ALBERTO RUIZ

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Journées Équations aux dérivées partielles (1993), p. 1-12
<http://www.numdam.org/item?id=JEDP_1993____A5_0>

REGULARIZING ESTIMATES FOR SCHRÖDINGER AND WAVE EQUATIONS

By Alberto Ruiz

§1. INTRODUCTION. Let us consider the initial value problems

$$(1\text{-}1)\qquad \begin{cases} L_1 u_1(x,t) = F(x,t), & (x,t) \in \mathbf{R}^n \times \mathbf{R}, \\ u(x,0) = 0. \end{cases}$$

where L_1 denotes the time dependent Schrödinger operator $i\partial_t + \Delta_x$, and

$$(1\text{-}2)\qquad \begin{cases} L_2 u_2(x,t) = F(x,t), & (x,t) \in \mathbf{R}^n \times \mathbf{R}, \\ u_2(x,0) = 0 \\ \partial_t u_2(x,0) = 0, \end{cases}$$

for L_2 the Wave operator $\partial_{tt} + \Delta_x$.

We define the Morrey-Campanato class $\mathbf{L}^{\alpha,p}, p \leq n/\alpha, \alpha > 0$ as

$$(1\text{-}3)\qquad \mathbf{L}^{\alpha,p} = \{V \in L^p_{loc}, \text{ such that } \|V\|_{\alpha,p} \equiv \sup_{r,x_0}(r^\alpha(r^{-n}\int_{B(x_0,r)} |V(x)|^p dx)^{1/p}) \leq \infty\}.$$

We prove weighted estimates for solutions of the problem (1-1), more precisely:

THEOREM 1

Let u_i be a solution of $(1-i), i = 1,2$ and V a non negative function such that $\sup_t V$ is in the class $\mathbf{L}^{2,p}$ with $p > (n-1)/2$, $n \geq 3$, then there exists a constant C only depending on n such that the following a priori estimate holds

$$(1\text{-}4)\qquad \sup_{x_0,R} R^{-1} \int_{B(x_0,R)} \int_{-\infty}^{+\infty} |D_x^{1/2} u_i|^2 dt dx \leq C \|\sup_t V\|_{2,p} \|F\|^2_{L^2(V^{-1}dxdt)}$$

The last inequality can be understood as a smoothing effect for the non homogeneous equation, with a gain of one half derivative and a gain of one half derivative in the L^p spaces gap. This can be easily seen in the weaker case $p = n/2$ which corresponds with $V \in L_x^{n/2}(L_t^\infty)$; by duality we can prove the following estimate (in the case of the Schródinger operator see **[RV2]**)

$$(1\text{-}5)\qquad |||D_x^{1/2} u_i|||^2_\infty \equiv \sup_{x_0,R} \frac{1}{R} \int_{B(x_0,R)} \int_{-\infty}^{\infty} |D_x^{1/2} u_i|^2 dt dx \leq C \|F\|^2_{L_x^q(L_t^2(\mathbf{R}))},$$

with $\frac{1}{q} - \frac{1}{2} = \frac{1}{n}$ and $n \geq 3$.

Similar estimates have been obtained in **[KPV]** for the wave equation, with gain of one derivative and with non homogeneous term F in $L^2(|x|dx)$. Also other kind of mixed norm inequalities has been obtained by **[H]**.

As in [RV2], these inequalities are consequence of a similar one for Helmholtz equation:

THEOREM 2 *Let u be a solution of*

$$\Delta u + (\tau + i\epsilon)u = f \qquad x \in \mathbf{R}^n,$$

where $\epsilon > 0$, *and let* $V(x) \in \mathbf{L}^{2,p}$ *with* $p > (n-1)/2, n \geq 3$.

Then there exists a constant $C > 0$, *independent of* τ *and* ϵ *such that*

$$\text{(1-5)} \qquad \sup_{x_0,R}\Big(\frac{1}{R}\int_{B(x_0,R)} |D_x^{1/2}u|^2\Big)^{1/2} \leq C\|V\|_{2,p}\|f\|^2_{L^2(V^{-1}dx)}.$$

Meanwhile the L^2 estimates are consequence of traze type lemmas (see [AH]), the present ones involve curvature of the zero set of the symbols and can be seen as consequence of some type of restriction theorems for the Fourier transform.

In the case of the Schrödinger equation, (1-2) can be used in a perturbation argument, obtaining the following theorem wich is an improvement of theorem 1.1 in [RV2]:

THEOREM 3 *Let V be a potential in* $\mathbf{R}^n \times \mathbf{R}, n > 2$, *which can be written as* $V(x,t) = V_1(x,t) + V_2(x,t)$ *with* $\sup_{t\in[-T,T]} |V_1| \in \mathbf{L}^{2,p}, p > (n-1)/2$, $V_2 \in L^r([-T,T] : L_x^\infty)$, $r > 1$

and $\| \sup_{t\in[-T,T]} |V_1| \|_{2,p}$ *small enough.*

Then there exists a unique solution $u(x,t)$ *of*

$$\text{(1-6)} \qquad \begin{cases} i\partial_t u + \Delta_x u + V(x,t)u = 0 & (x,t) \in \mathbf{R}^n \times \mathbf{R} \\ u(x,0) = u_0(x). \end{cases}$$

such that

$$\text{(1-7)} \qquad \|u\|_{L^2(\mathbf{R}^n\times[-T,T],|V_1(x,t)|dxdt)} + \sup_{|t|<T} \|u(.,t)\|_{L^2(\mathbf{R}^n)} \leq C(T)\|u_0\|_{L^2(\mathbf{R}^n)}.$$

Moreover,

$$\text{(1-8)} \qquad |||D_x^{1/2}u|||_T^2 \equiv \sup_{x_0,R} \frac{1}{R}\int_{B(x_0,R)}\int_{-T}^{T} |D_x^{1/2}u|^2 dtdx \leq C(T)\|u_0\|^2_{L^2(\mathbf{R}^n)}.$$

If $V_2 \equiv 0$, *T can be taken to be* ∞ *and* $C(T)$ *independent of T.*

Let us remark that this class of time independent potential contains the functions in the Lorentz spaces $L^{n/2,\infty}$ with small norm. Also some functions like $(1/|x|^2)f(x/|x|)$ for $f \in L^p(\mathbf{S}^{n-1})$ with $p > (n-1)/2$ and small norm and $V \in L_x^{n/2}(L_t^\infty)$, without any restriccion on the size of its norm, are included in the statement of the theorem.

Theorem 3 (smoothing effect for the initial value problem) has been obtained in the free case $V \equiv 0$ by [S], [V], [CS1] and for potential with more restrictive conditions that in our statement by [SSj], [CS2], and [RV2].

This kind of smoothing effect was firstly observed by **[K]** in the case of the non linear KDV equation, and plays an important role in the proof of well posedness of some linear and non linear equations (see [S], [KPV]). For similar identities see **[LP]**.

In section 2 we prove Theorem 1, as consequence of theorem 2 and some estimates for solutions of the the initial value problem for the homogeneous equation. In section 3 we outline the proof of theorem 2. In section 4 we prove theorem 3 .

These results are a summary of joint work with Luis Vega. An expanded version will appear elsewhere. **Notation**

$(F(x,.))\hat{}(\tau) = \int_{-\infty}^{\infty} e^{-it\tau} F(x,t)dt,$

$(F(.,t))\hat{}(\xi) = \int_{\mathbf{R}^n} e^{-ix\xi} F(x,t)dx.$

$\hat{F}(\xi,\tau)$ will be the whole Fourier transform.

S_r^{n-1} and $d\sigma_r$ the euclidean sphere of radius r and its measure.

$I^\alpha f$ will be the fractional integration defined by $(I^\alpha f)\hat{}(\xi) = |\xi|^{-\alpha}\hat{f}(\xi), 0 < \alpha < n.$

$D_x^s f$ will be the fractional derivative, $(D_x^s f)\hat{}(\xi) = |\xi|^s \hat{f}(\xi)$.. Sometimes we write $I^\alpha = D_x^{-\alpha}$

$J^s f$ will be the Bessel potential, $(J^s f)\hat{}(\xi) = (1+|\xi|^2)^{-s/2}\hat{f}(\xi)$, $0 < s < n.$

$|||F|||_T^2 \equiv \sup_{x_0,R} \frac{1}{R} \int_{B(x_0,R)} \int_{-T}^{T} |F|^2 dtdx.$

$|||F|||_\infty^2 \equiv \sup_{x_0,R} \frac{1}{R} \int_{B(x_0,R)} \int_{-\infty}^{\infty} |F|^2 dtdx.$

$|||f|||^2 \equiv \sup_{x_0,R} \frac{1}{R} \int_{B(x_0,R)} |f|^2 dx.$

§2. PROOF OF THEOREM 1

The proof is based upon representation formulas for solutions of problems (1-i) and some, more or less known, a priory estimates for solution of the initial value problems:

$$\text{(2-1)} \qquad \begin{cases} L_1 u_1(x,t) = 0 & x \in \mathbf{R}^n, t > 0, \\ u_1(x,0) = f_1, \end{cases}$$

and

$$\text{(2-2)} \qquad \begin{cases} L_2 u_2(x,t) = 0 & x \in \mathbf{R}^n, t > 0, \\ u_2(x,0) = f_1, \\ \partial_t u_2(x,0) = f_2. \end{cases}$$

Let us denote

$$\text{(2-3)} \qquad \begin{cases} Sf_1(x,t) = e^{it\Delta} f_1, \text{ the solution of } (2,1); \\ W_1 f_1(x,t) = \Re e^{it\sqrt{-\Delta}} f_1, \text{the solution of (2-2) for } f_2 = 0, \\ W_2 f_2(x,t) = \Im e^{it\sqrt{-\Delta}} D_x^{-1} f_2, \text{the solution of (2-2) for } f_1 = 0 \end{cases}$$

Proposition 2.1: *Let $d\sigma$ be the uniform measure on the unit sphere $\mathbf{S}^{n-1}$ and $\hat{d\sigma}$ its Fourier transform, let $V \in \mathbf{L}^{2,p}$ with $p > (n-1)/2$, $n \geq 3$, and consider the operator*

$$Tf(x) = \hat{d\sigma} * f(x).$$

Then there exists a constant C such that

$$\|Tf\|_{L^2(V)} \leq C\|V\|_{2,p}\|f\|_{L^2(V^{-1})}$$

for any f in C_0^∞.

Proof: See **[RV1]**.

Proposition 2.2: *Let μ be an L^2 density on $\mathbf{S}^{n-1}$, consider the extension operator*

$$E\mu(x) = (\mu d\sigma)\hat{}(x)$$

and V as in Proposition 2.1 then there exists a constant C, independent of μ such that

$$\|E\mu\|_{L^2(V)} \leq C(\|V\|_{2,p})^{1/2}\|\mu\|_{L^2(\mathbf{S}^{n-1})}$$

Proof: By duality it reduces to

$$\int_{\mathbf{S}^{n-1}} |E^*f|^2 d\sigma = \int_{\mathbf{S}^{n-1}} |\hat{f}|^2 d\sigma = \int f(f * (d\sigma)\hat{})$$

$$\leq C\|V\|_{2,p}\|f\|^2_{L^2(V(x)^{-1})}$$

where we have used Hölder inequality and Proposition 2.1.

Proposition 2.3: *Let $V(x,t)$ as in theorem 1, and $\gamma \geq -1/2$, then the following a priori estimates hold:*

(2-4) $$\|Sf_1\|_{L^2(Vdxdt)} \leq C\|\sup_t V\|^{1/2}_{2,p}\|f_1\|_{L^2},$$

(2-5) $$\|D_x^\gamma W_1 f_1\|_{L^2(Vdxdt)} \leq C\|\sup_t V\|^{1/2}_{2,p}\|D_x^{\gamma+1/2} f_1\|_{L^2},$$

(2-6.) $$\|D_x^\gamma W_2 f_2\|_{L^2(Vdxdt)} \leq C\|\sup_t V\|^{1/2}_{2,p}\|D_x^{\gamma-1/2} f_2\|_{L^2},$$

Proof: Let us consider first estimate (2-4). Using polar coordinates and a simple change of variable we can write,

(2–7) $$e^{it\Delta}u_0 = \int_0^\infty e^{itr^2}\int_{S_r^{n-1}} e^{ix\xi}\hat{u}_0(\xi)d\sigma_r(\xi)dr$$

$$= 1/2\int_0^\infty e^{its}\int_{S^{n-1}_{\sqrt{s}}} e^{ix\xi}\hat{u}_0(\xi)d\sigma_{\sqrt{s}}(\xi)s^{-1/2}ds$$

Taking supremun in t and using Plancherel, we have

$$
\begin{aligned}
\|Sf_1\|^2_{L^2(Vdxdt)} &\leq C\int_{\mathbf{R}^n}\left(\int_0^\infty \Big|\int_{S^{n-1}_{\sqrt{s}}} e^{ix.\xi}\hat{f}_1(\xi)d\sigma_{\sqrt{s}}(\xi)\Big|^2 s^{-1/2}ds\right)\sup_t V(x,t)dx \\
&\leq C\left(\int_0^\infty \int_{\mathbf{R}^n} \Big|\int_{S^{n-1}_r} e^{ix\xi}\hat{f}_1(\xi)d\sigma_r(\xi)\Big|^2 \sup_t V(x,t)dx r^{-1}dr\right) \\
&\leq C\|\sup_t V\|_{2,p}\left(\int_0^\infty \int_{S^{n-1}_r} |\hat{f}_1(\xi)|^2 d\sigma_r(\xi)dr\right) \\
&= C\|\sup_t V\|_{2,p}\|f_1\|^2_{L^2(\mathbf{R}^n)}
\end{aligned}
$$

The last inequality follows from Proposition (2.2). Let us go to (2.5) and (2.6). From (2-3), since all the operators in the statement commute, we may reduce to prove

$$\|D^\gamma_x e^{it\sqrt{-\Delta}} f\|_{L^2(Vdxdt)} \leq C\|\sup_t V\|^{1/2}_{2,p}\|D^{\gamma+1/2}_x f\|_{L^2}, \tag{2-8.}$$

As in the above case we may write

$$D^\gamma_x e^{it\sqrt{\Delta}} f = \int_0^\infty e^{itr} r^\gamma \int_{S^{n-1}_r} e^{ix\xi}\hat{f}(\xi)d\sigma_r(\xi)dr,$$

Then the proof follows as in the Schrödinger case.

Proposition 2.4: *The following inequalities hold*

$$\||D^{1/2}_x Sf_1\||_\infty \leq C\|f\|_{L^2}, \tag{2-9}$$

$$\||D^\gamma_x W_1 f_1\||_\infty \leq C\|D^\gamma_x f_1\|_{L^2}, \tag{2-10}$$

$$\||D^\gamma_x W_2 f_2\||_\infty \leq C\|D^{\gamma-1}_x f_2\|_{L^2}, \tag{2-11}$$

Proof: It is similar to the proof of Proposition 2.3 , just use theorem 2.1 in **[AH]** instead of Proposition 2.2.

Remark on dual operators.

We can obtain easily the following formal expresions for the dual of the above operators:

$$
\begin{aligned}
S^*F(x) &= \int_{-\infty}^\infty S(F(.,t))(s,x)\big|_{s=t}dt \\
W_i^*F(x) &= \int_{-\infty}^\infty W_i(F(.,t))(s,x)\big|_{s=t}dt, i = 1,2.
\end{aligned}
$$

By duality we can prove the following estimates, with C independent of T

(2-4*) $\| \int_0^T S(F(.,t))(s,x)\big|_{s=t} dt \|_{L^2(\mathbf{R}^n)} \leq C \| \sup_{t\in[0,T]} V\|_{2,p}^{1/2} \|F\|_{L^2(\mathbf{R}^n\times[0,T],V^{-1}dxdt)},$

(2-5*)

$$\|D_x^{-1/2} \int_0^T W_1(F(.,t))(s,x)\big|_{s=t} dt \|_{L^2(\mathbf{R}^n)} \leq C \| \sup_{t\in[0,T]} V\|_{2,p}^{1/2} \|F\|_{L^2(\mathbf{R}^n\times[0,T],V^{-1}dxdt)},$$

(2-6*)

$$\|D_x^{1/2} \int_0^T W_2(F(.,t))(s,x)\big|_{s=t} dt \|_{L^2(\mathbf{R}^n)} \leq C \| \sup_{t\in[0,T]} V\|_{2,p}^{1/2} \|F\|_{L^2(\mathbf{R}^n\times[0,T],V^{-1}dxdt)},$$

Next lemma gives a representation of the solutions of problems (1.1) and (1.2), in order to discribe it, let us take the solution to the corresponding equations obtained by taking whole Fourier transform :

$$v_1(x,t) = \lim_{\epsilon\to 0+} \iint e^{ix\xi+it\tau} \frac{\hat{F}(\xi,\tau)}{|\xi|^2 - \tau + i\epsilon} d\xi d\tau,$$

and

$$v_2(x,t) = \lim_{\epsilon\to 0+} \iint e^{ix\xi+it\tau} \frac{\hat{F}(\xi,\tau)}{|\xi|^2 - (\tau + i\epsilon)^2} d\xi d\tau.$$

Lemma 2.5: *The solutions of problems (1.1) and (1.2) can be written as:*

$$u_i(x,t) = v_i(x,t) + R_i(x,t)$$

where

$$R_1 = S(S^*(G)) \text{ and } R_2 = (W_2W_1^* + W_1W_2^*)(G),$$

for $G(x,t) = sigt F(x,t)$ *.*

Proof: The case of Schrödinger equation can by seen in **[RV2]**.

For the wave equation , since v_2 is a solution of the equation, the remainder term is given by

$$R_2(x,t) = W_1(v_2(.,0))(x,t) + W_2(\partial_t v_2(.,0))$$

But

$$v_2(x,0) = \lim_{\epsilon\to 0+} \iint e^{ix\xi} \frac{\hat{F}(\xi,\tau)}{|\xi|^2 - (\tau + i\epsilon)^2} d\xi d\tau$$

$$= \lim_{\epsilon\to 0+} \iint e^{ix\xi} \hat{F}(\xi,t) \int \frac{e^{it\tau}}{|\xi|^2 - (\tau + i\epsilon)^2} d\tau dt d\xi$$

$$= 1/2 \lim_{\epsilon\to 0+} \int e^{ix\xi} \hat{F}(\xi,t) |\xi|^{-1} \Big(\int \frac{e^{it\tau}}{|\xi| - (\tau + i\epsilon)} d\tau dt d\xi + \int \frac{e^{it\tau}}{|\xi| + (\tau + i\epsilon)} d\tau dt d\xi \Big) dt d\xi$$

$$= \int e^{ix\xi} |\xi|^{-1} \Im e^{it|\xi|} sigt \hat{F}(\xi,t) d\xi dt.$$

Similar calculations for $\partial_t v_2(x,0)$ give the result.

End of proof of theorem 1:

The bound for R_1 is a consequence of lemma 2.5 , (2.4*) and (2.9).

For R_2 use the lemma, (2.5*) and (2.11), (2.6*) and (2.10).

Let us proceed to bound v_i. Recall

$$D_x^{1/2} v_1(x,t) = \lim_{\epsilon \to 0+} \int e^{ix\xi + it\tau} \frac{|\xi|^{1/2} \hat{F}(\xi,\tau)}{|\xi|^2 - \tau + i\epsilon} d\xi d\tau,$$

then, by Minkowsky integral inequality, Plancherel identity in t, we have

$$\||| \iint \frac{|\xi|^{1/2}}{|\xi|^2 - \tau + i\epsilon} \hat{F}(\xi,\tau) e^{ix\xi + ir t} d\xi d\tau |||_\infty$$
$$= C \int_{-\infty}^{+\infty} ||| \int e^{ix\xi} \frac{|\xi|^{1/2}}{|\xi|^2 - 1 + i\epsilon} (F(.,\tau))\hat{}(\xi) d\xi |||^2 d\tau)^{1/2}.$$

Theorem 2 and Plancherel give

$$\left(\int_{-\infty}^{\infty} \| \sup_t V \|_{2,p} \int_{\mathbf{R}^n} |(F(x,.))\hat{}(\tau)|^2 (\sup_t V(x,t))^{-1} dx d\tau \right)^{1/2}$$

((2.12))

$$= \left(\int_{\mathbf{R}^n} \| \sup_t V \|_{2,p} \int_{-\infty}^{\infty} |F(x,t)|^2 dt (\sup_t V(x,t))^{-1} dx \right)^{1/2}$$
$$\leq \left(\int_{\mathbf{R}^n} \| \sup_t V \|_{2,p} \int_{-\infty}^{\infty} |F(x,t)|^2 V(x,t)^{-1} dt dx \right)^{1/2}$$

For v_2 similar argument work.

§3.

We are going to outline the proof of theorem 2. For the complete proof see **[RV3]**.

By homogeneity of the inequality we may reduce to the case $\tau = 1$.

Also, by dilation invariance we assume $x_0 = 0$.

Take

$$m(\xi) \equiv \frac{|\xi|^{1/2}}{|\xi|^2 - 1 + i\epsilon} = \sum_{j=0}^{\infty} a_j(\xi)\psi_j(\xi) + \psi_\infty(\xi),$$

where the functions $\psi_j, j = 1, \ldots$ are given by $\psi_j(\xi) = \psi(2^j|\xi|)$, for a cutoff function ψ supported on $\{t \in \mathbf{R} : 1/2 < t < 2\}$ and a_j is a symbol of zero order bounded by $C2^j$.

The terms $j = 0$ and $j = \infty$ can be bounded by using the results on fractional integrals and Hölder inequality (see **[FP]**,**[ChF]**).

A partition of unity in the Fourier transform side, together with the invariance by rotations, allow us to assume that

(3.1) $$supp\hat{f} \subset \{(\xi_1, \xi') \in \mathbf{R} \times \mathbf{R}^{n-1} : |\xi'| < 1/4\xi_1\}.$$

By taking $\delta = 2^j$ everything is reduced to the following

Lemma 3.1 *For δ a possitive number , $m(\xi)$ a C_0^∞ function supported on $\{\xi : 1-\delta < |\xi| < 1+\delta\}$ and $V \in \mathbf{L}^{2,p}$, then there exists a possitive η and a constant C, such that the following inequalities hold for any function satisfying (3.1):*

$$(3.2) \quad for\, p = (n-1)/2, \||\int_{\mathbf{R}^n} e^{ix.\xi} m(\xi)\hat{f}(\xi)d\xi\|| \le C\delta|log\delta|(\|V\|_{2,p})^{1/2}\|f\|_{L^2(V^{-1}dx)}.$$

$$(3.3) \quad for\, p > (n-1)/2, \||\int_{\mathbf{R}^n} e^{ix.\xi} m(\xi)\hat{f}(\xi)d\xi\|| \le C\delta^{1+\eta}(\|V\|_{2,p})^{1/2}\|f\|_{L^2(V^{-1}dx)}.$$

Proof: Let us denote $x = (x_1, x') \in \mathbf{R}^n, x' \in \mathbf{R}^{n-1}$.

By using Plancherel in x' we have

$$\int_0^R \int_{\mathbf{R}^{n-1}} |\int_{\mathbf{R}^n} e^{ix'.\xi' + ix_1\xi_1} m(\xi)\hat{f}(\xi)d\xi|^2 dx_1 dx'$$

$$= \int_0^R \int_{\mathbf{R}^{n-1}} |\int_{\mathbf{R}} e^{ix_1\xi_1} m(\xi)\hat{f}(\xi)d\xi_1|^2 d\xi_1' dx_1.$$

Define

$$(T_{x_1} f)(\xi') = \int_{\mathbf{R}} e^{ix_1\xi_1} m(\xi)\hat{f}(\xi)d\xi_1.$$

We prove that $T_{x_1} : L^2(V^{-1}dx) \to L^2(d\xi')$, and

$$(3.4) \qquad \|T_{x_1}\|_{L^2(V^{-1}dx)\to L^2(d\xi')} \le C\delta^{\eta+1}\|V\|_{2,p}, \text{ if } p > (n-1)/2,$$

$$(3.5) \qquad \|T_{x_1}\|_{L^2(V^{-1}dx)\to L^2(d\xi')} \le C\delta|log\delta|\|V\|_{2,p}, \text{ if } p = (n-1)/2,$$

with C independent of x_1.

$$(3.6) \qquad \int |T_{x_1} f(\xi')|^2 d\xi' = \int T_{x_1} f(\xi')\bar{T}_{x_1} f(\xi')d\xi' = \int_{\mathbf{R}^n} f(y)Q_{x_1}(y)dy,$$

where

$$Q_{x_1} f(y) \equiv T_{x_1}^*[\bar{T}_{x_1} f(\xi')](y)$$

$$(3.7) \qquad = \int e^{ix_1\eta_1 - iy_1\eta_1 - iy'.\xi'} m(\eta_1, \xi') \int\int_{\mathbf{R}} e^{ix_1\xi_1} m(\xi)\hat{f}(\xi)d\xi_1 d\eta_1 d\xi'.$$

Use Hölder inequality in (3.6) and obtain

$$\int |T_{x_1} f(\xi')|^2 d\xi' \le \left(\int |f(y)|^2 V^{-1}(y)dy\right)^{1/2} \left(\int |Q_{x_1} f|^2 V(y)dy\right)^{1/2},$$

hence it suffices to prove

$$\|Q_{x_1}f\|_{L^2(V(y)dy)} \leq C\|V\|_{2,p}\|f\|_{L^2(V^{-1}dy)}\delta^{2\eta+2}, \text{ for } p > (n-1)/2, \tag{3.8}$$

$$\leq C\|V\|_{2,p}\|f\|_{L^2(V^{-1}dy)}\delta^2|log\delta|, \text{ for } p = (n-1)/2, \tag{3.9}$$

We may write

$$Q_{x_1}f(y_1,y') = \int P(x_1-y_1,\xi_1,\xi')\hat{f}(\xi_1,\xi')e^{-iy_1\xi_1-iy'.\xi'}d\xi_1 d\xi',$$

where

$$P(t_1,\xi_1,\xi') = \bar{m}(\xi_1,\xi')e^{-it_1\xi_1}\int e^{it_1\eta_1}m(\eta_1,\xi')d\eta_1.$$

From the support propperties of m , we may write, for a function ϕ supported in $[-2,2]$ and identically 1 in $[-1,1]$

$$\begin{aligned} P(x_1-y_1,\xi_1,\xi') &= \bar{m}(\xi_1,\xi')\int \phi(\frac{\zeta_1}{2\delta})e^{i(x_1-y_1)\zeta_1}m(\zeta_1+\xi_1,\xi')d\zeta_1 \\ &= \bar{m}(\xi_1,\xi')\big(m((.)+\xi_1,\xi')\phi(\frac{(.)}{2\delta})\big)\hat{}(x_1-y_1) \\ &= C\bar{m}(\xi_1,\xi')\Big(e^{i\xi_1 z_1}(m((.),\xi'))\hat{}(z_1) *_{z_1} \delta\hat{\phi}(\delta z_1)\Big)(x_1-y_1), \end{aligned} \tag{3.10}$$

Now we take an appropriate decomposition of Q_{x_1}, and use real interpolation in each piece to obtain (3.9) and (3.8), in a similar way as we did in **[RV]**.

§4.PROOF OF THEOREM 3

We need the following

Proposition 4.1 *Let u be a solution of the Helmholtz equation*

$$\Delta u + (\tau + i\epsilon)u = f \quad \epsilon > 0. \tag{4.1}$$

and $V(x)$ non negative in the class $\mathbf{L}^{2,p}$, with $p > (n-1)/2$. Then

$$\|u\|_{L^2(Vdx)} \leq C\|V\|_{2,p}\|f\|_{L^2(V^{-1}dx)}$$

proof See **[CS]**, **[ChR]**.

Proposition 4.2*Let u be a solution of*

$$\begin{cases} i\partial_t u + \Delta u &= F \\ u(x,0) &= 0. \end{cases} \tag{4.2}$$

and $V(x,t)$ such that $\sup_{t\in[0,T"]} V(x,t) \in \mathbf{L}^{2,p}, p > (n-1)/2$ Then

$$\|u\|_{L^2(\mathbf{R}^n\times[0,T],Vdxdt)} \leq C\|\sup_{t\in[0,T]} V\|_{2,p}\|F\|_{L^2(\mathbf{R}^n\times[0,T],V^{-1}dxdt)}. \tag{4.3}$$

Proof:

We use the representation formula lemma 2.5. R_1 is bounded by (2.4) and (2.4*). The boundedness of the main term v_1 follows, as in the proof of theorem 1, by proposition 4.1

End of proof of theorem 3:

We must establish the solvability of (1.6), we make use of Duhamels formula,

$$u = e^{it\Delta}u_0 + i\int_0^t (e^{i(t-s)\Delta}V(.,s)u(.,s))(x)ds. \tag{4.4}$$

Define the operator $\mathcal{T}$ and the space of functions X_T by

$$\mathcal{T}F(x,t) = i\int_0^t (e^{i(t-s)\Delta}V(.,s)F(.,s))(x)ds, \tag{4.5}$$

$$X_T = \{F : \|F\|_{X_T} = \max(\|F\|_{L^2(\mathbf{R}^n\times[0,T],|V_1|dxdt)}, \sup_{|t|<T}\|F(.,t)\|_{L^2}) < \infty\}.$$

In order to establish the solvability of (1.6) it will be sufficient to prove that $e^{it\Delta}u_0 \in X_T$ provided that $u_0 \in L_2(\mathbf{R}^n)$ and to find and inverse in X_T of $(I - \mathcal{T})$. The bound of $e^{it\Delta}u_0$ is a consequence of a version for finite t-intervals of (2.4) and the fact that the $\| \ \|_2$ is preserved.

Now take $F \in X_T$. Use proposition 4.2 and 2.4 in (4.5) to obtain,

$$\begin{aligned}
&\|\mathcal{T}(F)\|_{L^2(\mathbf{R}^n\times[0,T],|V_1|dxdt)} \\
&\le C\|\sup_{t\in[0,T]}|V_1|\|_{2,p}\|V_1F\|_{L^2(\mathbf{R}^n\times[0,T],|V_1|^{-1}dxdt)} \\
&+ \|\sup_{t\in[0,T]}|V_1|\|_{2,p}^{1/2}\int_{-T}^{T}\|V_2(.,s)F(.,s)\|_{L^2(\mathbf{R}^n)}ds \\
&\le C\|\sup_{t\in[0,T]}|V_1|\|_{2,p}\|F\|_{L^2(\mathbf{R}^n\times[0,T],|V_1|dxdt)} \\
&+ \|\sup_{t\in[0,T]}|V_1|\|_{2,p}^{1/2}CT^{1/r'}\|V_2\|_{L^r([-T,T]:L_x^\infty)}\sup_{|s|<T}\|F(.,s)\|_{L^2(\mathbf{R}^n)} \\
&\le C(\|\sup_{t\in[0,T]}|V_1|\|_{2,p} + \|\sup_{t\in[0,T]}|V_1|\|_{2,p}^{1/2}T^{1/r'}\|V_2\|_{L^r([-T,T]:L_x^\infty)})\|F\|_{X_T},
\end{aligned} \tag{4.6}$$

Now using (2.4*) and that the $\| \ \|_2$ of the free propagator is preserved we have

$$\begin{aligned}
&\sup_{|t|\le T}\|\mathcal{T}F(.,t)\|_{L^2(\mathbf{R}^n)} \\
&\le C\|\sup_{t\in[0,T]}|V_1|\|_{2,p}^{1/2}\|V_1F\|_{L^2(\mathbf{R}^n\times[0,T],V_1^{-1}dxdt)} + \int_{-T}^{T}\|V_2(.,s)F(.,s)\|_{L^2(\mathbf{R}^n)}ds \\
&\le C(\|\sup_{t\in[0,T]}|V_1|\|_{2,p}^{1/2} + T^{1/r'}\|V_2\|_{L^r([-T,T]:L_x^\infty)})\|F\|_{X_T}.
\end{aligned}$$

Hence choosing $\| \sup_{t\in[0,T]} |V_1| \|_{2,p}$ and T small enough we conclude that the operator norm of $\mathcal{T}$ is less than one. Hence $(I - \mathcal{T})$ has an inverse. Repeating this procedure we establish the solvability of (4.4) for T arbitrarily large.

Now we are prepared to obtain the desired bound (1.8) for $|||D_x^{1/2}u|||$. Using in Duhamel's formula (4.4), (2.9) and theorem 1, we have

$$
\begin{aligned}
&|||D_x^{1/2}u|||_T \\
&\leq \|u_0\|_{L^2} + C\| \sup_{t\in[0,T]} |V_1| \|_{2,p}^{1/2} \|V_1 u\|_{L^2(\mathbf{R}^n\times[0,T],|V_1|^{-1}dxdt)} + \int_{-T}^{T} \|V_2(.,s)u(.,s)\|_{L^2(\mathbf{R}^n)} ds \\
&\leq \|u_0\|_{L^2} + C\| \sup_{t\in[0,T]} V_1 \|_{2,p}^{1/2} \|u\|_{L^2(\mathbf{R}^n\times[0,T],|V_1|dxdt)} \\
&+ CT^{1/r'} \|V_2\|_{L^r([-T,T]:L_x^\infty)} \sup_s \|u(.,s)\|_{L^2(\mathbf{R}^n)} \\
&\leq C(T)\|u\|_{X_T} \\
&\leq C\|u_o\|_{L^2(\mathbf{R}^n)}.
\end{aligned}
$$

The proof is over.

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