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CR MAPPINGS AND THEIR HOLOLOMORPHIC EXTENSION

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If M is a smooth manifold of real dimension $2n+1$, we say that M is a *CR manifold* of codimension one with *CR bundle* \mathcal{V} , if \mathcal{V} is a subbundle of CTM , the complexified tangent bundle of M , satisfying

$$\dim_{\mathbb{C}} \mathcal{V} = n, \quad \mathcal{V} \cap \bar{\mathcal{V}} = 0.$$

Any smooth real hypersurface M in \mathbb{C}^{n+1} is a *CR manifold* of codimension one, where \mathcal{V} is the subbundle of antiholomorphic tangent vectors to M .

Let (M, \mathcal{V}) and (M', \mathcal{V}') be two *CR manifolds* of codimension one. A smooth mapping from M into M' is called *CR* if for all $p \in M$

$$H'(\mathcal{V}_p) \subset \mathcal{V}'_{H(p)}.$$

We recall the following definition introduced in Baouendi-Jacobowitz-Treves [3]. If M is a real analytic hypersurface in \mathbb{C}^{n+1} containing the origin and defined locally by $\rho(z, \bar{z}) = 0$, $d\rho \neq 0$, we say that M is essentially finite at 0 if for any sufficiently small $z \in \mathbb{C}^{n+1} \setminus \{0\}$, there exists an arbitrarily small $\zeta \in \mathbb{C}^{n+1}$ satisfying: $\rho(z, \zeta) \neq 0$, $\rho(0, \zeta) = 0$.

Our main result is the following:

THEOREM 1. *Let M and M' be real analytic hypersurfaces in \mathbb{C}^{n+1} and $H : M \rightarrow M'$ a smooth *CR* mapping, defined near $p_0 \in M$ with $H(p_0) = p'_0$, and satisfying*

$$(1) \quad H'(CT_{p_0}M) \not\subset \mathcal{V}'_{p'_0} \oplus \bar{\mathcal{V}}'_{p'_0},$$

where \mathcal{V} is the CR bundle of M' . If M and M' are essentially finite at p_0 and p'_0 respectively then H extends as a holomorphic mapping from a neighborhood of p_0 in \mathbb{C}^{n+1} to \mathbb{C}^{n+1} .

Theorem 1 was first proved for $n = 1$ by S. Bell and the authors (see [1], [2]). It generalizes the result in the diffeomorphic case proved in [3]. We refer to the references of [2] and [3] for earlier works on holomorphic extendibility of CR mappings under stronger conditions.

The following is a key ingredient in the proof of Theorem 1. If j is a smooth CR function defined on M then there exists a unique formal (holomorphic) power series $J(z) = \sum a_\alpha z^\alpha$, $a_\alpha \in \mathbb{C}$, such that, if $U \ni u \mapsto Z(u) \in \mathbb{C}^{n+1}$ ($U \subset \mathbb{R}^{2n+1}$, $Z(0) = 0$) is a parametrization of M , then the Taylor series of $j(Z(u))$ at 0 is given by $J(Z(u))$. On the other hand it is clear that a CR mapping between two hypersurfaces M and M' in \mathbb{C}^{n+1} is given by $(n+1)$ CR functions (j_1, \dots, j_{n+1}) . Such a mapping is called of *finite multiplicity* at 0 if

$$\dim_{\mathbb{C}} \mathcal{O}[[Z]] / (J(Z)) < \infty,$$

where $\mathcal{O}[[Z]]$ is the ring of formal power series in $(n+1)$ indeterminates and $(J(Z))$ is the ideal generated by $(J_1(Z), \dots, J_{n+1}(Z))$. Here the dimension is taken in the sense of vector spaces. We have the following:

THEOREM 2. *If M and M' are essentially finite at p_0 and p'_0 respectively then a CR mapping $H : M \rightarrow M'$ is of finite multiplicity at p_0 if and only if condition (1) of Theorem 1 holds.*

We may restate Condition (1) in terms of local coordinates. We may assume $p_0 = H(p_0) = 0$ and M and M' are given locally by

$$(2) \quad \operatorname{Im} w = \varphi(z, \bar{z}, \operatorname{Re} w), \quad \operatorname{Im} w = \psi(z, \bar{z}, \operatorname{Re} w)$$

with $\varphi(z, 0, \operatorname{Re} w) = \psi(z, 0, \operatorname{Re} w) = 0$; $z \in \mathbb{C}^n$, $w \in \mathbb{C}$. The map H is then given by $n+1$ CR functions $(f_1, \dots, f_n, g) = (f, g)$ defined on M . Therefore we have

$$(3) \quad \frac{g - \bar{g}}{2i} = \psi\left(f, \bar{f}, \frac{g + \bar{g}}{2}\right).$$

With this notation Condition (1) is equivalent to

$$(4) \quad \frac{\partial g}{\partial s}(0) \neq 0,$$

with $s = \operatorname{Re} w$. (Here f_j and g are considered as smooth functions of z, \bar{z}, s).

Using Theorem 1 as well as Diederich-Fornaess [5], [6], Fornaess [7] and Bell-Catlin [4], we obtain the following

THEOREM 3. *Let D and D' be two bounded pseudoconvex domains in \mathbb{C}^{n+1} with real analytic boundaries and $H : D \rightarrow D'$ a proper, holomorphic mapping. Then H extends holomorphically to a neighborhood of \bar{D} , the closure of D .*

We give here an outline of the proof of Theorem 1. By solving (3) for \bar{g} we obtain a holomorphic function Q

$$(5) \quad \bar{g} = Q(f, \bar{f}, g).$$

As in [3] by writing

$$Q(f, \lambda, g) = \sum Q_{\zeta^\alpha}(f, \bar{f}, g) \frac{(\lambda - \bar{f})^\alpha}{\alpha!}$$

we are reduced to showing that for $z_0 \in \mathbb{C}^n$ fixed, $|z_0| < r$,

$$Q_{\zeta^\alpha}(f(z_0, \bar{z}_0, s), \bar{f}(z_0, \bar{z}_0, s), g(z_0, \bar{z}_0, s))$$

extends as a holomorphic function in $s + it$, $|s| < r$, $-R < t < 0$, for some r, R positive, and satisfies

$$(6) \quad |Q_{\zeta^\alpha}| \leq C^{\alpha+1} \alpha!, \quad C > 0.$$

The main ingredients used in proving the above are the following.

LEMMA 1. If j is a smooth CR function defined on M then the Taylor series of j in the coordinates (z, s) is given uniquely by

$$(7) \quad j \sim \sum a_{\alpha k} z^\alpha w^k |_{w=s+i\varphi(z, \bar{z}, s)}, \quad a_{\alpha k} \in \mathbb{C}.$$

A basis for the CR vector fields on M is given by

$$(8) \quad L_j = \frac{\partial}{\partial \bar{z}_j} - i \frac{\varphi_{\bar{z}_j}}{1 + i\varphi_s} \frac{\partial}{\partial s}, \quad 1 \leq j \leq n,$$

LEMMA 2. If $j(z, \bar{z}, s)$ is a CR function on M , then for all multi-indices α

$$\bar{L}^\alpha j(0) = \left(\frac{\partial}{\partial z} \right)^\alpha J(0, 0),$$

where $J(z, w) \sim \sum a_{\alpha k} z^\alpha w^k$ is as defined in Lemma 1.

Using the Nullstellensatz we may prove the following.

LEMMA 3. For $j = 1, \dots, n$ let $F_j(z, w)$ be the formal power series associated to f_j as in Lemma 1. Let I be the ideal generated by $F_j(z, 0)$, $1 \leq j \leq n$, the ring $\mathcal{O}[[Z]]$ of formal power series in the indeterminates z_1, \dots, z_n . Then under the assumptions of Theorem 1,

$$(9) \quad \dim_{\mathbb{C}} \mathcal{O}[[z]]/I < \infty,$$

and therefore

$$(10) \quad \det \left(\frac{\partial F_k}{\partial z_j}(z, 0) \right) \neq 0.$$

An immediate consequence of Lemmas 2 and 3 is that there exists a multi-index α such that

$$(11) \quad \bar{L}^\alpha (\det(\bar{L}_j f_k))(0) \neq 0.$$

LEMMA 4. For every multi-index α and every z_0 , $|z_0| < r$ there exist functions $a(s)$, $b(s)$ holomorphic in the domain $\mathcal{R} = \{s + it; |s| < r, -R < t < 0\}$, smooth in $\bar{\mathcal{R}}$ such that

$$Q_{\zeta^\alpha}(f, \bar{f}, g)(z_0, s) = \frac{a(s)}{b(s)}.$$

Lemma 4 is proved by applying successively \bar{L}^β to (5) and using (11).

LEMMA 5. For each j , $1 \leq j \leq n$, f_j satisfies a polynomial equation of the form

$$f_j^{N_j} + a_{N_j-1}^j f_j^{N_j-1} + \cdots + a_0^j = 0,$$

where $a_k^j = a_k^j(L^\gamma \bar{f}, L^\gamma \bar{g})$ is a holomorphic function of the $L^\gamma \bar{f}$, $L^\gamma \bar{g}$, for $|\gamma| \leq \gamma_0$.

The proof of Lemma 5 uses Lemma 3, as well as repeated applications of the Weierstrass Preparation theorem and the Nullstellensatz.

LEMMA 6. There exists N such that for each multi-index α , $Q_{\zeta^\alpha}(f, \bar{f}, g)(z, \bar{z}, s)$ is a root of a polynomial of the form

$$(12) \quad X^N + b_{N-1}^\alpha X^{N-1} + \cdots + b_0^\alpha = 0$$

where the b_k^α are holomorphic functions of $L^\gamma \bar{f}$ and $L^\gamma \bar{g}$, $|\gamma| \leq \gamma_0$, and satisfies

$$(13) \quad |b_j^\alpha(L^\gamma \bar{f}, L^\gamma \bar{g})| \leq (C^{\alpha+1} |\alpha|!)^{N-j}$$

at $(z, \bar{z}, s + it)$ for $|z| < r$, $|s| < r$ and $-R \leq t \leq 0$.

From Lemmas 4 and 6 it follows, using the Lemma in [2], that each $Q_{\zeta^\alpha}(f, \bar{f}, g)$ extends holomorphically to \mathcal{R} . Finally, the estimate (6) follows from (13).

For higher codimension, a slight modification of the proof of Theorem 1 yields the following.

THEOREM 4. Let M and M' be real analytic generic CR submanifolds of real codimensional ℓ in $\mathbb{C}^{n+\ell}$ and $H : M \rightarrow M'$ a smooth CR mapping defined near $p_0 \in M$, $H(p_0) = p'_0$, and satisfying

$$(14) \quad \dim_{\mathbb{C}}(H'(CT_{p_0} M) / \mathcal{V}'_{p_0} \oplus \bar{\mathcal{V}}'_{p'_0}) = \ell$$

where \mathcal{V}' is the CR bundle of M' . Assume that M and M' are essentially finite at p_0 , and that near p_0 , H extends holomorphically to a wedge of edge M . Then H extends as a holomorphic mapping from a neighborhood of p_0 in $\mathbb{C}^{n+\ell}$ to $\mathbb{C}^{n+\ell}$.

Complete details of the proofs will appear elsewhere.

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