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# Large time behaviour of heat kernels on Riemannian manifolds: fast and slow decays

Thierry Coulhon

## Abstract

Upper and lower estimates on the rate of decay of the heat kernel on a complete non-compact Riemannian manifold have recently been obtained in terms of the geometry at infinity of the manifold, more precisely in terms of a kind of  $L^2$  isoperimetric profile. We shall give an outline of these results and show how they can give some partial answers to the following question: given the volume growth of a manifold, e.g. polynomial or exponential, how fast and how slow can the heat kernel decay be? The connection between the volume growth and the  $L^2$  isoperimetric profile will be made through Poincaré type inequalities. A large part of the material presented here is the result of a joint work with A. Grigory'an.

## 1. Introduction.

Let  $M$  be a complete, connected, non-compact Riemannian manifold, and  $\Delta$  the Laplace-Beltrami operator on  $M$ . For  $y \in M$ , denote by  $p_t(\cdot, y)$  the smallest positive fundamental solution with initial value  $\delta_y$  of the heat equation

$$\frac{\partial u}{\partial t} = \Delta u.$$

Since  $p_t(x, y)$  is the kernel of the heat semi-group  $e^{t\Delta}$ ,  $t > 0$ , it is called the heat kernel.

Whereas for small time  $t > 0$ ,  $p_t(x, x)$  is comparable with  $t^{-n/2}$ , where  $n$  is the topological dimension of  $M$ , it is clear that the large time behaviour of  $\sup_{x \in M} p_t(x, x)$ , or  $p_t(x, x)$ , for fixed  $x \in M$ , should depend on the geometry at infinity of  $M$ .

More precisely, we shall see in §2 below that the on-diagonal heat kernel behaviour is controlled from above and below by an  $L^2$  isoperimetric profile. This relies on the conjunction of two results: on the one hand, an upper bound on the heat kernel decay is equivalent to a so-called Faber-Krahn inequality, that says that the  $\lambda_1$  of sets dominates a decreasing function of their volume. On the other hand, if

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this inequality is optimal (we then say that an anti-Faber-Krahn inequality holds), the corresponding lower bound on the heat kernel follows.

One may well consider that the most basic geometric information about a non-compact manifold is its volume growth, and ask about its connection with the heat kernel behaviour. By the above results, this means that one has to relate the  $L^2$  isoperimetric profile to the volume growth. This connection is certainly loose: one can easily imagine manifolds with a large volume growth, but that shrink from place to place, so that the heat diffusion is relatively slow.

Therefore the first question which makes sense is to ask for the kind of ingredient that allows to go from, say, a volume lower bound, to a good upper bound on the  $L^2$  isoperimetric profile, which will in turn give a good upper bound for the heat kernel. For instance, on a manifold such that  $V(x, r) \geq cr^D$ ,  $r > 0$  (here  $V(x, r)$  is the Riemannian volume of the geodesic ball of center  $x \in M$  and radius  $r > 0$ ), when can one say that  $\sup_{x \in M} p_t(x, x) \leq C' t^{-D/2}$ ,  $t > 0$ ? We shall see in §3.1 that the so-called pseudo-Poincaré inequalities provide such a tight connection between the volume growth and the heat kernel decay. They express a certain homogeneity of the geometry, and they hold on Lie groups, on manifolds with non-negative Ricci curvature, but they certainly do not hold in general.

The second question that arises naturally is therefore: what one can say about the heat kernel behaviour if the only information one has is in terms of volume growth? Consider, say, the class of manifolds

$$cr^D \leq V(x, r) \leq Cr^D, \quad r \geq 1.$$

What is the range of possible large time behaviours for  $\sup_{x \in M} p_t(x, x)$ ? If one has a uniform upper and lower bound on the volume growth, one can get a relaxed pseudo-Poincaré inequality. Together with the volume lower bound, it implies an upper estimate on the  $L^2$  isoperimetric profile, hence, again, an upper bound on the heat kernel decay. Applying the anti-Faber-Krahn inequality technology, one can then build manifolds where the heat kernel decay is that slow, or almost.

We shall not give here complete references, nor give an historical account of the subject. For this, see for example [7] and [17].

Note finally that Riemannian geometry is nothing but a convenient setting to present the interaction between the behaviour of solutions of an evolution equation and the geometry of the underlying space. A similar analysis can be performed for subelliptic second order operators on a manifold. A parallel theory can also be developed for random walks on graphs, see the survey [8].

## 2. Heat kernels and $L^2$ isoperimetric profile.

### 2.1 Profiles

Let  $\mathcal{K}(M)$  be the set of smooth relatively compact domains in  $M$  and, for  $\Omega \in \mathcal{K}(M)$ , let  $\text{Lip}(\Omega)$  be the space of Lipschitz functions with support in  $M$ . Denote by  $|\Omega|$  the Riemannian volume of  $\Omega$ . For  $1 \leq p \leq +\infty$ , define

$$\varphi_p(v) = \sup \left\{ \frac{\|f\|_p}{\|\|\nabla f\|\|_p}; f \in \text{Lip}(\Omega) \setminus \{0\}, \Omega \in \mathcal{K}(M), |\Omega| = v \right\}.$$

In other terms,  $\varphi_p$  is the smallest function  $\varphi$  such that the following Faber-Krahn inequality holds:

$$(F_\varphi^p) \quad \|f\|_p \leq \varphi(|\Omega|) \|\nabla f\|_p, \quad \forall f \in \text{Lip}(\Omega),$$

for every  $\Omega \in \mathcal{K}(M)$ . These inequalities were introduced in [5] and [4], see also [1], §10.3.

In particular,  $\varphi_1$  is what is sometimes called the isoperimetric profile of  $M$ , i.e. the smallest non-decreasing function such that

$$\frac{|\Omega|}{\varphi_1(|\Omega|)} \leq |\partial\Omega|,$$

for every  $\Omega \in \mathcal{K}(M)$ . This is an easy consequence of the co-area formula.

Also,  $\varphi_\infty$  is the inverse of the volume growth function  $\inf_{x \in M} V(x, r)$ , i.e.

$$\varphi_\infty(v) = \inf\{r > 0; V(x, r) \geq v, \forall x \in M\},$$

where  $V(x, r)$  is the Riemannian volume of the geodesic ball  $B(x, r)$  of center  $x \in M$  and radius  $r > 0$ . This was proved in [5], p. 89, in a particular case, but the proof adapts easily.

For  $1 < p < +\infty$ ,  $\varphi_p$  is relevant to questions concerning the  $p$ -Laplace operator, see [14], §3.3.

Using Hölder's inequality, one checks easily that, for  $1 \leq p < q < +\infty$ ,

$$\varphi_q(v) \leq C_{p,q} \varphi_p(v).$$

In some sense, the smaller  $p$  is, the more information is contained in  $\varphi_p$ . For example, the fact that  $\varphi_1$  dominates  $\varphi_2$  means, in view of Theorem 2.3 below, that the isoperimetric inequality controls from above the heat kernel decay.

The converse is false: for every  $\varepsilon > 0$ , there exists a manifold with bounded geometry such that  $\varphi_2(v) \leq C v^{1/D}$  for  $v \geq 1$ , but  $\varphi_1(v) \leq C v^{\frac{2}{D}-\varepsilon}$  is false, see [11]. This example was later improved in [2]. Again, in view of Theorem 2.3 below, this means that the heat diffusion on a manifold can stay relatively fast, even though the isoperimetric profile is relatively bad. Roughly speaking, one can say that the heat diffusion does not care if the  $L^1$  boundary of sets is not so large with respect to their volume. What matters for the heat to escape is that sets should have a relatively large  $L^2$  boundary.

Let us mention that if  $M$  has bounded geometry, one can show, using discretisation techniques as in [13], that

$$\varphi_1(v) \leq C \varphi_2^2(v), \quad v \geq 1.$$

The connection between  $\varphi_p$ ,  $1 \leq p < +\infty$  and  $\varphi_\infty$  is a special case. First, Carron noticed in [2], [3] that the estimate  $\varphi_2(v) \leq C v^{1/D}$  is equivalent for  $D > 2$  to the Sobolev inequality

$$\|f\|_{\frac{2D}{D-2}} \leq C \|\nabla f\|_2, \quad \forall f \in C_0^\infty(M),$$

and that the latter implies

$$V(x, r) \geq cr^D, \forall t > 0,$$

i.e.  $\varphi_\infty(v) \leq C'v^{1/D}$ . Again, given Theorem 2.3 below, this means that

$$\sup_{x \in M} p_t(x, x) \leq Ct^{-D/2}, \forall t > 0$$

implies

$$V(x, r) \geq cr^D, \forall x \in M, r > 0.$$

The heat flow cannot decrease quickly unless it has room to escape! However, in general, i.e. outside the polynomial growth range, the domination of  $\varphi_\infty$  by  $\varphi_2$  is no more optimal. There are heat kernel decays that are more rapid than one could predict from the volume growth. Indeed, in [18], Pittet and Saloff-Coste construct, for any  $n \in \mathbb{N}^*$ , manifolds with exponential volume growth whose heat kernels behave like  $e^{-ct^{\frac{n}{n+2}}}$  (the typical “good” heat kernel behaviour for exponential growth is  $e^{-ct^{\frac{1}{3}}}$ , whereas  $e^{-ct}$  corresponds to manifolds with a spectral gap).

Let us finally give another description of  $\varphi_2$  in terms of another formulation of the  $L^2$  Faber-Krahn inequalities (which is the way they were introduced by Grigory'an in [16]).

For  $\Omega \in \mathcal{K}(M)$ , let  $\lambda_1(\Omega)$  be the first eigenvalue of the Laplace operator in  $\Omega$  with the Dirichlet boundary condition, that is

$$\lambda_1(\Omega) = \inf_{f \in \text{Lip}(\Omega)} \frac{\|\nabla f\|_2^2}{\|f\|_2^2},$$

where  $\text{Lip}(\Omega)$  denotes the class of Lipschitz functions with support in  $\Omega$ .

Then  $\varphi_2$  is the smallest function  $\varphi$  such that the following Faber-Krahn type inequality holds:

$$\lambda_1(\Omega) \geq \frac{1}{\varphi^2(|\Omega|)},$$

for all  $\Omega \in \mathcal{K}(M)$ .

## 2.2 Faber-Krahn inequalities and upper bounds

The following theorem is due to Grigory'an in [16]. It also has a Markov semigroup version, see [6]. These abstract methods extend to discrete time, which enables one to treat by the same token random walks on graphs.

Let  $m$  be a decreasing differentiable function on  $\mathbb{R}_+^*$ . We shall say that  $m$  satisfies  $(\delta)$  if its logarithmic derivative has at most polynomial decay: there exists  $c > 0$  such that

$$f(u) \geq cf(t), \forall t > 0, u \in [t, 2t],$$

where  $f(t) = -\frac{m'(t)}{m(t)}$ . Condition  $(\delta)$  affects the regularity of the decay of  $m$  but not its rate: for example all functions  $C(\log t)^{-\alpha}$ ,  $Ct^{-\beta}$ ,  $C \exp(-ct^\gamma)$ ,  $\alpha, \beta, \gamma > 0$  satisfy  $(\delta)$ .

**Theorem 2.1** *Let  $m$  be a decreasing  $C^1$  bijection of  $\mathbb{R}_+^*$  satisfying  $(\delta)$ . Then*

$$\sup_{x \in M} p_t(x, x) \leq m(t), \forall t > 0,$$

*is equivalent to*

$$\lambda_1(\Omega) \geq \frac{1}{\varphi^2(|\Omega|)},$$

*for every  $\Omega \in \mathcal{K}(M)$ , where  $m$  and  $\varphi$  are related by  $-m'(t) = \frac{m(t)}{\varphi^2(1/m(t))}$  or  $t = \int_{m(t)}^{+\infty} \varphi^2(1/u) \frac{du}{u}$ , provided the integral converges.*

It may be helpful to remember that the smaller  $\varphi_2(v)$  for large  $v$ , the faster the decay of  $\sup_{x \in M} p_t(x, x)$ .

Note that the above equivalence has to be understood up to some multiplicative constants: one identifies  $m(\cdot)$  and  $C_\delta m(\delta)$ , for  $\delta \in ]0, 1[$ ,  $\varphi(\cdot)$  and  $C\varphi(\cdot)$ .

One can see the fact that the Faber-Krahn inequality implies the heat kernel decay in the following way. First

$$\lambda_1(\Omega) \geq \frac{1}{\varphi^2(|\Omega|)}, \forall \Omega \in \mathcal{K}(M)$$

is equivalent to the so-called Nash inequality

$$\|f\|_1^2 \theta \left( \frac{\|f\|_2^2}{\|f\|_1^2} \right) \leq (-\Delta f, f), \forall f \in C_0^\infty(M),$$

where  $\theta(x) = \frac{x}{\varphi^2(1/x)}$ . That Nash implies Faber-Krahn is fairly obvious (and not needed here); the converse relies on truncature techniques that come from [2], [3], and were developed in [5] and finally [1].

Now, if  $f$  is a function in  $C_0^\infty(M)$  such that  $\|f\|_1 = 1$ , applying the Nash inequality to  $e^{t\Delta} f$  yields a differential inequality on  $\|e^{t\Delta} f\|_2^2$ . By integration, this yields an estimate on  $\|e^{t\Delta}\|_{1 \rightarrow 2}$ , hence on

$$\|e^{t\Delta}\|_{1 \rightarrow \infty} = \sup_{x \in M} p_t(x, x),$$

which is nothing but the expected estimate. Note that this technique was essentially invented by John Nash in his seminal PDE paper of 1958!

The way to prove that a general heat kernel decay implies in turn the Faber-Krahn inequality, or the Nash inequality, was discovered more recently, though it relies on the very simple inequality

$$(2.2) \quad \sup_{x \in M} p_t(x, x) \geq \frac{e^{-\lambda_1(\Omega)t}}{|\Omega|}, \forall t > 0, \Omega \in \mathcal{K}(M).$$

As an example of the consequences of the above results, the following sequences of implications hold: for  $D > 1$ ,

$$|\Omega|^{\frac{D-1}{D}} \leq C|\partial\Omega| \implies \lambda_1(\Omega) \geq \frac{c}{|\Omega|^{2/D}} \implies \sup_{x \in M} p_t(x, x) \leq C' t^{-D/2}$$

and, for  $\alpha \in ]0, 1[$ , and  $\Omega, t$  large,

$$\frac{|\Omega|}{(\log |\Omega|)^{1/\alpha}} \leq C|\partial\Omega| \implies \lambda_1(\Omega) \geq \frac{c}{(\log |\Omega|)^{2/\alpha}} \implies \sup_{x \in M} p_t(x, x) \leq C' e^{-ct^{\frac{\alpha}{\alpha+2}}}.$$

### 2.3 Anti-Faber-Krahn inequalities and lower bounds

The following theorem has been proved in [9], §3, in a general operator semigroup setting. Again, the only non elementary ingredient in the proof is the inequality (2.2).

**Theorem 2.2** *Let  $m$  and  $\varphi$  be as in Theorem 2.1. Suppose that, for every  $v \in \mathbb{R}_+^*$ , there exists  $\Omega_v$  such that  $|\Omega_v| \leq v$  and  $\lambda_1(\Omega_v) \leq \frac{1}{\varphi^2(v)}$ . Then*

$$\sup_{x \in M} p_t(x, x) \geq m(Ct), \forall t > 0.$$

Take for instance  $M = \mathbb{H}^D$  the  $D$ -dimensional hyperbolic space, and embed in it arbitrary large euclidean balls, whose radii do not grow too fast. Then the above theorem shows that

$$\sup_{x \in M} p_t(x, x) \geq Ct^{-D/2}, \forall t > 0.$$

This bound is optimal.

Suppose now that  $\forall v \geq 1$ , there exists  $\Omega_v$  such that  $|\Omega_v| = v$  and  $|\partial\Omega_v| \leq C \frac{v}{(\log v)^{1/\alpha}}$ . This means that  $\varphi_1(v) \geq (\log v)^{1/\alpha}$ , therefore  $\varphi_2(v) \geq (\log v)^{1/2\alpha}$ ,  $v \geq 1$ , and one concludes that

$$\sup_{x \in M} p_t(x, x) \geq c' e^{-ct^{\frac{\alpha}{\alpha+1}}}, t \geq 1.$$

### 2.4 The main result

One can summarise Theorems 2.1 and 2.2 in the following way.

**Theorem 2.3** *Let  $\varphi_2$  be the  $L^2$  isoperimetric profile of  $M$ . Define  $m$  by  $-m'(t) = \frac{m(t)}{\varphi_2^2(1/m(t))}$ . If  $m$  satisfies  $(\delta)$ , then*

$$cm(Ct) \leq \sup_{x \in M} p_t(x, x) \leq C m(ct), \forall t > 0.$$

For example, if  $M$  is such that

$$\|f\|_{\frac{2D}{D-2}} \leq C \|\nabla f\|_2, \forall f \in C_0^\infty(M),$$

and if there exists  $x_0 \in M$  and  $C > 0$  such that

$$(2.3) \quad S(x_0, r) \leq C V(x_0, r)^{\frac{D-1}{D}}$$

(here  $S(x, r)$  is the  $n - 1$ -dimensional volume of the sphere in  $M$  of center  $x \in M$  and radius  $r > 0$ ), then

$$ct^{-D/2} \leq \sup_{x \in M} p_t(x, x) \leq Ct^{-D/2}.$$

Indeed, one can show ([9], §3), that the ‘‘anti-isoperimetric’’ inequality (2.3) implies the anti-Faber-Krahn inequality

$$\lambda_1(B(x_0, r)) \leq \frac{C'}{(V(x_0, r))^{2/D}}$$

i.e. a control from below on the  $L^2$  isoperimetric profile.

It would be very interesting to go further and understand in which situations more general anti-isoperimetric inequalities imply a good lower bound on the  $L^2$  isoperimetric profile, therefore a good on-diagonal lower bound for the heat kernel. For example, when can one improve  $e^{-ct^{\frac{\alpha}{\alpha+1}}}$  into  $e^{-ct^{\frac{\alpha}{\alpha+2}}}$  at the end of the previous section?

### 3. Heat kernels and volume growth.

One can consider that, contrary to the volume growth or to the  $L^1$  isoperimetric profile, the  $L^2$  isoperimetric profile is not really a simple geometric datum. We have seen that the  $L^1$  isoperimetric profile controls from above the  $L^2$  isoperimetric profile, but also that it can be much bigger. Which upper bound on the  $L^2$  isoperimetric profile, therefore on the heat kernel decay, can one deduce from the volume growth, according to whether one is or is not prepared to assume a certain regularity of the geometry? in both cases, we shall see that the answer is to be found in some Poincaré type inequalities. Let us mention that from upper bounds on the volume one can deduce fairly sharp lower bounds on the heat kernel, but this is another story (see [9], §6,9,10).

#### 3.1 Pseudo-Poincaré inequalities and fast decays

Say that  $M$  satisfies the pseudo-Poincaré inequality ( $PP_p$ ) if

$$\|f - f_r\|_p \leq Cr \|\nabla f\|_p, \quad \forall f \in C_0^\infty(M), r > 0,$$

where  $f_r(x) = \frac{1}{V(x,r)} \int_{B(x,r)} f(y) dy$ .

The following Proposition is a variation on [1], Prop. 10.6. Its ancestors are in [12], §3, [4], p.340. It might however be useful to write a proof here.

**Proposition 3.1** *If ( $PP_p$ ) holds, then there exists  $C$  such that*

$$\varphi_p(v) \leq C\varphi_\infty(Cv), \quad \forall v > 0.$$

**Proof:** Suppose  $p > 1$ . Take  $\Omega \in \mathcal{K}(M)$  and  $f \geq 0$  in  $\text{Lip}(\Omega)$ . Write

$$\|f\|_p^p = (f - f_r, f^{p-1}) + (f_r, f^{p-1}).$$

By Hölder and ( $PP_p$ ),

$$(f - f_r, f^{p-1}) \leq \|f - f_r\|_p \|f\|_p^{p-1} \leq Cr \|\nabla f\|_p \|f\|_p^{p-1}.$$

Let  $v > 0$  to be chosen later. Set  $r = \varphi_\infty(v)$ . By continuity,  $V(x, r) \geq v$ , for all  $x \in M$ , therefore  $\|f_r\|_\infty \leq v^{-1} \|f\|_1$ , and

$$(f_r, f^{p-1}) \leq \|f_r\|_\infty \|f^{p-1}\|_1 \leq v^{-1} \|f\|_1 \|f^{p-1}\|_1 \leq v^{-1} |\Omega| \|f\|_p^p,$$

since  $f$  is supported in  $\Omega$ . Choosing  $v = 2|\Omega|$  yields

$$\|f\|_p \leq 2C\varphi_\infty(2|\Omega|) \|\nabla f\|_p, \quad \forall f \in \text{Lip}(\Omega),$$

for all  $f \geq 0$  in  $\text{Lip}(\Omega)$ . The inequality holds for all  $f$  in  $\text{Lip}(\Omega)$  since  $|\nabla|f|| \leq |\nabla f|$ . Therefore  $(F_\varphi^p)$  holds with  $\varphi(v) = 2C\varphi_\infty(2v)$ , which proves the claim.

The proof for  $p = 1$  can be found in [12], §3.

Now Theorem 2.1 together with Proposition 3.1 yields the following.

**Theorem 3.2** *If  $(PP_2)$  holds, and if  $m$  defined by  $-m'(t) = \frac{m(t)}{\varphi_\infty^2(1/m(t))}$  satisfies  $(\delta)$ , then there exist  $C, c$  such that*

$$\sup_{x \in M} p_t(x, x) \leq C m(ct) \quad \forall t > 0.$$

For instance, if  $(PP_2)$  is valid on  $M$ , and if  $V(x, r) \geq cr^D$ , for all  $x \in M$ ,  $r \geq 1$ , one gets

$$\sup_{x \in M} p_t(x, x) \leq C t^{-D/2}, \quad \forall t \geq 1.$$

If  $V(x, r) \geq ce^r$ , for all  $x \in M$ ,  $r \geq 1$ , one gets

$$\sup_{x \in M} p_t(x, x) \leq C e^{-ct^{1/3}}, \quad \forall t \geq 1.$$

There are two important classes of manifolds where  $(PP_1)$  (and in fact  $(PP_p)$  for all  $1 \leq p < +\infty$ ) holds: Lie groups, and manifolds with non-negative Ricci curvature. The above results are optimal for these classes of manifolds. Since the analogue of  $(PP_p)$  also holds on finitely generated groups, one can also get in this way, through discretisation techniques, an upper estimate of  $\sup_{x \in M} p_t(x, x)$  for covering manifolds. For all this, see [12].

### 3.2 Relaxed pseudo-Poincaré inequalities and slow decays

We shall say that a manifold has bounded geometry if it has Ricci curvature bounded from below, but much weaker conditions suffice to apply our techniques, see [14].

It was noticed in [12], §5, in a discrete setting, that the conjunction of an upper bound and of a lower bound of the volume growth imply a relaxed pseudo-Poincaré inequality, of the form

$$\|f - f_r\|_p \leq \eta(r) \|\nabla f\|_p, \quad \forall f \in C_0^\infty(M), r > 0,$$

where  $\eta(r)$  may be larger than  $Cr$ . The following general version of this fact was proved in [10].

**Proposition 3.3** *Suppose that  $M$  has bounded geometry and that*

$$V_1(r) \leq V(x, r) \leq V_2(r)$$

*for all  $x \in M$  and  $r > 0$ , where  $V_1$  is strictly positive, continuous and strictly increasing. Then, for every  $\varepsilon > 0$  small enough, there exists  $C_\varepsilon$  such that*

$$\|f - f_r\|_2 \leq C_\varepsilon \left( \frac{r + 2\varepsilon}{V_1(r)} \right)^{1/2} V_2(r + \varepsilon) \|\nabla f\|_2, \quad \forall f \in C_0^\infty(M), \forall r > 0,$$

*where  $f_r(x) = \frac{1}{V(x, r)} \int_{B(x, r)} f$ .*

Assuming that  $V_2(r + \varepsilon) \geq C_\varepsilon V_2(r)$ ,  $r \geq 1$ , and replacing in the proof of Proposition 3.1 the full pseudo-Poincaré inequality by the relaxed pseudo-Poincaré inequality of Proposition 3.3, one gets the following estimate on  $\varphi_2$ :

$$\varphi_2(v) \leq C \frac{(V_2 \circ V_1^{-1}(Cv)) \sqrt{V_1^{-1}(Cv)}}{\sqrt{v}},$$

$V_1^{-1}$  being the reciprocal function of  $V_1$ . Now Theorem 2.1 yields the following (see [6], [10]).

**Theorem 3.4** *Suppose that  $M$  has bounded geometry and that*

$$V_1(r) \leq V(x, r) \leq V_2(r)$$

*for all  $x \in M$  and  $r > 0$ , where  $V_1$  is strictly positive, continuous and strictly increasing. Assume also that  $V_2(r + \varepsilon) \leq C_\varepsilon V_2(r)$ ,  $r \geq 1$ . Define  $m$  by*

$$t = \int_{m(t)}^{+\infty} (V_2 \circ V_1^{-1}(1/u))^2 V_1^{-1}(1/u) du.$$

*Then, if  $m$  satisfies  $(\delta)$ ,*

$$\sup_{x \in M} p_t(x, x) \leq C m(ct), \quad \forall t \geq 1.$$

In the case where  $V_1(r) = cV(r)$  and  $V_2(r) = CV(r)$ , one gets

$$t = \int_{m(t)}^{+\infty} V^{-1}(1/u) \frac{du}{u^2},$$

whereas the standard behaviour, corresponding to the case where the optimal pseudo-Poincaré inequality holds, is governed by  $M(t)$ , where, according to Theorem 3.2,

$$t = \int_{M(t)}^{+\infty} (V^{-1}(1/u))^2 \frac{du}{u}.$$

If  $V(r) = r^D$ ,  $M(t) = t^{-D/2}$  and  $m(t) = t^{-\frac{D}{D+1}}$ . If  $V(r) = e^r$ ,  $M(t) = e^{-ct^{1/3}}$  and  $m(t) = \frac{\log t}{t}$ .

More generally, in the case where  $V_1$  and  $V_2$  are both exponentials, one gets the following heat kernel estimate.

**Proposition 3.5** *Suppose that  $M$  has weak bounded geometry and that*

$$ce^{\alpha r} \leq V(x, r) \leq Ce^{\beta r}, \quad \forall x \in M, r \geq 1. \quad (3.1)$$

*Then*

$$\sup_{x \in M} p_t(x, x) \leq C \left( \frac{\log t}{t} \right)^{\frac{1}{2\theta-1}}, \quad \forall t \geq 2, \quad (3.2)$$

*where  $\theta = \beta/\alpha$ .*

Note that since on any non-compact manifold with bounded geometry one always has

$$\sup_{x \in M} p_t(x, x) \leq \frac{C}{\sqrt{t}}$$

(this is due to Grigo'ryan, Varopoulos, and Chavel-Feldman, see [10] for explanations and references), the above result is meaningful only if  $\frac{1}{2\theta-1} > 1/2$ , i.e.  $\theta < 3/2$ , which means that the volume growth estimate is sufficiently pinched.

### 3.3 Getting anti-Faber-Krahn inequalities

Finally, given the volume growth of a manifold, how fast and how slow can the heat kernel decay be? Some partial answers are given in [10], including a complete treatment of the polynomial case.

**Theorem 3.6** *Let  $M$  be a manifold with bounded geometry such that*

$$c r^D \leq V(x, r) \leq C r^D, \quad \forall r \geq 1.$$

*Then*

$$c' t^{-\frac{D}{2}} \leq \sup p_t(x, x) \leq C' t^{-\frac{D}{D+1}}, \quad \forall t \geq 1$$

*and both bounds are sharp.*

The upper bound follows from Theorem 3.4 (it was first proved in [12]), and the lower bound by proving

$$\lambda_1(B(x, r)) \leq \frac{C}{r^2}, \quad \forall r > 0,$$

and applying Theorem 2.2 (or directly (2.2)!). Let us indicate how one builds a manifold satisfying the assumptions and such that

$$\sup p_t(x, x) \geq c t^{-\frac{D}{D+1}}, \quad \forall t > 0.$$

The manifold  $M$  will be a perturbation of  $\mathbb{R}^D$ . Let us fix a point  $o \in \mathbb{R}^D$  and introduce a new Riemannian metric in the polar coordinates  $(r, \theta)$  centred at  $o$ :

$$ds^2 = dr^2 + h^2(r)d\theta^2.$$

The function  $h(r)$  will be smooth and positive on  $(0, \infty)$  and  $h(r) = r$  for small  $r$ . Denote  $B(r) = B(o, r)$ , and let  $S(r) = \omega_D h(r)^{D-1} = |\partial B(r)|$ .

Choosing the function  $S(r)$  is equivalent to choosing  $h(r)$ , therefore the metric on  $M$ . The function  $S(r)$  will be chosen to make  $\lambda_1(B(r))$  as small as possible. The control from above of  $\lambda_1(B(r))$  yields then a lower bound of  $\sup_{x \in M} p_t(x, x)$  by Theorem 2.2.

Here is the way one reaches a small  $\lambda_1(B(r))$ : the manifold has from time to time very narrow parts, i.e.  $S(r)$  is small (the difficulty is to ensure that without destroying the volume bounds nor the boundedness of the geometry). One then builds a radial function  $f(x) = g(d(o, x))$ , which is zero if  $d(o, x) \geq r$ , and increases only in the narrow parts. It follows that  $\int_{B(r)} |\nabla f|^2$  is relatively small with respect to  $\int_{B(r)} |f|^2$ .

Similar constructions, starting with the hyperbolic space instead of  $\mathbb{R}^D$ , can also be made in the exponential growth range.

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