangle_T,$$

$$A_- := \sup_{T>0} \inf_{p^{-1}(1)} \langle a \rangle_T = \lim_{T \to \infty} \inf_{p^{-1}(1)} \langle a \rangle_T,$$

where the second equalities follow from subadditivity arguments. The following result is essentially due to [4]:

Theorem 1.1 *For every $\epsilon > 0$ all except possibly finitely many eigenfrequencies belong to $\mathbf{R} + i]A_- - \epsilon, A_+ + \epsilon[$*

We refer to M. Asch, G. Lebeau [1] for two interesting refinements of this result and to P. Freitas [2] for various estimates. Both papers contain interesting numerical results.

The results presented here concern the asymptotic distribution of eigenfrequencies inside the bands appearing in Theorem 1.1. Two of them are analogous to results concerning the distribution of resonances for strictly convex obstacles obtained by the author [8] and by M. Zworski and the author [10]. The third result has not yet any corresponding analogue, and Zworski and the author intend to look into that question, as well as the question of getting remainder estimates in the main result of [10]. The first result gives the standard Weyl asymptotics, and can probably be deduced from [5].

Theorem 1.2 *The number of eigenfrequencies τ, with $0 \leq \operatorname{Re}\tau \leq \lambda$ is equal to*

$$(\frac{\lambda}{2\pi})^n \int\int_{p(x,\xi)\leq 1} dx d\xi + \mathcal{O}(\lambda^{n-1}),\ \lambda \to \infty.$$

During this conference M. Solomjak indicted to us some work of A.S. Markus and V.I. Matseev, and the most relevant paper seems to be [5]. Theorem 2.1 in that paper looks very much like a generalization of the preceding result. The proof in that paper seems to be very close to the one we give below (in the case when $f = 1$) and uses finite rank perturbations to open a gap in the spectrum as well as considerations of relative determinants. We have not found any result like Theorem 1.4 below and it would be interesting to see what our proof gives for more general elliptic operators.

Notice that Theorem 1.2 is the standard (and in general optimal) result in the selfadjoint case, $a = 0$. Also notice that it implies that the number of eigenfrequencies with $\lambda \leq \operatorname{Re}\tau \leq \lambda + 1$ is $\mathcal{O}(\lambda^{n-1})$, when $\lambda \to +\infty$.

In order to state the second result, we introduce the almost everywhere limit on $p^{-1}(1)$, given by the Birkhoff ergodic theorem:

$$\langle a\rangle_\infty = \lim_{T\to\infty} \langle a\rangle_T.$$

Then

$$A_- \leq \operatorname{ess\,inf} \langle a\rangle_\infty \leq \operatorname{ess\,sup} \langle a\rangle_\infty \leq A_+. \tag{1.4}$$

At each place the inequality may be strict. If the geodesic flow is ergodic, we have equality in the middle.

Theorem 1.3 *For every $\epsilon > 0$, the number of eigenfrequencies in $[\lambda, \lambda+1] + i(\mathbf{R} \setminus [\operatorname{ess\,inf} \langle a\rangle_\infty - \epsilon, \operatorname{ess\,sup} \langle a\rangle_\infty + \epsilon])$ is $o(\lambda^{n-1})$, $\lambda \to \infty$.*

Somewhat vaguely, one can say that the relative density of the eigenfrequencies outside $\mathbf{R} + i[\operatorname{ess\,inf} \langle a\rangle_\infty - \epsilon, \operatorname{ess\,sup} \langle a\rangle_\infty + \epsilon]$ is equal to 0.

The last result concerns the meanvalue distribution of the imaginary parts.

Theorem 1.4 *Fix some $C_0 > 1$ and let $\lambda_1, \lambda_2 \in \mathbf{R}$ satisfy*

$$1 \ll \lambda_1 < \lambda_2,\ \frac{\lambda_2}{\lambda_1} \leq C_0,\ \lambda_2 - \lambda_1 \geq \log \lambda_1. \tag{1.5}$$

Let $N(\lambda_1, \lambda_2)$ be the number of eigenfrequencies τ with $\lambda_1 \leq \operatorname{Re}\tau \leq \lambda_2$. Then

$$\frac{1}{N(\lambda_1,\lambda_2)} \sum_{\substack{\tau\in\sigma(P)\\ \lambda_1\leq\operatorname{Re}\tau\leq\lambda_2}} \operatorname{Im}\tau = \frac{1}{\operatorname{vol}(M)} \int_M a(x)dx + \mathcal{O}(1)\frac{\log\lambda_1}{\lambda_2-\lambda_1} \tag{1.6}$$

M. Hitrik (personal communication) has obtained further results in the case when the geodesic flow is periodic. A more detailed study of the distribution of the imaginary parts may be possible in this case.

In the following, we outline the proofs of the theorems 1.2,1.4 which use some recent trace formula techniques, here in the semiclassical setting of [9]. The proof of Theorem 1.3 is more technical. For more details, see [7].

2. Ideas of the proofs of Theorem 1 and 3.

Write $\tau = \sqrt{z}/h$, $0 < h \ll 1$, where z belongs to the fixed domain $\Omega := e^{i]-\theta_0,\theta_0[}]\alpha,\beta[$, with $0 < \alpha < 1 < \beta < \infty$, $0 < \theta_0 < \frac{\pi}{4}$. From (1.2), we get $(\mathcal{P} - z)v = 0$, where $\mathcal{P} = P + ihQ(z)$, $P = -h^2\Delta$, $Q(z) = 2a(x)\sqrt{z}$. Everything works in a more general h-pseudodifferential framework and the essential features are that P is elliptic selfadjoint, $dp \neq 0$ on $p^{-1}([\alpha,\beta])$ and that Q is holomorphic in z and formally selfadjoint for $z > 0$. Let $\alpha < E_1 < E_2 < \beta$ with $E_2 - E_1 \geq 4h$, $E_1 - \alpha, \beta - E_2 \geq$ Const. > 0. Put $E_0 = (E_1 + E_2)/2$, $r_0 = (E_2 - E_1)/2$.

Lemma 2.1 *For every $C > 0$ there exists a selfadjoint operator $\widetilde{P} = P + h\delta P$ with the same domain as P such that $(E_j + [-Ch, Ch]) \cap \sigma(\widetilde{P}) = \emptyset$, $j = 1, 2$, $\|\delta P\| \leq C$, $\|\delta P\|_{\mathrm{tr}} \leq \widetilde{C}(C)h^{1-n}$.*

Here $\|\cdot\|_{\mathrm{tr}}$ denotes the trace norm. This lemma follows easily form the fact that P has $\leq \mathcal{O}(h^{1-n})$ eigenvalues in intervals of length h.

Put $\widetilde{\mathcal{P}} = \widetilde{P} + ihQ(z) = \mathcal{P} + h\delta P$. We have

$$\|(z - \widetilde{\mathcal{P}})^{-1}\| \leq \frac{\mathcal{O}(1)}{h + |\mathrm{Im}\, z|},$$

for

$$z \in D(E_0, r_0 + 2h) \setminus \{z \in D(E_0, r_0 - 2h)\ ;\ |\mathrm{Im}\, z| \leq Ch\},$$

where $D(E, r)$ denotes the open disc in $\mathbf{C}$ with center E and radius r, and $C > 0$ is sufficiently large. Write

$$z - \mathcal{P} = (z - \widetilde{\mathcal{P}})(1 + h(z - \widetilde{\mathcal{P}})^{-1}\delta P),\ D(z) = \det(1 + h(z - \widetilde{\mathcal{P}})^{-1}\delta P).$$

Using a convexity estimate of H. Weyl (see [3]), we get

$$|D(z)| \leq \exp \|h(z - \widetilde{\mathcal{P}})^{-1}\delta P\|_{\mathrm{tr}} \leq \exp(\mathcal{O}(1)\frac{h^{2-n}}{h + |\mathrm{Im}\, z|})$$

In the subset where $|\mathrm{Im}\, z| \geq Ch$ (for a sufficiently large constant C), we also have

$$\|(z - \mathcal{P})^{-1}\| \leq \frac{\mathcal{O}(1)}{h + |\mathrm{Im}\, z|},$$

and writing

$$(1 + h(z - \widetilde{\mathcal{P}})^{-1}\delta P)^{-1} = (z - \mathcal{P})^{-1}(z - \widetilde{\mathcal{P}}) = 1 - h(z - \mathcal{P})^{-1}\delta P,$$

we get a similar upper bound for the inverse of D, i.e. a lower bound for D:

$$|D(z)| \geq \exp(-\mathcal{O}(1)\frac{h^{2-n}}{h+|\mathrm{Im}\, z|}).$$

Using Jensen's formula in a standard way, we see that the number of eigenvalues $z_1, .., z_N$ of $\mathcal{P}$ in

$$D(E_0, r_0+h) \setminus \{z \in D(E_0, r_0-h)\ ;\ |\mathrm{Im}\, z| < Ch\} \tag{2.1}$$

is $\mathcal{O}(h^{1-n})$. Let $\tilde{b}_{z_j}(z)$ be the Blaschke factor associated to z_j and $D(E_0, r_0+\frac{3}{2}h)$ (having z_j as its only zero inside the disc and being of modulus one on the boundary). Write

$$D(z) = G(z)D_b(z),\ D_b(z) = \prod_{j=1}^{N} \tilde{b}_{z_j}(z),$$

so that $G(z)$ is holomorphic and non-vanishing on the domain (2.1). Standard arguments (used in a similar context in [9]), involving Harnack's inequality, show that

$$|\log|G(z)|| \leq \mathcal{O}(1)\frac{h^{2-n}}{h+|\mathrm{Im}\, z|}, \tag{2.2}$$

$$|\frac{d}{dz}\log G(z)| \leq \frac{\mathcal{O}(1)h^{2-n}}{(h+|\mathrm{Im}\, z|)\min(h+|\mathrm{Im}\, z|, r_0+\frac{2h}{3}-|z|)}, \tag{2.3}$$

in

$$D(E_0, r_0+\frac{2}{3}h) \setminus \{z \in D(E_0, r_0-\frac{2h}{3})\ ;\ |\mathrm{Im}\, z| < Ch\}. \tag{2.4}$$

Let γ be a hexagonal positively oriented contour in the domain (2.4) with vertices at the points $E_0 \pm \frac{ir_0}{2}$, $E_0 \pm r(h) \pm i2Ch$, $j=1,2$ with $r(h) = r_0 + \alpha(h)h$ where $\alpha(h)$ is small and chosen so that γ avoids the z_j. One can verify that if f is holomorphic in $D(E_0, r_0+2h)$, then

$$\sum_{\lambda \in \sigma(\tilde{\mathcal{P}}) \cap \mathrm{int}\,(\gamma)} f(\lambda) = \mathrm{tr}\,(\frac{1}{2\pi i}\int_\gamma f(z)(1-\partial_z\tilde{\mathcal{P}})(z-\tilde{\mathcal{P}})^{-1}dz), \mathrm{tr} \tag{2.5}$$

where $\sigma(\tilde{\mathcal{P}})$ denotes the set of eigenvalues of $\tilde{\mathcal{P}}$, i.e. the complex numbers z such that $\mathrm{Ker}\,(\tilde{\mathcal{P}}-z) \neq 0$, and $\mathrm{int}\,\gamma$ denotes the interior of γ. We have the analogous relation for $\mathcal{P}$. Moreover,

$$\begin{aligned} &\mathrm{tr}\,(\frac{1}{2\pi i}(\int_\gamma f(z)(1-\partial_z\tilde{\mathcal{P}})(z-\tilde{\mathcal{P}})^{-1}dz - \int_\gamma f(z)(1-\partial_z\mathcal{P})(z-\mathcal{P})^{-1}dz)) \qquad (2.6)\\ &= \frac{1}{2\pi i}\int_\gamma f(z)\frac{d}{dz}(\log D(z))dz \\ &= \sum_{z_j \in \mathrm{int}\,(\gamma)} f(z_j) + \frac{1}{2\pi i}\int_\gamma f(z)\frac{d}{dz}\log G(z)dz. \end{aligned}$$

Assume that $|f| \leq 1$. Then

$$\sum_{z_j \in \mathrm{int}\,(\gamma)} f(z_j) = \mathcal{O}(h^{1-n}),$$

$$\frac{1}{2\pi i}\int_\gamma f(z)\frac{d}{dz}\log G(z) = \mathcal{O}(1)h^{2-n}\int_0^1 \frac{1}{(h+t)^2}dt = \mathcal{O}(h^{1-n}).$$

Choosing $f=1$, we get

$$\#(\sigma(\mathcal{P})\cap \mathrm{int}\,(\gamma)) = \#(\sigma(\widetilde{\mathcal{P}})\cap \mathrm{int}\,(\gamma)) + \mathcal{O}(h^{1-n}).$$

By an easy deformation argument and a well known result on spectral asymptotics:

$$\#(\sigma(\widetilde{\mathcal{P}})\cap \mathrm{int}\,(\gamma)) = \#(\sigma(\widetilde{P})\cap \mathrm{int}\,(\gamma)) = \frac{1}{(2\pi h)^n}(\iint_{E_1\le p\le E_2} dxd\xi + \mathcal{O}(h)),$$

which implies Theorem 1.2.

To obtain Theorem 1.4, we take f holomorphic in $D(E_0; r_0+2h)$ with $f'=\mathcal{O}(1)$ and with $f(z)$ real when z is real. Since $\mathrm{Im}\, f(z_j) = \mathcal{O}(h)$, we get

$$\mathrm{Im}\,\frac{1}{2\pi i}\int_\gamma f(z)\frac{d}{dz}\log D_b(z)dz = \mathcal{O}(h^{2-n}),$$

$$\begin{gathered}\mathrm{Im}\,\tfrac{1}{2\pi i}\textstyle\int_\gamma f(z)\frac{d}{dz}(\log G(z))dz = \\ \mathrm{Im}\,\tfrac{1}{2\pi i}[f(z)\log G(z)]_{E_0+r(h)+i0}^{E_0+r(h)-i0} - \mathrm{Im}\,\tfrac{1}{2\pi i}\textstyle\int_\gamma f'(z)\log(G(z))dz,\end{gathered}$$

where we choose a branch of the logarithm of $G(z)$ with a cut along $E_0+[0,+\infty[$. Here we notice that the first term in the last expression vanishes, since $f(E_0+r(h))$ is real and the two limiting branches of $\log G$ differ by a multiple of $2\pi i$ at that point.

On the other hand, we have on γ:

$$|\mathrm{Re}\,\log G(z)| = |\log|G(z)|| \le \mathcal{O}(1)\frac{h^{2-n}}{h+|\mathrm{Im}\,z|}. \tag{2.7}$$

By (2.2), we have on γ:

$$|\frac{d}{dz}\log G(z)| \le \frac{\mathcal{O}(1)h^{2-n}}{(h+|\mathrm{Im}\,z|)^2}.$$

We integrate this from $E_0 \pm ir_0/2$ to $z\in\gamma$, when $\pm\mathrm{Im}\,z\ge 0$, and get

$$|\mathrm{Im}\,(\log G(z)) - C_\pm(h)| \le \mathcal{O}(1)\frac{h^{2-n}}{h+|\mathrm{Im}\,z|},\quad \pm\mathrm{Im}\,z\ge 0,$$

where $C_\pm(h)\in\mathbf{R}$. Hence, using also (2.7) for $\mathrm{Re}\,\log G(z)$, we get:

$$\log(G(z)) = iC_\pm(h) + \mathcal{O}(1)\frac{h^{2-n}}{h+|\mathrm{Im}\,z|}.$$

Here

$$\frac{1}{2\pi i}\int_{\gamma\cap\{\pm\mathrm{Im}\,z>0\}} iC_\pm(h)f'(z)dz$$

is real, so

$$\operatorname{Im} \frac{1}{2\pi i} \int_\gamma f'(z) \log G(z) dz = \mathcal{O}(1) h^{2-n} \log \frac{1}{h}.$$

Taking imaginary parts of (2.5), (2.6), we get:

$$\sum_{\lambda \in \sigma(P) \cap \operatorname{int}(\gamma)} \operatorname{Im} f(\lambda) = \operatorname{Im} \operatorname{tr} \frac{1}{2\pi i} \int_\gamma f(z)(1 - \partial_z \widetilde{\mathcal{P}})(z - \widetilde{\mathcal{P}})^{-1} dz + \mathcal{O}(1) h^{2-n} \log \frac{1}{h}.$$

Thanks to the fact that we have a sufficiently good control over $(z - \widetilde{\mathcal{P}})^{-1}$ along *all of* γ (which is not the case for $(z - \mathcal{P})^{-1}$), we can analyze the integral in the preceding equation and show that it is equal to

$$\frac{h}{(2\pi h)^n} \iint_{E_1 \le p(x,\xi) \le E_2} f'(p(x,\xi)) q(x,\xi,p(x,\xi)) dx d\xi + \mathcal{O}(h^{2-n} \log \frac{1}{h}),$$

with $q(x,\xi,z) = 2\sqrt{z} a(x)$. Taking $f(z) = 2\sqrt{z}$, leads to Theorem 1.4.

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