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The resolvent expansion for the signature operator on a manifold with a conic singular stratum

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This is a report on a joint project with Jochen Brüning, concerning the signature operator D_S on a manifold M with a conic singular stratum Σ. Locally near Σ, M is represented as

$$(0, \epsilon) \times N;$$

N is a fibre bundle with base Σ^h of dimension h and fibre Y^v of dimension v:

$$N^{h+v} \xrightarrow{\pi} \Sigma^h, \ \ \pi^{-1}(point) \cong Y^v.$$

This fibration induces a vertical bundle of tangents to the fibres, $T_V N \subset TN$, and a horizontal cotangent sub-bundle $T_H^* N = \pi^* T^* \Sigma$ of cotangents annihilating $T_V N$. We take a complementary tangent sub-bundle $T_H N \subset TN$, inducing a vertical $T_V^* N$ that annihilates $T_H N$, such that

$$TN = T_V N \oplus T_H N, \ \ T^* N = T_V^* N \oplus T_H^* N.$$

Then we give M a metric defined, for x in $(0, \epsilon)$, α in $T_V^* N$, β in $T_H^* N$, by

$$g_M(adx + \alpha + \beta) = a^2 dx^2 + x^2 g_V(\alpha) + g_H(\beta) \tag{1}$$

The horizontal part g_H is induced from a metric on Σ, and the vertical part g_V forms a family of metrics on the fibres of N.

The metric determines a signature operator on $C^\infty \cap L^2(\Lambda M)$. Near Σ, it has a representation

$$D \sim (\partial_x + x^{-1} A(s) + B + x\Psi)\gamma \tag{2}$$

acting on functions from $(0,\epsilon)$ to forms on N. Here s is in Σ; Ψ and γ are morphisms, with $\gamma^2 = 1$; B is essentially a Dirac-type operator on the singular stratum Σ, and $A(s)$ is a Dirac-type operator in the fibre $\pi^{-1}(s)$. Generally, the operator D has many closed extensions, and we need to pick one. When the middle cohomology of the fibres is 0 (the "nonsingular" case), this can be determined locally in Σ. For each operator $\partial_x + x^{-1}A(s)$ we choose a domain that varies nicely with the metric, and thus we define a closed extension, denoted D_S. In the singular case, when the fibres have middle cohomolgy, these forms define a "middle cohomology" bundle over Σ in which $A(s) \equiv 0$. There $D = (\partial_x + B + x\Psi)\gamma$ is an operator requiring a boundary condition of Atiyah-Patodi-Singer type; with this condition, we have again a closed signature operator D_S.

In the nonsingular case, we construct the resolvent for the "full" signature operator,

$$\widetilde{D}_S(\mu)^{-1} = \begin{bmatrix} \mu & \text{-}D_S^* \\ D_S & \mu \end{bmatrix}^{-1},$$

and obtain its asymptotic expansion as $\mu \to \infty$. The supertrace of an appropriate term in this expansion then gives an index formula

$$\operatorname{ind}(D_S) = \int_M \omega + \int_\Sigma \alpha. \tag{3}$$

The form ω is the usual index form. In principal, it might be singular along Σ, but in any case the integral can be regularized by analytic continuation. The form α is determined locally in Σ, but $\alpha(s)$ involves spectral data from $A(s)$. In the case that $\dim\Sigma = 1$, it turns out to be a constant times the famous adiabatic limit [3]

$$\frac{1}{2\pi}\int_\Sigma \int_0^\infty tr(\beta\partial A/\partial s)Ae^{-tA^2}dt\,ds = \frac{1}{2\pi}\int_\Sigma tr[(\beta\partial A/\partial s)A(A^2)^{-z}]_{z=1}ds$$

where $\beta\partial/\partial s$ is the leading term in B.(1)

In the singular case, we expect an added term in the index formula (3), essentially the eta invariant of B acting in the "middle cohomology" bundle.

Here are some details of the derivation of the represention (2) for the signature operator. The map $\alpha\otimes\beta \longmapsto \alpha\wedge\beta$ defines an isomorphism

$$\bigoplus \Lambda_V^p \otimes \Lambda_H^q \cong \Lambda(N)$$

(1) Note that the A in this formula differs from the A in [3] by the term $-\nu$ introduced below.

and hence a bigrading $\Lambda(N) = \bigoplus \Lambda^{p,q}(N)$; p is the "vertical degree" and q the "horizontal degree". The C^∞ sections of these bundles are denoted $\lambda^{p,q}(N)$. Then the exterior derivative on N is a sum

$$d_N = d_V + d_H + \psi$$

mapping

$$\lambda^{p,q} \to \lambda^{p+1,q} \bigoplus \lambda^{p,q+1} \bigoplus_{0 \le j \le p} \lambda^{p-j-1,q+j+2}.$$

Here d_V is a "vertical" operator, in the sense that on any fibre, $d_V\omega$ depends only on the restriction of ω to that fibre. Moreover, $d_V^2 = 0$, $d_V d_H + d_H d_V = 0$, d_H^2 is a vertical first order operator, and ψ is a bundle morphism raising the horizontal degree by at least 2. Adjoints $\delta_V = d_V^*$ and $\delta_N = d_N^*$ of these operators are taken with respect to a metric on N derived from (1),

$$g_N(\alpha + \beta) = g_V(\alpha) + g_H(\beta). \tag{4}$$

Again, δ_V and δ_H^2 are vertical operators, and $\delta_V^2 = 0$, $\delta_V\delta_H + \delta_H\delta_V = 0$.

We define operators (vd) and (td) in $\lambda(N)$ which, in $\lambda^{p,q}$, are simply multiplication by p and $p+q$, respectively; and a further operator $\nu = (vd) - v/2$, where v is the dimension of the fibre Y. We map the pair $\alpha \oplus \beta$ in $\lambda(N) \bigoplus \lambda(N)$ to $x^\nu\alpha + dx \wedge x^\nu\beta$ in $\lambda(M)$. Then, for forms supported in $0 < x < \epsilon$,

$$\int [\|\alpha(x)\|_N^2 + \|\beta(x)\|_N^2] dx = \|x^\nu\alpha + dx \wedge x^\nu\beta\|_M^2 .$$

We have also a "Clifford volume element" on N,

$$\gamma = - *_N i^{m/2+(td)(td+1)} \quad (m = \dim(M))$$

satisfying $\gamma^2 = 1$, $\gamma^* = \gamma$, $\gamma\nu = -\nu\gamma$. Then we find that the signature operator D_S on M can be represented in $C_c^\infty((0,\epsilon), \lambda(N))$ as

$$\begin{aligned} D_S &\sim \partial_x\gamma + x^{-1}(d_V + \delta_V - \nu\gamma) + (d_H + \delta_H) + x\Psi\gamma \\ &= : (\partial_x + x^{-1}A + B + x\Psi)\gamma. \end{aligned}$$

The various relations above imply that

$$\begin{aligned} A^* &= A,\ B^* = B,\ AB + BA \text{ is vertical and first order} \\ &\quad \text{and } (A+B)^2 - \Delta_N \text{ has order less than 2.} \end{aligned}$$

All these operators are defined independent of any coordinate system, which is important when we want to split off, say, the part acting in the middle cohomology bundle of the fibres. However, to construct the pseudo-differential parametrix for the resolvent we need a representation of D_S locally in Σ, as a differential operator on Σ with coefficients which are operators in a fixed Hilbert space. To this end, we introduce local coordinates in a neighborhood $U \subset \Sigma$, and a map

$$T: \; U \times [\Lambda(Y) \bigotimes \Lambda R^h] \to \Lambda(\pi^{-1}(U))$$

preserving fibres and bigradings. We introduce a norm on the left such that T induces an isometry of the Hilbert space structures on the forms. Then, using a subscript T to indicate an operator conjugated by T, we have

$$T^{-1} D_S T = (\partial_x + x^{-1} A_T(s) + B_T + x\Psi_T)\gamma_T$$

where A_T and B_T have the properties of A and B. Moreover

$$B_T = \sum_k \beta_k(s)\, \partial/\partial s_k + R_T(s) \tag{5}$$

with a vertical operator R_T and bundle morphisms β_k satisfying

$$\beta_k A_T + A_T \beta_k = 0 \;\text{ and }\; (\sum \beta_k \sigma_k)^2 =: -|\sigma|_s^2 I. \tag{6}$$

Although T is defined only locally in Σ, it is defined globally in the fibres.

Essential to our analysis is the spectral resolution of

$$A(s) = (d_V + \delta_V(s))\gamma(s) - \nu$$

where $\nu = (vd) - v/2$. Using the appropriate metric in each fibre, we have

$$A \cong \gamma \cdot (\delta_Y + d_Y) \otimes 1 \quad - \nu \otimes 1 \;\text{ in }\; \lambda(Y) \bigotimes \Lambda T_s^* \Sigma.$$

We decompose $\lambda(Y)$ into harmonic, closed, and coclosed forms, and obtain

$$\begin{aligned} A &= A_h \oplus A_{cl} \oplus A_{ccl} \\ &= -\nu \oplus [\gamma(\delta_Y \otimes 1) - \nu] \oplus [\gamma(d_Y \otimes 1) - \nu]. \end{aligned}$$

The eigenvalues turn out to be

$$\mu_{h,j} = j - v/2 \; (0 \le j \le v) \text{ with eigenforms in } H^j(Y, \Lambda_s^*)$$

$$\begin{aligned}
\mu^{\pm}_{cd,j} &= -\frac{1}{2} \pm \sqrt{\lambda + (\frac{v+1}{2} - j)^2} \text{ with } \lambda \text{ in } spec(d\delta)_j(Y, \Lambda^*_s) \\
\mu^{\pm}_{ccd,j} &= \frac{1}{2} \pm \sqrt{\lambda + (\frac{v-1}{2} - j)^2} \text{ with } \lambda \text{ in } spec(\delta d)_j(Y, \Lambda^*_s) \\
\mu^{\pm}_{cl} &= -\frac{1}{2} \pm \sqrt{\lambda} \text{ with } \lambda \text{ in } spec(d\delta)_{(v+1)/2}(Y, \Lambda^*_s)
\end{aligned}$$

Note: 1) The "harmonic eigenvalues" $j - v/2$ are independent of the metric on Y. 2) The low eigenvalues, those in the interval $(-1/2, 1/2)$, come only from dimension j between $v/2 - 1$ and $v/2 + 1$. 3) If the metric on Y is small enough, then the Laplace eigenvalues λ are large enough that all the μ^+ eigenvalues are $\geq 1/2$, and all the μ^- eigenvalues are $\leq -1/2$.

We use this spectral decomposition to split our forms into three parts. Let $Y_s = \pi^{-1}(s)$ denote the fibre over s in Σ. Then

$$L^2(\Lambda(N)|Y_s) = E_<(s) \bigoplus E_0(s) \bigoplus E_>(s)$$

where

$$\begin{aligned}
E_<(s) &= \text{ sum of eigenspaces of } A(s) \text{ corresponding to the } \mu^- \text{ eigenvalues} \\
&\quad \text{together with the } H^j(Y, \Lambda^*_s) \text{ for } j \text{ less than } v/2, \\
E_0(s) &\cong H^{v/2}(Y_s) \bigotimes \Lambda T^*_s \Sigma \\
E_>(s) &= \text{ sum of eigenspaces of } A(s) \text{ corresponding to the } \mu^+ \text{ eigenvalues} \\
&\quad \text{together with the } H^j(Y, \Lambda^*_s) \text{ for } j \text{ greater than } v/2.
\end{aligned}$$

Then in the local coordinate representation (5), the β_k preserve E_0, and exchange $E_<$ with $E_>$.

The domain of D_S is defined with respect to this splitting. Let $\pi_>(s)$ denote projection on $E_>(s)$, and so on. Supppose for now that $E_0 = 0$. Then we define the domain of D_S by

$$\begin{aligned}
\|\pi_> \gamma u(x)\|_{L^2\Lambda(N)} &= O(x^{1/2-\delta}) \text{ for all } \delta > 0 \\
\|\pi_< \gamma u(x)\|_{L^2\Lambda(N)} &= O(x^{-1/2+\delta}) \text{ for some } \delta > 0
\end{aligned} \tag{7}$$

With this domain, D_S is closed, and the domain of its adjoint is defined by

$$\begin{aligned}
\|\pi_> w(x)\|_{L^2\Lambda(N)} &= O(x^{-1/2+\delta}) \text{ for some } \delta > 0 \\
\|\pi_< w(x)\|_{L^2\Lambda(N)} &= O(x^{1/2-\delta}) \text{ for all } \delta > 0.
\end{aligned} \tag{8}$$

Now we come to the resolvent parametrix. For simplicity of notation, we drop the subscript S on the signature operator. Rather than treat the Laplacians $\Delta_+ = D^*D$ and $\Delta_- = DD^*$ directly, we consider the first-order operator

$$\widetilde{D}(\mu) = \begin{bmatrix} \mu & -D^* \\ D & \mu \end{bmatrix}$$

with inverse

$$\widetilde{D}(\mu)^{-1} = \begin{bmatrix} \mu(\Delta_+ + \mu^2)^{-1} & D^*(\Delta_- + \mu^2)^{-1} \\ -D(\Delta_+ + \mu^2)^{-1} & \mu(\Delta_- + \mu^2)^{-1} \end{bmatrix} \tag{9}$$

This will give us the various zeta ... functions for D. We treat D as a differential operator on Σ with operator coefficients, and principal symbol

$$\hat{D} = (\partial_x + x^{-1}A(s) + i\sum \beta_k \sigma_k)\gamma.$$

The domain of $\partial_x + x^{-1}A(s)$ is defined by the growth relations (7). In view of the relations (6) involving β_k and A, the corresponding symbol Laplacians are

$$\mu^2 + \hat{\Delta}_\pm = -\partial_x^2 + x^{-2}(A^2 \pm A) + (|\sigma|_s^2 + \mu^2), \tag{10}$$

a family of regular singular operators on the cones over Σ. In this symbol, the dual transform variable σ combines with the resolvent parameter μ. When we compute the trace, we take the asymptotics of such symbols and integrate over σ and s.

In view of (9), we take as leading term in our parametrix the operator on Σ with symbol

$$\hat{P} = \begin{bmatrix} \mu(\hat{\Delta}_+ + \mu^2)^{-1} & \hat{D}^*(\hat{\Delta}_- + \mu^2)^{-1} \\ -\hat{D}(\hat{\Delta}_+ + \mu^2)^{-1} & \mu(\hat{\Delta}_- + \mu^2)^{-1} \end{bmatrix}$$

The symbol resolvents $(\hat{\Delta}_\pm + \mu^2)^{-1}$ have kernels which can be viewed as direct sums of Bessel functions with orders determined by the eigenvalues of A. They can be represented as

$$(xy)^{1/2} I_{p\pm}(xz) K_{p\pm}(yz) \quad (x < y)$$

with $z^2 = |\sigma|_s^2 + \mu^2$, and the orders $p_\pm$ given by

$$p_\pm = \begin{cases} A \pm 1/2 \text{ in } E_> \\ -(A \pm 1/2) \text{ in } E_< \end{cases}$$

A very tedious check, using relations among the Bessel functions, shows that $Op(\hat{P})$ maps into the the domain of $D \bigoplus D^*$, as defined in (7) and (8). Locally

$$\widetilde{D}Op(\hat{P}) = I + Op\left(\begin{bmatrix} 0 & -\gamma\Sigma\beta_k\partial/\partial s_k \\ \Sigma\beta_k\partial/\partial s_k\gamma & 0 \end{bmatrix}\hat{P}\right) + ROp(\hat{P}) \quad (11)$$

where R is a vertical first-order operator. The remainder terms $\to 0$ as $\mu \to \infty$, so the Neumann series

$$\widetilde{D}(\mu)^{-1} = Op(\hat{P})\left\{I - Op\left(\begin{bmatrix} 0 & -\gamma\Sigma\beta_k\partial/\partial s_k \\ \Sigma\beta_k\partial/\partial s_k\gamma & 0 \end{bmatrix}\hat{P}\right) - ROp(\hat{P}) + \ldots\right\} \quad (12)$$

gives an expansion for large μ. In view of (9) we can extract from this the powers of the individual resolvents $(\Delta_\pm + \mu^2)^{-m}$ and obtain the asymptotic expansion of their traces, using the Singular Asymptotics Lemma of [[1]]. Let $\mathcal{G}^m(x, y, s, t; \mu)$ denote the kernel of $(\Delta + \mu^2)^{-m}$,and tr_H denote the trace in $H = L^2(\Lambda Y) \otimes \Lambda R^h$. The function

$$\sigma(x, s; \mu) = tr_H \mathcal{G}^m(x, x, s, s; \mu/x)$$

has the usual resolvent kernel expansion in decreasing powers of μ: $\sigma(x, s; \mu) \sim \sum \sigma_\alpha(x, s)\mu^\alpha$. And for functions $\phi(x)$ and $\psi(s)$ with suitable small support, we have

$$tr\left(\phi\psi(\Delta + \mu^2)^{-m}\right) = \int_0^\infty \int_{R^h} \phi(x)\psi(s)\sigma(x, s; x\mu) ds\, dx.$$

As in [2], this integral has an aymptotic expansion as $\mu \to \infty$, with three types of terms:

$$\begin{aligned} &\sum_\alpha \int_0^\infty \int_{R^h} \phi(x)\psi(s)\sigma_\alpha(x, s)(x\mu)^\alpha ds\, dx \qquad (13) \\ &+ \sum_{l \geq 0} \mu^{-l-1} \int_0^\infty \frac{\zeta^l}{l!} \int_{R^h} \phi(0)\psi(s)\sigma^{(l)}(0, s; \zeta)\, ds\, d\zeta \\ &+ \sum_{\alpha=-1}^{-\infty} \phi(0) \int_{R^h} \psi(s)\sigma_\alpha^{(-\alpha-1)}(0, s)\, ds \frac{\mu^\alpha \log \mu}{(-\alpha - 1)!}. \end{aligned}$$

The first sum comes from the resolvent expansion away from the singular set Σ; when we compute the index from (13, Sec 6), those terms will give the usual index form, regularized (if necessary) by analytic continuation. The other two sums give the contribution from the singular stratum Σ. Note that these "singular" terms depend on the Taylor expansion of the operator in powers of x. As we go farther out in the Neumann series (12), the terms vanish to higher order as $x \to 0$. Similarly, lower order terms in the expansion of ψdo products vanish to higher order. (This vanishing is checked by rescaling $(x, s) \to (tx, ts)$.)

In particular, to get the index, we need the term in μ^{-2m} from

$$Str(\frac{d}{d(\mu^2)})^{m-1}\frac{1}{\mu}\tilde{D}(\mu)^{-1}$$

taking m large enough that the trace exists. When $\dim\Sigma = 1$, in view of the vanishing just noted, the only contributions to the singular term come from the first two terms in the Neumann series (12). The supertrace contribution from $Op(\hat{P})$ is zero, because of symmetry. For the next term we need just the leading term in the ψdo expansion, $Op\left(\hat{P}\begin{bmatrix} 0 & -\gamma\beta\partial/\partial s \\ \beta\partial/\partial s\gamma & 0 \end{bmatrix}\hat{P}\right)$. Juggling with commutators reduces this to the consideration of

$$x^{-1}\beta\,\partial A/\partial s[G_+^2 - G_-^2],$$

where $G_\pm$ is the kernel of $Op(\hat{\Delta}_\pm + \mu^2)^{-1}$. Since the kernels are given by Bessel functions, it is a question of finding the right Bessel asymptotics. This tedious calculation is justified by the resulting clean index formula

$$\mathrm{ind}(D_S) = \int_M \omega + c\int_\Sigma tr[(\beta\partial A/\partial s)A(A^2)^{-z}]_{z=1}ds.$$

for a constant c.

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