JOURNÉES ÉQUATIONS AUX DÉRIVÉES PARTIELLES

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Journées Équations aux dérivées partielles (1991), p. 1-10

<http://www.numdam.org/item?id=JEDP_1991____A4_0>

On Large Amplitude Solutions of the Yang-Mills Equations in Minkowski Space-Time

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June 2, 1991

Abstract

We consider Yang-Mills fields in Minkowski space-time. We present a survey of results about the global existence and asymptotic properties of large-amplitude solutions. We point out recent results concerning the time decay of spherically symmetric solutions in the large amplitude sector. Some open problems are also discussed .

1 Introduction

In this paper we discuss the role of the large-amplitude solutions of the classical Yang-Mills equations in Minkowski space-time. The aim is to give an updated account of the known results concerning the global existence and asymptotic behavior of these solutions.

We shall consider Minkowski space-time R^{3+1} with coordinates $(t, x) = (x^0, ..., x^3)$ and endowed with the flat metric $\eta = -dt^2 + dr^2 + r^2 d\Omega_{S^2}$. We use Einstein 's convention of raising and lowering indices thoroughly. The gauge group is a Lie group G and we denote its Lie algebra by $\mathcal{G}$ and the Lie algebra commutator by [.,.]. The gauge group G is asssumed to be compact and semi-simple. In particular , the Lie algebra $\mathcal{G}$ admits a Killing form, namely a bilinear symmetric positive definite form that is invariant under the Ad action . In the sequel we will often write '.' for this bilinear form. Also, we fix a basis T_a,a=1,2,...,N of $\mathcal{G}$ which is orthonormal with respect the Killing form.

*To appear in Seminaire de L'Ecole Politechnique , Proceedings of Colloques de Equations aux Derivees Partielles, Saint-Jean-de-Monts , jun 1991. This work has been supported by Sonderforschungsbereich 256 of the Deutsche Forschungsgemeinschaft.

The Yang-Mills equations in Minkowski Space-Time R^{3+1} for a $\mathcal{G}$-valued Yang-Mills potential $A : R^{3+1} \longrightarrow \Lambda^1 \mathcal{G}$ are:

$$F_{\alpha\beta}^{\ \ ;\beta} = 0 \tag{1.1}$$
$$^{*}F_{\alpha\beta}^{\ \ ;\beta} = 0 \tag{1.2}$$

Here $A = A_\mu dx^\mu = (A_\mu^a T_a)dx^\mu$ and F denotes the Yang-Mills curvature of A, namely the 2-form $F_A : R^{3+1} \longrightarrow \Lambda^2 \mathcal{G}$ defined by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$. D denotes the covariant derivative $D_\mu = \partial_\mu + [A_\mu, .]$ and we will also refer to it by the use of semicollon ";". The tensor $^*F$ denotes the Hodge dual of F.

There has been a lot of research on the associated euclidean version of the system . In that case the Yang-Mills equations become a non-linear elliptic system. Here the underlying metric has Lorentz signature and the equations are hyperbolic. We look for the dynamical developments of initial data defined on a space-like submanifold. The Yang-Mills equations allow an infinite-dimensional group of transformations known as gauge transformations and are degenerate. After fixing the gauge system 1.1-1.2 becomes a non-linear hyperbolic system of partial differential equations containing quadractic non-linearities. In three space dimensions this leads to the development of singularities unless a certain algebraic condition, called null condition, is satisfied by the non-linearities.

The Yang-Mills equations are supposed to be a model for the fundamental par ticles that form matter. The main interest in such system is that one believes that classical solutions can help explain the full quantum theory , at least in the semi-classical limit, and that many of the quantum features would survive in the classical level. The importance lies in the fact that this leads to predictions that cannot be reached by standard perturbative approaches like for example, the problem of color confinement . Here one deals with configurations which , due to the presence of Yang-Mills color charges , strongly interact in the infrared region. This means that in this region a significant role will be played by field oscilations of large amplitude, for which the non-linear character of the Yang-Mills equations is dominant. Thus, the problem of color confinenment is closely connected with the quantization of large-amplitude oscillations (see [4] for more details). From the mathematical point of view one asks if the solutions to the Yang-Mills equations exist globally in time and investigates their asymptotic behavior. The question of global existence has been solved already and one obtains global solutions in H^s (cf. [3]) but no information about the asymptotic behavior of its solutions could be inferred from the proof. The advantadge of the method is that it allows general configurations admitting Coulomb charges .

It was proved later in [1] (see also [2]) the existence of global large solutions together with the characterization of the asymptotic behavior in time. The proof relies on the use of conformal compactification method of Penrose and a careful analysis of the conformally-invariant Yang-Mills equations on the so-called Einstein Cylinder $[-\pi/2, \pi/2] \times S^3$. When translated back to Minkowski space the conditions on the initial data imply a fall-off at space-like infinity which amounts to $H^{3,2}$ regularity . Here $H^{s,\delta}$ denotes the weighted Sobolev spaces $H^{s,\delta}$:

$$||F||_{s,\delta} = (\sum_{i=0}^{s} \int \sigma^{2\delta+2i} |D^i F(x)|^2 dx)^{1/2}$$

where $\sigma^2 = 1 + |x|^2$. The major drawback though is the strong fall-off rate of the Cauchy data, requiring for example that the electric field decays like $E(0,x) = O(|x|^{-4})$ as $|x| \to +\infty$. This excludes configurations containing Coulomb charges , dipoles and quadrupoles.

Other results , which correspond to configurations decaying as slowly as to allow dipoles, require spherical symmetry requirements. We shall review the results in the next section. æ

2 The Spherically Symmetric Case

A remarkable feature of the Yang-Mills equations is that they admit a large class of non-trivial spherically symmetric solutions. By this we mean invariance under the combined effect of a rotation and a compensating gauge transformation. Remark that the latter is not the case of classical electrodynamics so that the existence of such solutions is strictly tied to the non-abelian character of the theory. In fact these solutions present a remarkable non-trivial mixing of internal and external degrees of freedom. For the gauge group $SU(2)$ one can write down the solution very easily. In the so-called canonical gauge the solution is:

$$\begin{aligned} A_0^a &= \phi \frac{x^a}{r} \\ A_i^a &= \psi \frac{x_i x^a}{r^2} + \frac{f_1}{r}(\delta_i^a - \frac{x_i x^a}{r^2}) + \frac{f_2 - 1}{r} \epsilon^{iab} \frac{x^b}{r} \end{aligned}$$

where ϕ,ψ,f_1 and f_2 are functions of t and r. This class is called the class of canonical gauges. Its elements are completely regular except at the central line r=0.

The Yang-Mills equations written in terms of the constitutive functions of the Ansatz becomes a very complicated system of equations: [1]

$$\begin{aligned} \Box_{(1)} f_1 - (\dot{\phi f_1}) - \phi(\dot{f_1} + f_2\phi) - \frac{1}{r^2} f_1(1 - f_1^2 - f_2^2) &= 0 \\ \Box_{(1)} f_2 + (\dot{\phi f_2}) + \phi(\dot{f_2} - f_1\phi) - \frac{1}{r^2} f_2(1 - f_1^2 - f_2^2) &= 0 \\ r^2 \dot{\phi}' + 2(f_1 f_2' - f_2 f_1') &= 0 \end{aligned}$$

[1] We denote by $\Box_{(1)} = \partial_t^2 - \partial_r^2$ the D'Alembertian in 1-d , by $\Delta_{(3)}$ the standard Laplace operator in 3-d and by $\Box_{(3)}$ the D'Alembertian in 3-d.

$$\Delta_{(3)}\phi - \frac{2\phi}{r^2}(f_1^2 + f_2^2) + \frac{2}{r^2}(f_1\dot{f}_2 - f_2\dot{f}_1) \quad = \quad 0$$

One would like to study such systems of equations. Not much is know in this important sector of non-abelian gauge theory except for a result due to Glassey and Strauss [5] . They consider in their paper special solutions which, after using all gauge degrees of freedom, have the form:

$$A_0^a = 0$$
$$A_i^a = \alpha(t,r)\epsilon^{iab}\frac{x^b}{r}$$

The Yang-Mills equations reduce then to a single scalar wave equation for α:

$$\Box_{(3)}\alpha + \frac{2}{r^2}\alpha - \frac{3}{r}\alpha^2 + \alpha^3 = 0$$

The equation presents a singularity along the central line r=0 and much of the analysis contained in [5] concerns this problem. They also obtain time-decay estimates for configurations allowing dipoles. The main defficiency though is that it adresses a small class of solutions and rely heavily in the special form of the Ansatz.

What we would like to show now is how it is possible to extend the result by Glassey and Strauss so that the entire spherical symmetric sector of the Yang-Mills theory is covered. One observes again that the field equations cannot be significantly reduced then and a more elaborate geometric analysis is needed. Before we proceed we would like to recall some facts concerning the notion of symmetry in gauge theories (see [8] and references therein for details).

Definition 2.1 *Consider the principal bundle $E = R^{3+1} \times G$ and an action $SO(3) \times E \longrightarrow E$ of the rotation group. If ω is a connection 1-form on E then we say that ω is spherically symmetric iff $s^*\omega = \omega$ for every element s in SO(3), where s^* is the pull-back induced by the bundle automorphism $s. : x \mapsto s.x$*

In terms of the coordinates of the base space, this amounts to the fact that ,after the application of the symmetry generator, the connection can be brought to the original form by means of a compensating gauge transformation.

Consider now the canonical action of SO(3) on the base space R^{3+1}. The problem one encounters is that there is no canonical procedure to uniquely lift the $SO(3)$-action on Minkowski space to the whole of E. It can proved that all possible lifts of the action will be in correspondence with homomorphisms $\lambda : SU(2) \longrightarrow G$. One says that this mapping determines the type of spherical symmetry of the gauge field F_A . Degenerate cases will correspond to configurations which are either reducible to abelian $U(1)$-gauge fields or to classical configurations for which no compensating gauge transformation is required. This is the case when λ is the trivial homomorphism. The non-abelian configurations described here correspond to the case when λ is an embedding of the rotation group into the gauge group. This can only happen of course when G admits an SU(2) subgroup. The symmetry definition can be unravelled to produce an explicit Ansatz for the gauge potentials. The usual construction displays the potentials in the so-called abelian gauge. Despite its structural advantadges, the abelian gauge is unfortunately a singular gauge. Besides the obvious problem at the origin, the potential has on this gauge string singularities . For the sake of the global existence argument one needs a gauge in which the potentials have good space-regularity . In particular, one must be assured that there exists a gauge in which the string-singularities disappear. This gauge is called in the monopole literature the canonical or the no-string gauge. The existence of the canonical gauge is tied to the existence of su(2)- subalgebras. It can be written down as follows:

$$A_0 = \sum_{l=1}^{N} \phi_l(t,r)\wp_l \tag{2.1}$$

$$\begin{aligned} A_i &= \sum_{l=1}^{N} \psi_l(t,r)\wp_l \\ &+ \frac{1}{r}[\sum_{l=1}^{N}(a_{1l}[T_m,\wp_l]\epsilon_{mjn}\frac{x_n}{r} + a_{2l}[T_j,\wp_l]) - T_j]\epsilon_{ijk}\frac{x_k}{r} \end{aligned} \tag{2.2}$$

Here a_{1l} and a_{2l} are functions of t,r alone , $T_i = \lambda_*(O_{(i)})$ with $\lambda : SU(2) \to G$ a group homomorphism defining the spherical symmetry type of the Yang-Mills potential and $\wp_l$ is defined in terms of su(2) representation matrices :

$$\wp_j = \sum_{m=-l_j}^{m=l_j} Y^{l_j}_{-m}(\frac{x}{r})\mathcal{Y}^{l_j}_m$$

The functions $Y^{l_j}_{-m}$ are the standard spherical harmonic functions on the sphere and $\mathcal{Y}^l_m$ are a basis for an su(2) representation of dimension $2l+1$ labeled by the third eigenvalue. [2]

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[2]i.e. the matrices $\mathcal{Y}^l_m$ are defined by:

1.
$$[L_3, \mathcal{Y}^l_m] = -im\mathcal{Y}^l_m$$

2.
$$\sum_{k=1}^{3}[L_k,[L_k,\mathcal{Y}^l_m]] = -l(l+1)\mathcal{Y}^l_m$$

There are some subtleties associated with this kind of Ansatz. Observe that the solution is everywhere regular except at the origin $r = 0$, the question of regularity at the central line becoming therefore very important. It will actually follow from our theorem that the solution will be bounded near $r = 0$.

Despite its good regularity properties the Ansatz 2.1-2.2 is not adequate for the point of view of partial differential equations. One needs a characterization of the invariance condition in terms of gauge-invariant differential operators. By commuting these operators with the equations of motion one can set-up energy estimates that will lead to the preservation of the Ansatz by the non-linear flow. Define the operators:

$$\hat{\mathcal{L}}_{O_{(i)}} = \mathcal{L}_{O_{(i)}} + [T_i, .]$$

It can be proved that the symmetry condition is equivalent then to:

$$\mathcal{L}_{O_{(i)}} F + [T_i, F] = 0$$

The operators satisfy the commutation rules $[\hat{\mathcal{L}}_{O_{(i)}}, \hat{\mathcal{L}}_{O_{(j)}}] - \epsilon_{ijk}\hat{\mathcal{L}}_{O_{(k)}} = 0$ when applied to the spherical symmetric configurations. Physically, $\mathcal{L}_{O_{(i)}}$ measures the orbital angular momentum, while $[T_i, .]$ measures the isospin contribution to the total angular momentum.

Finally, we remark that the Yang-Mills equations contain quadratic terms. In three space dimensions this kind of terms could lead to singularities, unless a certain algebraic condition, called the null condition, is present. On this case this condition is satisfied and is a consequence of the tensorial covariant nature of the equations .

Our main result ([6]) (in colaboration with V. Georgiev) consists of:

THEOREM *Let (E(0),A(0)) initial data for the Yang-Mills equations satisfying the constraint equations and the spherically symmetric Ansatz 2.1-2.2. Assume that the conformal energy:*

$$E_0 = \int_{R^3} (1 + |x|^2)|F(0, x)|^2 dx \tag{2.3}$$

is finite $E_0 < +\infty$ and the initial data satisfy the estimate:

$$|F(0, .)|_1 := sup_{|\alpha| \leq 1}(sup_x(1 + |x|)^{5/2+|\alpha|}|\partial_x^\alpha F(0, x)|) < +\infty \tag{2.4}$$

It follows that the solution exists globally and satisfies the decay rate:

$$|F(t, .)|_\infty \leq C_0(1 + t)^{-1} \tag{2.5}$$

with C_0 depending only on the conformal energy E_0 and the norms $|F(0, .)|_1$.

Remarks :

1. We assume that the initial data satisfies the constraints $Div_A E = 0$, $Div_A H = 0$. These equations are preserved by the non-linear flow.

2. The global existence part of the theorem follows from [3]. One has only to remark that the Yang-Mills flow preserves the canonical class of potentials 2.1-2.2 (see [8] for details).

3. Configurations containing Coulomb charges cannot be acomodated in the hyphoteses of this theorem.

The proof is long and will not be presented here. We shall give some ideas though. The decay estimates are obtained by a careful decomposition of Minkowski Space-Time into two different regions. The first part consists of the exterior of a small cone of aperture ϵ around the central line and includes the wave zone , where we must exploit the null condition in the non-linearities. For that matter we use the representation of the Yang-Mills equations in light-cone coordinates (see [8]):

$$\begin{aligned}
(D_4 + \frac{1}{r})\underline{\alpha}_A + \not{D}_A\rho - \epsilon_{AB}\,\not{D}_B\sigma &= 0 && (2.6)\\
(D_3 - \frac{1}{r})\alpha_A - \not{D}_A\rho - \epsilon_{AB}\,\not{D}_B\sigma &= 0 && (2.7)\\
(D_4 + \frac{2}{r})\rho + \not{D}\cdot\alpha &= 0 && (2.8)\\
(D_3 - \frac{2}{r})\rho - \not{D}\cdot\underline{\alpha} &= 0 && (2.9)\\
(D_4 + \frac{2}{r})\sigma + \not{D}\times\alpha &= 0 && (2.10)\\
(D_3 - \frac{2}{r})\sigma + \not{D}\times\underline{\alpha} &= 0 && (2.11)
\end{aligned}$$

In these equations $\{\underline{\alpha}_A, \alpha_A, \rho, \sigma\}$ are the null components of the Yang-Mills curvature tensor, D_3 and D_4 denote the derivatives along the null directions and $\not{D}$ are angular derivatives. These can be estimated by using the spherically symmetric Ansatz 2.1- 2.2. One arrives then at:

Proposition 2.1 *There exists a constant $C = cE_0^{1/4}(1 + cE_0^{1/2}|F(0)|_1)^{1/2} + |F(0)|_1$ depending only on the initial data such that the following estimates for the null components $\{\underline{\alpha}_A, \alpha_A, \rho, \sigma\}$ of the Yang-Mills curvature tensor F are verified for all points (t,x) in the exterior region $|x| \geq 1$, $|x| + 1 \geq \epsilon t$:*

$$\begin{aligned}
|\rho(t,x)| &\leq C|x|^{-2} && (2.12)\\
|\sigma(t,x)| &\leq C|x|^{-2} && (2.13)\\
|\alpha(t,x)| &\leq C(1 + t_0 + |x_0|)^{-1}|x_0|^{-3/2} && (2.14)\\
|\underline{\alpha}(t,x)| &\leq C|x|^{-1}(1 + |t - |x||)^{-1}(1 + [F(t)]_\infty) && (2.15)
\end{aligned}$$

where

$$[F(t)]_\infty = sup_{\Sigma_i}(1+|x|)(1+\tau+|x|)|F(\tau,x)| \tag{2.16}$$

measures the curvature tensor F in the interior region $1+|x| \leq \epsilon t$.

These estimates are obtained by integration along the characteristic curves and by using the energy bounds to obtain a Gronwall exponent lower than one in the bootstrapping inequalities. Different components must be integrated along different characteristic directions. The essential difficulty here is that the component $\underline{\alpha}$ must be integrated past the central line $r = 0$ where the field F has not yet been estimated. This is expressed by the appearance of the interior norm for F in the estimates of $\underline{\alpha}$. The interior norm is is controlled in turn by looking to the fundamental solution of the wave operator. By straightforward differentiation we get from equations 1.1-1.2 :

$$\Box_A F_{\alpha\beta} = 2[F^{\gamma}_{\alpha}, F_{\gamma\beta}] \tag{2.17}$$

where $\Box_A = -D_\lambda D^\lambda$ denotes the wave operator relative to the Yang-Mills potential A. Writing 2.17 explicitly in terms of the gauge potential , we find:

$$\begin{aligned} \Box F_{\alpha\beta} &= -2\partial_\gamma([A^\gamma, F_{\alpha\beta}]) + [\partial_\gamma A_\gamma, F_{\alpha\beta}] \\ &- [A^\gamma, [A_\gamma, F_{\alpha\beta}]] + 2[F^{\gamma}_{\alpha}, F_{\gamma\beta}] \end{aligned} \tag{2.18}$$

Let us consider now a fixed point $p = (t_0, x_0)$ with $|x_0| + 1 \leq \epsilon t_0$ and $t_0 \geq \frac{2}{\epsilon}$. Using the fundamental solution representation of the wave equation we may write (cf. [3]):

$$\begin{aligned} F_{\alpha\beta}(p) &= F^{LIN}_{\alpha\beta}(p) - \frac{1}{4\pi}\int_{K_p} r dr d\omega(-2\partial_\gamma([A^\gamma, F_{\alpha\beta}]) \\ &+ [\partial_\gamma A_\gamma, F_{\alpha\beta}] - [A^\gamma, [A_\gamma, F_{\alpha\beta}]] + 2[F^{\gamma}_{\alpha}, F_{\gamma\beta}]) \\ &:= F^{LIN}_{\alpha\beta}(p) - \frac{1}{4\pi}(I_1 + I_2 + I_3 + I_4) \end{aligned} \tag{2.19}$$

where $F^{LIN}_{\alpha\beta}(p)$ is the solution of the wave equation $\Box F^{LIN}_{\alpha\beta} = 0$ with the same initial data as $F_{\alpha\beta}$. and $I_1, ..., I_4$ denote the terms containing the non-linearities. The linear term is bound trivially. In order to estimate the non-linear terms I_1, I_2, I_3, I_4 in the right-hand side of equation 2.19 one resorts to using a local light-cone reference system and the conformal energy measured in the cone K_p. Here, a fundamental role is played by the so-called Cronström gauge adapted to the point p, that is to say:

$$(t - t_0)A_0(t,x) + \sum_{j=1}^{3}(x^j - x^j_0)A_j(t,x) = 0 \tag{2.20}$$

This allows us to integrate by parts in the cone K_p and stop the loss of derivatives. The integrals I_1, I_2 and I_3 are estimated in this gauge. The key property is that on this gauge one can represent the potentials A in terms of the curvature as:

$$\begin{aligned} A_\nu(x) &= \int_0^1 \lambda d\lambda(^{(p)}x^3 F_{3\nu}(p+\lambda(x-p))) \\ &= \int_0^1 \lambda d\lambda r F_{3\nu}(t_0-\lambda r, x_0+\lambda r\omega) \\ \partial^\nu A_\nu(x) &= \int_0^1 \lambda^2 d\lambda^{(p)}x^3[F_{3\nu}(p+\lambda(x-p)), A^\nu(p+\lambda(x-p)) \\ &= \int_0^1 \lambda^2 d\lambda r[F_{3\nu}, A^\nu](t_0-\lambda r, x_0+\lambda r\omega) \end{aligned}$$

for all x in the cone K_p (see [3]). To estimate the last term one appeals again to the conformal energy and to the estimates in the exterior region. Here one exploits the null condition which says that in the estimation of the commutator terms one can always take one factor in the L^2-norm. This completes the estimate of the curvature tensor near the central line which combined with the estimates in the exterior region conclude the proof of the main theorem.

3 Final Remarks

We would like to finish the lecture by remarking that none of the results shown here describe the asymptotic behavior of Yang-Mills charges. The fact that one insists to use the conformal energy to get a strong a priori estimate is the source of the problem. A proper understanding of the meaning of such charges should lead to a way to subtract the infrared fields. Another problem consists of the heavy use of the spherically symmetric Ansatz. One would like to extend these results to the general case. Both problems are still open and represent an obstacle to the solution of the more difficult problem of the interaction of monopoles.

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References

[1] *The Yang-Mills Equations on the Universal Cosmos* , Y. Choquet-Bruhat , S. Paneitz , I. Segal, J. Functional Analysis **53** ,p. 112 (1983)

[2] *Lectures on Global Solutions of Hyperbolic Equations of Gauge Theories* , Y. Choquet-Bruhat , Proceedings of Symposium at Simon Bolivar University, Caracas , Ed. C. Aragone.

[3] *The Global Existence of Yang-Mills-Higgs Fields in 4-dimensional Minkowski Space* , D. Eardley and V. Moncrief , Comm. Math. Phys. **83** , p. 171 (1982)

[4] *Quantization of Non-abelian Gauge Theories* , V.N. Gribov, Nuclear Phys. B **139**,1,(1978)

[5] *Decay of Classical Yang-Mills Fields*, R.T. Glassey, W.A. Strauss, Commun. Math. Phys. **65**,1, (1979)

[6] *The Asymptotic Behavior of Yang-Mills Fields in the Large* , V. Georgiev and P.P. Schirmer, preprint, (1991).

[7] *Vortices and Monopoles* , Arthur Jaffe , Clifford Taubes, Birkhäuser Progress in Physics PPh2, Birkhäuser 1980

[8] *Global Existence for Spherically Symmetric Yang-Mills Fields on 3+1 Space-Time Dimensions*, P.P. Schirmer, Doctoral dissertation, New York University, 1990.