

VICTOR IVRII

Estimates for a number of negative eigenvalues of the Schrödinger operator with intensive magnetic field

Journées Équations aux dérivées partielles (1987), p. 1-7

http://www.numdam.org/item?id=JEDP_1987___A20_0

© Journées Équations aux dérivées partielles, 1987, tous droits réservés.

L'accès aux archives de la revue « Journées Équations aux dérivées partielles » (<http://www.math.sciences.univ-nantes.fr/edpa/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
<http://www.numdam.org/>

Estimates for a number of negative eigenvalues of the Schrödinger operator with intensive magnetic field.

Victor Ivrii

1. In this lecture I give estimates from above and from below for a maximal dimension $N \leq \infty$ of a linear subspace $\mathcal{L} \subset C_0^\infty(X)$ on which a quadratic form

$$Q(u) = \int \left[g^{jk} (D_j - V_j)u \cdot \overline{(D_k - V_k)u} + V|u|^2 \right] dx \quad (1)$$

is negative definite; here X is a domain in \mathbb{R}^d , $d = 3$, $g^{jk} = g^{kj}$, V_j, V are real-valued and $g^{jk}, g^{jk}V_j, g^{jk}V_jV_k + V \in L_{loc}^1(X)$; we use Einstein's summation rule. Then Q is correctly defined on $C_0^\infty(X)$. Let us assume that

$$(H_1) \quad c^{-1} \leq |\xi|^{-2} g^{jk}(x) \xi_j \xi_k \leq c \quad \forall x \in X, \xi \in \mathbb{R}^d \setminus 0$$

and there are given functions $\delta, \rho = \rho_0, \rho_1, \dots$ on X such that

$$(H_2) \quad \rho_m(x) \geq 0, \delta(x) \geq 0, |\delta(x) - \delta(y)| \leq |x - y|$$

and for every point $y \in X' = \{x, \rho\delta > 1\}$ in $X \cap B(y, \delta(y))$ the following conditions are fulfilled:

$$(H_3) \quad c^{-1} \leq \rho_m(x)/\rho_m(y) \leq c, \quad |D_j \rho_m| \leq c \rho_m / \delta,$$

$$m = 0, 1, \dots, \rho_1 \geq \rho / \delta, \rho \geq \rho_2 \geq 1 / \delta,$$

$$(H_4) \quad |D^B g^{jk}| \leq c \delta^{-|B|},$$

$$|D^B F^j| \leq c \rho_1 \delta^{-|B|},$$

$$|D^B V| \leq c \rho^2 \delta^{-|B|} \quad \forall B: |B| \leq K < \infty,$$

$$(H_5) \quad \partial X \cap B(y, \delta(y)) = \{x_k = z(x_k)\} \cap B(y, \delta(y)) \quad \text{with}$$

$$|D^B z| \leq c \delta^{1-|B|} \quad \forall B: |B| \leq K$$

where $B(y, \delta)$ is an open ball with a center y and a radius δ ,

$x_k = (x_1, x_3)$ etc, $k = k(y) = 1, 2, 3$ in (H_5) , $F^j = i \varepsilon^{jk1} D_k V_1$

are components of the vector intensity of the magnetic field, ξ^{jkl} is absolutely skew-symmetric pseudo-tensor with $\xi^{123} = 1/\sqrt{g}$, $g = \det(g_{jk})$, $(g_{jk}) = (g^{jk})^{-1}$. Let $F = (g_{jk} F^j F^k)^{1/2}$ be a scalar intensity of the magnetic field. Let us further assume that

$$(H_6) \quad \forall y \in \{X', \varrho_1 > c \varrho / \gamma\} \quad \text{in } X \cap B(y, \gamma(y)) \\ F \geq c^{-1} \varrho_1,$$

$$(H_5)' \quad \forall y \in \{X', \varrho_1 > c \varrho / \gamma, v + F \leq \varepsilon \varrho^2\} \\ B(y, \gamma(y)) \subset X$$

with $\varepsilon > 0$.

Moreover, let us assume that for every $y \in X''' = \{X', \varrho_1 > c^{-1} \varrho^2, \varrho_2 < c_1^{-1} \varrho, v + F \leq \varepsilon \varrho^2\}$ in $B(y, \gamma(y))$ the following inequalities are fulfilled:

$$(H_7) \quad |D^B(v + (2j+1)F)| \leq c \varrho_2^2 \gamma^{-|B|} \quad \forall B: |B| \leq K$$

with $j = j(y) \in \mathbb{Z}^+$, $c_1 > 8c$; then X''' is a union of the disjoint domains X_j''' .

Let

$$\gamma_1 = \gamma^2 |\nabla(v/F)| \varrho_1 / \varrho^2 + \\ \gamma \min_{j \in \mathbb{Z}^+} |v + (2j+1)\mu h F|^{1/2} / \varrho \quad (\mu = h = 1 \text{ here})$$

on X' and $\gamma_2 = \gamma_1 \varrho / \varrho_2$ on X''' .

Finally, let us assume that

$$(H_8) \quad Q(u) \geq c^{-1} \int (|\nabla u|^2 - w|u|^2) dx \quad \forall u \in C_0^\infty(X'')$$

where $X'' = \{X, \varrho \gamma < 2\}$, $w \in L_{loc}^1(X)$, $w \geq 0$.

Our the first principal result is

Theorem 1. Let conditions (H_1) - (H_8) be fulfilled. Then

$$\mathcal{N} - CR_1 - C'R_2 \leq N \leq \mathcal{N} + C(R_1 + R_3) + C'R_2$$

where

$$N = (1/2 \pi^2) \sum_{j=0}^{\infty} \int_{X'} (V + (2j+1)F)^{-1/2} F \sqrt{g} dx ,$$

$$z_{\pm} = \max(\pm z, 0), \quad R_1 = \sum_{t=1}^5 R_{1t}, \quad R_2 = \sum_{t=1}^2 R_{2t},$$

$$R_{11} = \int_{\{X' \setminus X_0''', V+F \leq \varepsilon \rho^2 \gamma\}} \rho^2 \gamma^{-1} dx ,$$

$$R_{12} = \int_{\{X' \setminus X''', V+F \leq \varepsilon \rho^2, \gamma_1 \geq \rho^{-1/2+\sigma} \gamma^{1/2+\sigma}\}} \rho_1 \gamma_1^{-1} dx ,$$

$$R_{13} = \int_{\{X' \setminus X'''; V+F \leq \varepsilon \rho^2, \gamma_1 \leq \rho^{-1/2+\sigma} \gamma^{1/2+\sigma}\}} \rho_1 \rho^{1/2+\sigma} \gamma^{-1/2+\sigma} dx ,$$

$$R_{14} = \sum_{j=0}^{\infty} \int_{\{X_j''', V+(2j+1)F \leq \varepsilon \rho_2^2, \gamma_2 \geq \rho_2^{-1/2+\sigma} \gamma^{1/2+\sigma}\}} \rho^2 \gamma_2^{-1} dx ,$$

$$R_{15} = \sum_{j=0}^{\infty} \int_{\{X_j''', V+(2j+1)F \leq \varepsilon \rho_2^2, \gamma_2 \leq \rho_2^{-1/2+\sigma} \gamma^{1/2+\sigma}\}} \rho^2 \rho_2^{1/2+\sigma} \gamma^{-1/2+\sigma} dx ,$$

$$R_{21} = \int_{X'} \rho^3 \rho_1^{-s} \gamma^{-2s} dx ,$$

$$R_{22} = \int_{X'''} \rho^2 \rho_2^{-s} \gamma^{-1-s} dx ,$$

$$R_3 = \int_{X''} W^{3/2} dx ,$$

here and in what follows $\xi > 0$, $\sigma > 0$, s are arbitrary and $C = C(c)$, $C' = C'(c, c_2, \xi, \sigma, s)$, $K = K(\sigma, s)$ in $(H_4), (H_7)$.

Remark 2. If conditions $(H_1)-(H_8)$ are fulfilled and if $\mathcal{N} + R_1 + R_2 + R_3 < \infty$ then Q is semi-bounded from below on $L^2(X)$ and hence it generates a self-adjoint Schrödinger operator $A = (D_j - V_j)g^{jk}(D_k - V_k) + V$ on X with the Dirichlet boundary condition; then N is a dimension of its invariant negative subspace.

Theorem 1 is a more refined and general version of the principal theorems announced in [1,2]. Moreover, under a certain condition of a global nature concerning integral curves of the vector field (F^1, F^2, F^3) one can derive a more precise estimates. If A depends on parameters then theorem 1 implies asymptotics of N with respect to these parameters (see e.g. [1,2]).

2. The following assertion is the crucial step in the proof of theorem 1:

Theorem 3. Let

$$A_{\mu, h} = (hD_j - \mu V_j)g^{jk}(hD_k - \mu V_k) + V \quad (2)$$

with the Dirichlet boundary condition be a self-adjoint semi-bounded Schrödinger operator with the discrete spectrum and with the polynomial growth of the eigenvalue counting function $N(\lambda)$ as $\lambda \rightarrow \infty$; here $h \in (0, 1]$, $\mu \geq 1$; let $e(x, y, \lambda, \mu, h)$ be a Schwartz kernel of its spectral projector. Let $y \in X$ and in $B(y, 1) \subset X$ conditions $(H_1)-(H_6)$ be fulfilled with $\gamma = \rho = \rho_1 = 1$; moreover, let $\psi \in C_0^K(B(y, 1/2))$, $0 \leq \psi \leq 1$, $|D^B \psi| \leq c_2 \gamma^{-|B|} \quad \forall B: |B| \leq K$. Then

(i) The following estimate holds:

$$\mathcal{R} = \left| \int (e(x, x, 0, \mu, h) - S(x, \mu h)h^{-d}) \psi^2(x) dx \right| \leq$$

$$\begin{aligned}
& ch^{-2} \left(1 + \mu h \int \gamma_1^{-1} dx \right) + \\
& \quad \left\{ B(y,1), \gamma_1 \geq h^{1/2-\sigma} \right\} \\
& \mu h^{1/2-\sigma} \text{ mes } \left\{ B(y,1), \gamma_1 \leq h^{1/2-\sigma} \right\} + c'h^{-1}
\end{aligned}$$

where

$$S(x, \mu h) = (1/2 \pi^{d-1}) \sum_{j=0}^{\infty} (V + (2j+1) \mu h F)^{(d-2)/2} \mu h F \sqrt{g}.$$

(ii) If $V + \mu h F \geq \varepsilon$ in $X \cap B(y,1)$ (and not necessarily $B(y,1) \subset X$ here) then

$$e(x, x, 0, \mu, h) \leq c'h^s \mu^{-s} \quad \forall x \in X \cap B(y, 1/2).$$

(iii) If $V = -(2j+1)F + \zeta^2 V'$ with $\zeta \in (h, 1]$ and $j \in \mathbb{Z}^+$ and if V' satisfies $(H_4)_3$ with $\gamma = \varrho = 1$ then for $\mu = h^{-1}$

$$\begin{aligned}
\mathcal{R} \leq & ch^{-2} \left(1 + \int \gamma_2^{-1} dx \right) + \\
& \left\{ B(y,1), \gamma_2 \geq (h/\zeta)^{1/2-\sigma} \right\}
\end{aligned}$$

$$(h/\zeta)^{-1/2-\sigma} \text{ mes } \left\{ B(y,1), \gamma_2 \leq (h/\zeta)^{1/2-\sigma} \right\} + c'h^{-1} \zeta^{-1};$$

here $\varrho_2 = \zeta$ in the definition of γ_2 .

(iv) Moreover, if $V' \geq \varepsilon$ in $B(y,1)$ then

$$e(x, x, 0, \mu = h^{-1}, h) \leq c'h^{-2} (h/\zeta)^s \quad \forall x \in B(y, 1/2).$$

The proof of theorem 3 is complicated and it is based on the quasiclassical microlocal analysis of the non-stationary Schrödinger equation with parameters μ, h . Without conditions $(H_5)'$, (H_6) the similar assertion holds for $\mu \leq c, d \geq 2$ and it is the basis of the proof of the principal theorems of [3]. When all these asymptotics are established we generalize them first to arbitrary balls and then complete the proof of theorem 1 by means of an appropriate partition of unity and Rosenblyum variational estimate for an eigen-

value counting function for operator generated by Q in $L^2(X'', Jdx)$ with an admissible weight function J . One can find the similar procedure in [4].

3. Let us consider now the case $d = 2$. Certainly, now $F = F^3$, $F^3 = i(D_1V_2 - D_2V_1)/\sqrt{g}$. This case is not completely investigated yet. However I have proved the following

Theorem 4. Let $d = 2$ and all the conditions of theorem 3 be fulfilled. Then

(i) If $\gamma_1 \geq c^{-1}$ then

$$\mathcal{R} \leq c \mu^{-1} h^{-1} + c' \mu^{-2}.$$

(ii) Assertion (ii) of theorem 3 holds.

(iii) If $V = -(2j+1)F + \zeta^2 V'$ with $\zeta \in (h, 1]$ and $j \in \mathbb{Z}^+$ and if V' satisfies $(H_4)_3$ with $\gamma = \rho = 1$ and if $\gamma_2 \geq c^{-1}$ (here $\rho_2 = \zeta$ in the definition of γ_2) then

$$\mathcal{R} \leq c \zeta^{-2} + c' h \zeta^{-1}.$$

(iv) On the other hand, if

$$|V + (2j+1) \mu h F| \geq \zeta^2 \geq c \mu^{-2} \quad \forall j \in \mathbb{Z}^+ \quad \forall x \in B(y, 1)$$

then

$$\begin{aligned} |e(x, x, \lambda_2, \mu, h) - e(x, x, \lambda_1, \mu, h)| &\leq c' (h/\zeta)^s \\ \forall x \in B(y, 1/2) \quad \forall \lambda_1, \lambda_2 \in (-\zeta^2/c, \zeta^2/c) &\end{aligned} \quad (3).$$

(v) Moreover, if

$$\begin{aligned} |V + (2j+1) \mu h F + \mu^{-2} R V^2 / 8F| &\geq \zeta^2 \geq \varepsilon \mu^{-2} \\ \forall j \in \mathbb{Z}^+ \quad \forall x \in B(y, 1) &\end{aligned}$$

where R is a scalar curvature associated with the metrics (g_{jk}/F) and if $\mu^{-1} + \mu h \leq \delta = \delta(c, \varepsilon)$ then inequality (3) holds.

Let us note that in the two-dimensional case the presence of the intensive magnetic field can improve the remainder estimate; on the

other hand, in this case the gaps in the quasiclassical limit of the spectrum can appear. The role of Landau levels $E_j = V + (2j+1)\mu hF$ is more important in the two-dimensional case than in three-dimensional one; there is a correction $\Delta E_j = \mu^{-2} R(E_j - V)^2 / 8F$ to these levels.

4. Finally, it is well-known that if $X = \mathbb{R}^d$, $d = 2, 3$, g^{jk} , F^j are constant and $V = 0$ then

$$e(x, x, \lambda, \mu, h) =$$

$$(1/2 \pi^{d-1}) \sum_{j=0}^{\infty} (\lambda - (2j+1)\mu hF)_+^{(d-1)/2} \mu h^{1-d} F \sqrt{g};$$

in particular,

$$\sigma(A) = \sigma_{ac}(A) = [\mu hF, \infty) \quad \text{for } d = 3,$$

$$\sigma(A) = \sigma_{ess}(A) = \sigma_{pp}(A) = \{(2j+1)\mu hF, j \in \mathbb{Z}^+\}$$

for $d = 2$.

References

1. V.Ivrii, Proc. ICM-86, Berkeley (to appear)
2. V.Ivrii, Soviet Math. Dokl., 1987 (to appear)
3. V.Ivrii, C.R.A.S. Paris, 1986, sér. 1, t. 302, NN 13, 14, 15, pp. 467-470, 491-494, 535-538.
4. V.Ivrii, S.Fedorova, Funkts. Analis i Ego Prilozeniya, 1986, 20, N 4, p. 29-34.
5. Y.Colin de Verdiere, Comm. Math. Phys., 1986, 105, p. 327-335.
6. Y.Colin de Verdiere, Prépublications de l'Institut Fourier, N 33 (1985).
7. H.Tamura, Preprint of Nagoya University (1985).