JOURNÉES ÉQUATIONS AUX DÉRIVÉES PARTIELLES

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Journées Équations aux dérivées partielles (1989), p. 1-16
<http://www.numdam.org/item?id=JEDP_1989____A8_0>

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by

Waichiro MATSUMOTO and Hideo YAMAHARA

§0. Introduction, definitions and theorems.

On higher order scalar equations, the strong hyperbolicity is well characterized. (See O. A. Oleinik [13], V. Ja. Ivrii and V. M. Petkov [3], V. Ja. Ivrii [2], L. Hörmander [1], N. Iwasaki [4], [5], [6], etc.) On the other hand, on first order systems, if their coefficients are constant, we also have a complete result. (See K. Kasahara and M. Yamaguti [7]). In case of first order systems with variable coefficients, we have some results, but they are not satisfactory. (See, for example, N. D. Koutev and V. M. Petkov [8], T. Nishitani [10], [11], [12], H. Yamahara [14], [15] etc.).

In this note, we give some necessary conditions for the strong hyperbolicity of first order systems with variable coefficients, assuming that coefficients depend only on the time variable. This is a further developed results of H. Yamahara [14] and [15]. On the other hand, these become sufficient under a reasonable supplementary condition.

Let us consider the following Cauchy problem.

$$
(1)\qquad \begin{cases} P\,u \equiv (P_p - B)u \equiv \{D_t - \sum_{i=1}^{\ell} A_i(t,x)\, D_{x_i} - B(t,x)\}u = f(t,x), \\ \\ u(t_0,x) = u_0(x), \end{cases}
$$

where $u(t,x)$, $u_0(x)$, $f(t,x)$ are vectors of dimension N and $A_i(t,x)$, $B(t,x)$ are square matrices of order N with elements in $C^{\infty}(\Omega)$, (Ω is an open set in $\mathbb{R}^{1+\ell}_{t,x}$). We say that the Cauchy problem (1) is uniformly well-posed in Ω if the following holds :

$$(2)\qquad \begin{cases} \forall K = [T_1,T_2]\times K_0\ ,\ \forall K' \subset\subset K \\ \exists\omega : \text{a lens-shaped neighborhood of the origin,} \\ \forall (t_0,x_0)\in K',\ \forall u_0\in C^\infty(K_0),\ \forall f\in C^\infty(K),\ \exists\, !\, u \text{ solution of (1) in } (t_0,x_0)+\omega. \end{cases}$$

Proposition 0.1. *If* (1) *is uniformly well-posed in* Ω, *the following holds :*

$$(3)\qquad \begin{cases} \forall (\hat t_0,\hat x_0)\in K',\ \forall M\in\mathbb{N},\ \exists M'\in\mathbb{N},\ \exists\delta>0,\ \exists C>0 \\ \forall (t_0,x_0)\in K' \text{ s.t. } |t_0-\hat t_0|\leqslant\delta,\ \forall U_0\in C^M(K) \\ \forall f\in C^{M-1}(K),\ \exists u \textit{ solution of (1) in } K'',\ (K''=\{|t-t_0|\leqslant\delta\}\times\{|x-x_0|\leqslant\delta\}) \end{cases}$$

and u *satisfies*

$$(4)\qquad |u|_{M,K''}\leqslant C\{|u_0|_{M',K_0}+|f|_{M'-1,K}\},$$

where $|u|_{M,K} \overset{\text{def}}{=} \sum_{|\alpha|\leqslant M}\ \max_{(t,x)\in K} |D^\alpha_{t,x}\, u(t,x)|.$

By the estimate (4), we have the following theorem.

Theorem 0. *(P. D. Lax and S. Mizohata)*

If (1) *is uniformly well posed, all characteristic roots of* P_p *are real in* $\Omega\times\mathbb{R}^\ell_\xi\backslash 0$.

From now on, we always suppose the conclusion of the above theorem.

Definition 0.1. (Strong hyperbolicity)

We say that P_p is strongly hyperbolic when the Cauchy problem (1) of P_p+B *is* uniformly well posed in Ω for arbitrary choice of $B(t,x)$.

Throughout this note, we assume the following :

Assumption. A_i depends only on t, $(1\leqslant i\leqslant \ell)$.

Let $\{\lambda_j\}_{j=1}^d$ be the different characteristic roots of P_p at $t=t_0$ and $\xi=\xi_0\neq 0$.

We set

$$A^{(0)}=\Sigma_{i=1}^{\ell} A_i(t_0)\,\xi_{0i}\ ,$$

$A^{(1)} = \sum_{i=1}^{\ell} \frac{\partial}{\partial t} A_i(t_0)\, \xi_{0i}$,

$\mathcal{P}_j$: the projection to the generalized eigenspace of λ_j ,

$A_j^{(i)} = A^{(i)} \mathcal{P}_j$, $(0 \leqslant i \leqslant 1, 1 \leqslant j \leqslant d)$.

Theorem 1. *If* P_p *is strongly hyperbolic in* Ω, *the following holds*

(5) $\mathcal{P}_j(A_j^{(0)} - \lambda_j I_N)(A_j^{(1)})^k (A_j^{(0)} - \lambda_j I_N) = 0$

for $1 \leqslant j \leqslant d$ *and* $k \in \mathbf{Z}_+ = \{0,1,\ldots\}$.

<u>Remark.</u> Let m^j be the multiplicity of λ_j. At least for $k \geqslant m^j$, Condition (5) becomes trivial.

Corollary 2. *The lengths of Jordan chains of* $A^{(0)}$ *are at most* 2.

By virtue of Bronshtein-Mandai's theorem, the characteristic roots $\lambda^{(j)}(t)$ $(1 \leqslant j \leqslant N)$ of $P_p(t_i\, ; \xi_0)$ belong to C_t^∞. (See T. Mandai [17] and M. D. Bronshtein [16]). Let us set

$$\lambda_0^{(j)}(t) = \lambda^{(j)}(t_0) + (t - t_0) \frac{\partial}{\partial t} \lambda^{(j)}(t_0).$$

Theorem 3. *If* $\ell = 1$ *and* $\{\lambda_0^{(j)}(t)\}_{j=1}^N$ *are distinct for* $0 < |t - t_0| \leqslant {}^{\exists}\delta_0$, *condition (5) is suficient for the strong hyperbolicity of* P_p *near* t_0.

<u>Remark.</u> $\lambda_0^{(j)}(t)$ is obtained by $\sum_i (\frac{\partial}{\partial t})^k A_i(t_0)\xi_{0i}$ with $0 \leqslant k \leqslant 2$.

In the following sections 1, 2 and 3, we give a proof of Theorem 1 for k = 0 and 1. The proof of Theorem 1 for $k \geqslant 2$ and that of Theorem 3 will be given in the forthcoming paper [19].

§I. Reduction.

We may assume $t_0 = 0$. We take B as constant matrix and f = 0. Let us take Fourier image of (1) on the variable x ;

$$(1.1)\quad \begin{cases} \{D_t - \sum_{i=0}^{\ell} A_i(t)\xi_i - B(t)\}\hat{u} = 0 \\ \hat{u}(0,\xi) = \hat{u}_0(\xi). \end{cases}$$

Setting $\xi = n\xi_0$, we expand $\sum_{i=1}^{\ell} A_i(t)\xi_{0_i}$ as $A^{(0)} + t\,A^{(1)} + t^2\,A^2(t)$. Further, we transform $A^{(0)}$ to Jordan's normal form Λ :

$$\Lambda = \begin{pmatrix} \Lambda_1 & & \\ & \cdot & \\ & & \Lambda_d \end{pmatrix}, \qquad \Lambda_j = \lambda_j I_{m^j} + J^j ,$$

$$J^j = J^j(r_j,1) \oplus J^j(r_j,2) \oplus \ldots \oplus J^j(r_j, m^j_{r_j}) \oplus \ldots \oplus J^j(1, m^j_1),$$

$$J^j(k,h) = \begin{pmatrix} 0 & 1 & & \\ & \cdot & \cdot & \\ & & \cdot & 1 \\ & & & 0 \end{pmatrix}; \; k \times k, \; (1 \leq j \leq d).$$

Thus, we arrive at

$$(1.2)\quad \begin{cases} \{D_t - n(\Lambda + t\,\tilde{A}^{(1)} + t^2\,\tilde{A}^{(2)}) - \tilde{B}(t)\}\hat{u}_1 = 0, \\ \hat{u}_1(0) = \hat{u}_{10}. \end{cases}$$

Corresponding to Λ, we can transform (1.2) by the similar transformation by $N(t) = I + t\,N_1$:

$$(1.3)\quad \begin{cases} \hat{P}\,\hat{u} = \{D_t - n(\Lambda + t\,\tilde{\tilde{A}}^{(1)} + t^2\,\tilde{\tilde{A}}^{(2)}(t)) - \tilde{\tilde{B}}(t)\}\hat{u}_2 = 0, \\ \hat{u}_2(0) = \hat{u}_{x_0} , \end{cases}$$

where, decomposing in blocks $\tilde{A}^{(1)}$ and $\tilde{\tilde{A}}^{(1)}$ corresponding to Λ, say, $(\tilde{A}^{(1)}(j,j'))_{1\leq j,j'\leq d}$ and $(\tilde{\tilde{A}}^{(1)}(j,j'))_{1\leq j,j'\leq d}$, it holds that $\tilde{\tilde{A}}(j,j) = \tilde{A}(j,j)$ and $\tilde{\tilde{A}}(j,j') = 0$ for $j \neq j'$.

Ex.

$$\Lambda = \begin{matrix} \Lambda_1 & \\ & \Lambda_2 \end{matrix}, \qquad A = \begin{matrix} A(1,1) & A(1,2) \\ A(2,1) & A(2,2) \end{matrix}$$

As our consideration becomes independent of the part which has the factor $t^2 n$, from now on, we take out (j,j) block and omit the subscript "j". We may assume $\lambda = 0$. Further, we set $t = n^{-\sigma} s$ $(\sigma > 0)$. Thus, we arrive at

$$(1.4) \quad \begin{cases} P_0 v \equiv \{n^{\sigma} D_s - (n J + n^{1-\sigma} s A_1 + n^{1-2\sigma} s^2 A_2(s) + B\} v \\ \quad = 0, \\ v(0) = v_0 , \end{cases}$$

where v, v_0 are vectors of dimension m, J, A_1, A_2, B are square matrix of order m and

$J = J(r,1) \oplus \ldots \oplus J(r,m_r) \oplus J(r-1,1) \oplus \ldots \oplus J(r-1,m_{r-1}) \oplus J(r-2,1) \oplus \ldots \oplus J(1,m_1)$,

$\sum_{j-1}^{r} j\, m_j = m$.

Here, condition (5) for j in §0 is equivalent to

$$(1.5) \quad J(A_1)^k J = 0 \text{ for } k \in \mathbb{Z}_+ .$$

Proposition 1.1. *We assume that* (1) *is uniformly well posed in* Ω. *If, for* P_0 *in* (1.4), *there exists an invertible matrix* $N(s,n)$ *for* $0 < |s| \leq {}^{\exists}\delta$ *and* $\ell \geq 2$, $(\ell \in \mathbb{N})$ *such that*

$$\tilde{P} = N^{-1} L N = n^{\sigma} D_s - n^{\mu}(\tilde{J}(s) + \tilde{K}(s)) - n^{\mu'} C(s,n),$$

$\mu > \mu'$, $\mu > \sigma$, $C(s,n)$ *is bounded,*

$$\tilde{J} = \bigoplus_{\substack{1 \leq k \leq R \\ 1 \leq h \leq M_R}} \tilde{J}(k,h), \quad \tilde{J}(k,h) = \begin{pmatrix} 0 & a_1^{k,h} & & \\ & 0 & \ddots & \\ & & \ddots & a_{k-1}^{k,h} \\ & & & 0 \end{pmatrix},$$

$a_i^{k,h}$ *is not identically zero, and is analytic for* $s \neq 0$,

$\tilde{K} = (K(k,h,k',h'))_{\substack{1 \leq k,k' \leq R \\ 1 \leq h \leq M_k \\ 1 \leq h' \leq M_{k'}}}$: *block decomposed with respect to* $\tilde{J}$,

with

$$K(k,h,k',h')=\begin{pmatrix} \alpha_1^{k,h,k',h'} & & 0 \\ . & & 0 \\ \alpha_k^{k,h,k',h'} & & 0 \end{pmatrix} \quad ; \quad k\times k'$$

$$\alpha_i^{k,h,k',h'} = 0 \text{ for } i \neq 0 \bmod \ell,$$

then, we have the following ;

1) *If* $\ell \geqslant 3$, $\tilde{J}+\tilde{K}$ *is nilpotent.*

2) *Let* $\det(\lambda I-(\tilde{J}+\tilde{K}))$ *be* $\sum_{i=0}^{m} C_i(s)\,\lambda^{m-i}$.

If $\ell=2$ *and* $C_{2i}(s)$ ($C_{2i+1}(s)$, resp.) *is even function (odd function, resp.),* $\tilde{J}+\tilde{K}$ *is nilpotent.*

Now, we assume (5) does not hold for k = 0 or k = 1.

In order to make B stronger than $n^{1-2\sigma} s^2 A_2(s)$, we take $1-2\sigma<0$ ie $\sigma>\frac{1}{2}$.

§2. Maximal connection.

Let us consider

$$\tilde{J}= \bigoplus_{\substack{1\leqslant k\leqslant R\\ 1\leqslant h\leqslant M_R}} \tilde{J}(k,h),\ \tilde{J}(k,h) \text{ is that in Prop. 1.1, } M=\sum_{j=1}^{R} j\,M_j\ .$$

Corresponding to the blocks of $\tilde{J}$, we decompose M×M matrix K to $(K(k,h,k',h'))_{\substack{1\leqslant k,k'\leqslant R\\ 1\leqslant h\leqslant M_k\\ 1\leqslant h'\leqslant M_{k'}}}$.

Ex.

$$\tilde{J}=\begin{pmatrix} \tilde{J}(2,1) & & \\ & \tilde{J}(2,2) & \\ & & \tilde{J}(1,1)\end{pmatrix}, \quad K=\begin{pmatrix} \tilde{J}(2,1,2,1) & \tilde{J}(2,1,2,2) & \tilde{J}(2,1,1,1) \\ \tilde{J}(2,2,2,1) & \tilde{J}(2,2,2,2) & \tilde{J}(2,2,1,1) \\ \tilde{J}(1,1,2,1) & \tilde{J}(1,1,2,2) & \tilde{J}(1,1,1,1)\end{pmatrix}$$

We call (K(k,h,k',h')) the block decomposition of K with respect to $\tilde{J}$.

The following notions are important.

Definition 2.1. (Maximal connection of Jordan chain).

Let $(K(k,h,k',h'))$ be the block decomposition of K with respect to $\tilde{J}$. If

1)

$$(2.1)\quad \begin{cases} K(R,h,k',h') = \begin{pmatrix} 0 & & 0 \\ \cdot & & \\ 0 & & 0 \\ \alpha^{R,h,k',h'} & & 0 \end{pmatrix}, \\ K(k,h,k',h') = 0 \qquad (k < R) \end{cases}$$

for arbitrary h, k' and h',

(2) $\quad K \neq 0,$

(3) $\quad \mathcal{A} = (\alpha^{R,h,R,h'})_{1 \leqslant h,h' \leqslant M_R}$ is nilpotent,

$\tilde{J}+K$ is again nilpotent. We say that in $\tilde{J}+K$, the Jordan chains of $\tilde{J}$ are maximally connected by K, or that K brings a maximal connection (of Jordan chain) to $\tilde{J}$.

Definition 2.2. (Selfsimilar matrix).

Let us take $1 < R_0 < R_1 < \ldots < R_p < R_{p+1}$, such that

$$R_{j+1} = k_j\, R_j + R_j^0,\ k_j \geqslant 1,\ 0 \leqslant R_j^0 < R_j,\ k_j,\ R_j \in \mathbb{N}.$$

We set

$$A_0 = \begin{pmatrix} 0 & 1 & & \\ & 0 & \cdot & \\ & & \cdot & 1 \\ & & & 0 \end{pmatrix};\ R_0 \times R_0,$$

$$A_{j+1} = \underbrace{A_j \oplus \ldots \oplus A_j}_{k_j} \oplus A_j^0 + K_j\ ;\ R_{j+1} \times R_{j+1}$$

A_j^0 = the first R_j^0 rows and R_j^0 columns part of A_j,

(2.2) $\quad K_j = (K_j(k,k'))$; block decomposition w.r.t. A_{j+1}

$$K_j(h,h+1) = \begin{pmatrix} 0 & 0 \\ \cdot & 0 \\ 0 & \\ s & 0 \end{pmatrix},\ 1 \leqslant h \leqslant k_j,$$

$\hat{J}(i) : i \times i$ = the first i rows and i columns part of A_{p+1}.

We call

$$\hat{J} = \bigoplus_{1 \leq i \leq R_{p+1}} (\underbrace{\hat{J}(i) \oplus \dots \oplus \hat{J}(i)}_{M_i})$$

a self similar matrix of step p+1 and A_j and A_j^0 the factors of step j.

Let $\hat{J}$ be M×M selfsimilar matrix and K is a M×M matrix block decomposed w.r.t $\hat{J}$. Let an element of a block of K belong to $q_p^{(r)}$-th A_p in the direction of row and to $q_p^{(c)}$-th A_p in the direction of column. ((k_p+1)-th $A_p = A_p^0$). Further, in A_p, let it belong to $q_{p-1}^{(r)}$-th A_{p-1} in the direction of row and to $q_{p-1}^{(c)}$-th A_{p-1} in the direction of column. We continue this procedure up to $q_0^{(*)}$. At last, let it be the $(q_{-1}^{(r)}, q_{-1}^{(c)})$ element of A_0. We set $q_h = q_h^{(r)} - q_h^{(c)} + 1$ $(-1 \leq h \leq p)$.

Definition 2.3. (Address)

We call $q = (q_p, q_{p-1}, \dots, q_{-1})$ the address of the element.

To the set of addresses, we give the dictionary order.

Definition 2.4. (Acceptable matrix).

Let us take $1 > \nu_{-1} > \nu_0 > \dots > \nu_p > 0$, $\nu = \sum_{j=-1}^{p} \nu_j$, and $\sigma > 0$ $(\nu_j, \sigma \in \mathbb{R}_+)$. For a block decomposed matrix K w.r.t. a selfsimilar matrix $\hat{J}$, if the adress q of its element has a q_j such that $q_j = k_j + 1$ and $\sum_{h=0}^{j-1} (q_h - 1) R_h + q_{-1} = R_j^0$ (that is, the element is found at the left-down corner of $R_{j+1} \times R$ matrix in step of j+1, $R(\leq R_j+1)$; free), the element has the form $c(s) n^{1-\nu'}$, $\nu' = \nu'(q) = 2\sigma - \sum_{j=-1}^{p} (q_j - 1)\nu_j$ and otherwise, it has the form $c\, n^{1-\nu'}$, $\nu' = \nu'(q) = \sigma - \sum_{j=-1}^{p} (q_j - 1)\nu_j$ and c is constant. Further, if all $\nu(q)$ are greater than ν, we say that K is acceptable w.r.t. $n^{1-\nu}\hat{J}$. We call $\varepsilon(q) = (\nu' - \nu)/(\sum_{j=0}^{p} (q_j - 1) R_j + q_{-1})$ the descent index of the element with the address q.

When the descent index is smaller, we say that it is more effective.

<u>Remark.</u> Corresponding to the above $\hat{J}$, we take a shearing operator with weight ε

$$W = \bigoplus_{\substack{1 \leq k \leq R_{p+1} \\ 1 \leq h \leq M_k}} W(k,h,\varepsilon),$$

$W(k,h,\varepsilon) = \mathrm{diag}(1, n^{\varepsilon}, n^{2\varepsilon}, \dots, n^{(k-1)\varepsilon})$, $\varepsilon = \nu(q)$. Then, the element with the adress q obtain the order $1-\nu-\varepsilon$ of $W^{-1}(n^{1-\nu}\hat{J})W$ by the shearing transformation $W^{-1}KW$.

Now, we return to the equation (1.4). We assume that (1) is uniformly well-posed.

Let us set $W = \bigoplus_{\substack{1\leq k\leq r \\ 1\leq h\leq m_k}} W(k,h,\frac{\sigma}{r})$.

setting $w_1 = W^{-1}w$, w_1 satisfys $P_1 w_1 = 0$,

(2.3) $\quad P_1 = W^{-1}P_0 W = n^{\sigma} D_s - (n^{1-\sigma/r} J + n^{1-\sigma/r} s\, K_1 + s\, A_1^1(n) + s^2 A_2^1(s\,;\,n) + B^1(n))$,

where $s(n^{1-\sigma/r} K_1 + A_1^1(n))$ is brought from $n^{1-\sigma} s\, A^1$ and the order of $A_1^1(n)$ is less than $1-\sigma/r$. $s\, A_1^1(n) + s^2 A_2^1(s\,;\,n)$ is acceptable w.r.t. $n^{1-\sigma/r} J$.

In $K_1 = (K_1(k,h,k',h'))_{\substack{1\leq k,k'\leq r \\ 1\leq h\leq m_k \\ 1\leq h'\leq m_{k'}}}$, $\quad K_1(\ ,k,k',h') = \begin{pmatrix} & 0 & \\ & \cdot & 0 \\ & 0 & \\ \alpha^{h,k',h'} & & \end{pmatrix}$

and $K_1(k,h,k',h') = 0$ for $k<r$. By virtue of Proposition 1.1, $(\alpha^{h,R,h'})_{1\leq h,h'\leq m_r}$ must be nilpotent, and then, K_1 brings a maximal connection to J if $K_1 \neq 0$. We can take each Jordan chain in $J+K_1$ composed by vectors of $s^{\mu} v$, v : constant vector. Replacing $s^{\mu} v$ by vn we can have a constant matrix N which transform J to $\hat{J}_1$, a selfsimilar matrix. We have

(2.4) $\quad \tilde{P}_1 = N^{-1} P_1 N = n^{\sigma} D_s - (n^{1-\sigma/r}\hat{J}^1(s) + s\,\tilde{A}_1^1(n) + s^2\tilde{A}_2^1(s\,;\,n) + \tilde{B}^1(n))$.

Let us set the length of the longest Jordan chain of $\hat{J}_1(s)$ as $R_1 = k_0 R_0 + \ell$, $R_0 = r$, $0 \leq \ell < R_0$. In $s\,\tilde{A}_1^1(n) + s^2\tilde{A}_2^1(s\,;\,n)$, the highest order on n is given only by the elements with the address $(k_0+1, r-1)$ if $\ell \geq R_0 - 1$, and by those with the adress $(k_0, r-1)$ (and also by those with $(k_0+1, r-2)$ in cas e of $k_0 = 1$) if $\ell < R_0 - 1$. In the former case, if an element with the address $(k_0+1, r-1)$ does not vanish, after the shearing transformation with weight $\frac{\sigma}{R_0 R_1}$, a maximal connection occurs by virtue of Proposition 1.1. In the latter case, no maximal connection occurs. Continuing this procedure, we arrive at the following proposition.

Proposition 2.1. *Let us set* $R_{-1}=1$, $R_0=R$, $R_{j+1}=k_j R_j+R_j-R_{j-1}$ $(0\leq j\leq p+1)$ *and* $\hat{R}=k_p R_p+\ell$, $(k_j\in\mathbb{N}=\{1,2,\ldots\},\ 0\leq \ell<R_p)$.

(1) *In the above procedure, if* p *times maximal connections occur, the highest order part must be the selfsimilar matrix of step* $p+1$ *replacing* R_{p+1} *by* $\hat{R}$ *and has the order* $1-\nu$ *on* n, $\nu=\sum_{j=0}^{p}\nu_j$, $\nu_j=\frac{\sigma}{R_{j-1}R_j}$.

(2) *The operator* $\tilde{P}_{p+1}$ *has the following form ;*

(2.5) $\tilde{P}_{p+1}=n^{\sigma}D_s-(n^{1-\nu}\tilde{J}_{p+1}(s)+s A_1^{p+1}(n)+s^2 A_2^{p+1}(s\ ;\ n)+B^{p+1})$,
where $s A_1^{p+1}(n)+s^2 A_2^{p+1}(s\ ;\ n)$ *is acceptable w.r.t.* $n^{1-\nu}\tilde{J}_{p+1}$.

(3) *In* $s A_1^{p+1}(n)+s^2 A_2^{p+1}(s\ ;\ n)$, *if* $\ell\geq R_p-R_{p-1}$, *the highest order is given only by the elements with address* $(k_p+1,k_{p-1},\ldots,k_0,r-1)$ *and if* $\ell<R_p-R_{p-1}$, *it is given by those with the address* $(k_p,k_{p-1},\ldots,k_0,r-1)$ *(and also by those with* $(1,\ldots,1,2,k_\ell-1,k_{\ell-1},\ldots,k_0,r-1)$ *in case of* $k_{\ell+1}=\ldots=k_p=1$ *and* $k_\ell\geq 2$ *and also by those with* $(1,\ldots,1,2,1,\ldots,1,r-2)$ *in case of* $k_0=k_1=\ldots=k_p=1$*).*

Proof By the induction on p.

§3 Proof of Theorem 1, case of $k\leq 1$.

The maximal connections can occur at most $[\frac{m-r}{r-1}]$ times. Let no maximal connection occur on $\tilde{P}_{p+1}$, that is, in $\hat{R}=k_p R_p+\ell$, $\ell\neq R_p-R_{p-1}$ or $\ell=R_p-R_{p-1}$ but all elements with the address $(k_p+1, k_{p-1}, \ldots, k_0, r-1)$ vanish.
Let W be the shearing operator corresponding to $\tilde{J}_{p+1}$ in (2.5) with weight ε ($\varepsilon=\frac{\sigma}{R_p R_{p+1}}$ in case of $\ell>R_p-R_{p-1}$ and $\varepsilon=\frac{\sigma}{R_p(R_{p+1}-R_p)}$ in case of $\ell\leq R_p-R_{p-1}$).
We set

(3.1) $\hat{P}_{p+2}\equiv W^{-1}\tilde{P}_{p+1}W=$
$n^{\sigma}D_s-\{n^{1-\nu-\varepsilon}(\tilde{J}_{p+1}(s)+s K_{p+2})+s A_1^{p+2}(n)+s^2 A_2^{p+2}(s\ ;\ n)+B^{p+2}(n)\}$,

where the orders of A_1^{p+2} and A_2^{p+2} are less than $1-\nu-\varepsilon$. Here the highest order in $B^{p+2}(n)$ is σ and it is given by the elements with the address $(k_p+1,k_{p-1},\ldots,k_0,r)$ in

case of $\ell > R_p - R_{p-1}$ and $(k_p, k_{p-1}, \ldots, k_0, r)$ in case of $\ell \leqslant R_p - R_{p-1}$.
By a suitable choice of B in the original operator P, we can take B^{p+2} such that it has only one non-zero element, (g,1)-element $c_0 n^{\sigma}$ (c_0 is a large constant), where $g = k_p R_p + \sum_{j=0}^{p-1}(k_j - 1)R_j + r$ if $\ell > R_p - R_{p-1}$ and $g = \sum_{j=0}^{p}(k_j - 1)R_j + r$ if $\ell \leqslant R_p - R_{p-1}$.
We consider the characteristic polynomial of the full operator $\hat{P}_{p+2}$:

$$\det(\lambda I - \{n^{1-\nu-\varepsilon}(\tilde{J}_{p+1}(s) + s\, K_{p+2}) + s\, A_1^{p+2}(n) + s^2 A_2^{p+2} + B^{p+2}(n)\})$$
$$= \sum_{j=0}^{m} \alpha_j(s\,;\,n)\lambda^{m-j}.$$

$\alpha_g(s\,;\,n)$ has the form $c_0\, n^{\delta} s^{\mu}(1 + o(1))$, $\delta - (g-1)(1-\nu-\varepsilon) + \sigma$ and $\exists \mu \in \mathbb{Z}_+$ $(\nu = \sum_{j=0}^{p} \frac{\sigma}{R_{j-1} \; R_j})$. Here, we cannot find Jordan chains which are composed the vector of type $s^{\mu'}$ v, v : constant vector.

By virtue of Proposition 1.1, $\tilde{J}_{p+1} + K_{p+2}$ is nilpotent. Let us take N(s) which transforms $\tilde{J}_{p+1} + s\, K_{p+2}$ to Jordan's normal form and set

(3.2) $\quad \tilde{P}_{p+2} \equiv N^{-1} \cdot \hat{P}_{p+2} \cdot N = n^{\sigma} D_s - n^{1-\nu-\varepsilon} J_{p+2} - C(s\,;\,n).$

Here, the commutator $n^{\sigma} N^{-1}(s)\, D_s\, N(s)$ has the same order σ as $B^{p+2}(n)$ and it can give an influence on $\alpha_g(s\,;\,n)$. That is, setting

$$\det(\lambda I - n^{1-\nu-\varepsilon} J_{p+2} - C(s\,;\,n)) = \sum_{j=0}^{m} \alpha'_j(s\,;\,n)\lambda^{m-j},$$

$\alpha'_g(s\,;\,n)$ may have the form $(c_0 + c'_0(s))n^{\delta}(1 + o(1))$. However, $c'_0(s)$ is decided by the principal part part of the original operator P and independent of $B^{p+2}(n)$. Thus, $\alpha'_g(s,n) \neq 0$ and it has the order σ, if we take c_0 sufficiently large.

Let σ be i_1/i_0 ($i_0, i_1 \in \mathbb{N}$). $1-\nu-\varepsilon$ is also expressed as i_2/i_0 ($i_2 \in \mathbb{N}$). If $1-\nu-\varepsilon > \sigma$, we can find a matrix $N' \sim I + \sum_{h \in \mathbb{N}} n^{h/i_0} N_h(s)$ such that

(3.3) $\quad Q \equiv N'^{-1} \circ \tilde{P}_{p+2} \circ N' = n^{\sigma} D_s - n^{1-\nu-\varepsilon} J_{p+2} - C'(s\,;\,n),$

where $C'(s\,;\,n) = (C'(k,h,k',h')(s\,;\,n))$; block decomposition w.r.t. J_{p+2},

$$C'(k,h,k',h') = \begin{pmatrix} \gamma_1^{khk'h'} & 0 \\ . & 0 \\ \gamma_k^{khk'h'} & 0 \end{pmatrix} \quad \text{and}$$

$$\gamma_j^{khk'h'} \sim \sum_{\substack{i\in \mathbf{Z} \\ i/i_0+j(\nu+\varepsilon)<1}} n^{i/i_0+(j-1)(\nu+\varepsilon)} \gamma_{ji}^{khk'h'}(s).$$

(See, for example, V.M. Petkov [18] or rather its proof).

We say that a matrix which has the form as C' is admissible to $n^{1-\nu-\varepsilon}$ J. By this transformation, the principal part of $\alpha'_g(s\,;n)$ is preserved. From now on, we assume that $1-\nu-\varepsilon>\sigma$.

We introduce a notion :

Definition 3.1. (Stable coefficient of characteristic polynomial)

Let $\tilde{C}$ be admissible to $n^{\tilde{\nu}}$ J. We set

$\det(\lambda\ I-n^{\tilde{\nu}}\ J-\tilde{C}(s\,;n)) = \Sigma_{j=0}^{m}\tilde{\alpha}_j(s\,;n)\lambda^{m-j}$.

When the principal part of $\tilde{\alpha}_j$ is preserved by any perturbation of order at most σ, we say that α_j is a stable coefficient of the characteristic polynomial of full operator.

On the stable coefficients, the following proposition was obtained by W. Matsumoto [9].

Proposition 3.1. *If the original Cauchy problem is uniformly well-posed, the characteristic polynomial of full operator has no stable coefficient.*

We transform Q by shearing operator W' corresponding to J with weight $\varepsilon_0>0$, ε_0 : very small.

(3.4) $\quad Q'\equiv W'^{-1}\,Q\,W' = n^{\sigma}\,D_s - n^{1-\nu-\varepsilon-\varepsilon_0}\,J - C''(s\,;n)$,

where the order of $C''(s\,;n)$ is less than $1-\nu-\varepsilon-\varepsilon_0$ and $C''(s,n)$ is admissible to $n^{1-\nu-\varepsilon-\varepsilon_0}$ J. Further, the elements which concern $\alpha'_g(s\,;n)$ has the order $\sigma+(g-1)\varepsilon_0$ in $C''(s,n)$. This implies that $\alpha'_g(s\,;n)$ is stable in the characteristic polynomial of the full operator Q', if we can find σ such that $1-\nu-\varepsilon>\sigma>\frac{1}{2}$. Then, when we can find a σ such that $1-\nu-\varepsilon>\sigma>\frac{1}{2}$, we arrive at a contradiction. Here, the existence of such σ is equivalent to "$g \geqslant 3$" and further equivalent to "$r \geqslant 2$ and if $r = 2$,

$$\text{in } A_1 = (A_1(k,h,k',h'))_{\substack{1\leqslant k,k'\leqslant 2\\ 1\leqslant h\leqslant m_k\\ 1\leqslant h'\leqslant m_{k'}}} \text{ in (1.4) } \left(A_1(2,h,2,h') = \begin{pmatrix} * & * \\ \alpha(h,h') & * \end{pmatrix}\right),$$

$(\alpha_{hh'})_{1\leqslant h,h'\leqslant M_2}$ vanishes". "$r \leqslant 2$" is equivalent to condition (1.5) with $k = 0$ (Corollary 2) and the rest is equivalent to (1.5) with $k = 1$.

Q.E.D.

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