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On the Spectral Theory of the Laplacian  
on non-compact Hyperbolic Manifolds

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In this lecture we shall discuss some problems which arise in the spectral theory of the Laplacian on non-compact hyperbolic manifolds obtained as the quotient of hyperbolic  $n$ -space by discrete groups of hyperbolic isometries. In particular we shall be interested in objects such as the resolvent kernel, the generalized eigenfunctions and the scattering matrix. The relations that exist among these objects are of a considerable interest and are basic in the study of the continuous spectrum of the Laplacian.

The problems we discuss here have their counterparts in Euclidean space, in the theory of Schrödinger operators on  $\mathbb{R}^n$ . In some respects the spectral theory of the Laplacian on hyperbolic manifolds is simpler and more complete than the corresponding theory for general Schrödinger operators. In fact, the theory we discuss here should be compared with the theory of Schrödinger operators  $-\Delta + V$  on  $\mathbb{R}^n$  for potentials  $V$  having a compact support.

## 1. The Laplacian on the hyperbolic space $B^n$

We take as a model of the hyperbolic  $n$ -space the unit ball:  
 $B^n = \{x \in R^n : |x| < 1\}$  with the Poincare metric:

$$ds^2 = 4(1 - |x|^2)^{-2} |dx|^2.$$

The volume element is

$$d\mu = 2^n (1 - |x|^2)^{-n} dx_1 \cdots dx_n.$$

$B^n$  is a complete Riemannian manifold with all sectional curvatures equal  $-1$ . It has an ideal boundary  $\partial B^n$  identified with the sphere  $S^{n-1} = \{x : x \in R^n, |x| = 1\}$ . One refers to points  $\omega \in \partial B^n$  as points at infinity.

We denote by  $\rho(x, x')$  the geodesic distance between two points in  $B^n$  and set:  $\sigma = (\cosh \rho + 1)/2$ .  $\sigma(x, x')$  is a "two point invariant" with respect to the non-Euclidean motions. It is given by:

$$\sigma(x, x') = |x - x'|^2 (1 - |x|^2)^{-1} (1 - |x'|^2)^{-1} + 1. \quad (1.1)$$

The basic invariant differential operator is the Laplace-Beltrami operator  $\Delta$ . In cartesian coordinates;

$$\Delta = \frac{1}{4} (1 - |x|^2)^2 \sum_{i=1}^n \partial_i^2 + \frac{n-2}{2} (1 - |x|^2) \sum_{i=1}^n x_i \partial_i,$$

$\partial_i = \partial/\partial x_i$ . It will be convenient to consider the modified Laplacian:

$$P = -\Delta - \left(\frac{n-1}{2}\right)^2.$$

$P$  is an elliptic operator, formally self-adjoint with respect to  $d\mu$ . It has a unique self-adjoint realization in  $L^2(B^n; d\mu)$  also denoted by  $P$ . It is a non-negative operator with a purely continuous spectrum consisting of the positive real axis.

We describe briefly the main mathematical objects associated with  $P$ , all of which can be given explicitly.

*I. The resolvent kernel.* This is the kernel of the operator  $(P-\lambda)^{-1}$  defined for  $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$ . We set  $\lambda = k^2$ ,  $\text{Im } k > 0$ , and denote the kernel by  $G(x, x'; k)$ . One has the following explicit expression:

$$G(x, x'; k) = c_n(s) \sigma^{-s} F\left(s, s - \frac{n-2}{2}; 2s - (n-2); \sigma^{-1}\right) \quad (1.2)$$

where  $s = \frac{n-1}{2} - ik$ ,  $\sigma = \sigma(x, x')$  is given by (1.1);  $F(a, b; c; t)$  is the hypergeometric function, and

$$c_n(s) = \pi^{-(2s+1)} \pi^{-(n-1)/2} \Gamma(s) / \Gamma\left(s - \frac{n-3}{2}\right).$$

*II. The generalized eigenfunctions.* These functions, defined in  $B^n$ , play in hyperbolic space the role that exponential functions play in  $\mathbb{R}^n$ . Denoted by  $E(x, \omega; k)$  they depend on an arbitrary point  $\omega \in \partial B^n$  and on the eigenvalue parameter  $k$ . They are distinguished solutions of the equation:  $Pu = k^2 u$  in  $B^n$ . With a chosen normalization they are given explicitly by

$$E(x, \omega; k) = \pi^{-(n-1)/2} \left( \frac{1-|x|^2}{2|\omega-x|^2} \right)^s, \quad s = \frac{n-1}{2} - ik. \quad (1.3)$$

*III. The scattering matrix.* This is an operator valued function, denoted by  $S(k)$ , defined for  $k \in \mathbb{C}$ .  $S(k)$  acts on functions, distributions and hyperfunctions on  $\partial B^n$ . It has a distributional kernel  $S(\omega, \omega'; k)$ ,  $\omega, \omega' \in \partial B^n$ , given by

$$S(\omega, \omega'; k) = \pi^{-\frac{n-1}{2}} \frac{\Gamma(\frac{n-1}{2} + ik)}{\Gamma(ik)} |\omega - \omega'|^{-(n-1) + 2ik}$$

$S(k)$  is meromorphic function of  $k$  in a weak sense with poles on the positive imaginary axis. It verifies the functional equation:  $S(k)S(-k) = id$ .

$S(k)$  is related to the generalized eigenfunctions through the functional equation:

$$E(x, \omega; k) = S(k)E(x, \cdot; -k).$$

## 2. The Laplacian on $B^n/\Gamma$

We consider a Riemannian hyperbolic manifold  $M^n$  of the form:

$$M^n = B^n/\Gamma \tag{2.1}$$

where  $\Gamma$  is a discrete group of non-Euclidean isometries acting on  $B^n$ . We assume that  $M^n$  is not compact. We also assume that  $\Gamma$  has no elements of finite order (elliptic elements). This ensures that  $M^n$  is a smooth Riemannian manifold. (The last assumption may be relaxed).

We denote by  $\Delta_\Gamma$  the Laplace-Beltrami operator on  $M^n$ . We shall be interested in studying spectral properties of  $\Delta_\Gamma$  related to the continuous spectrum. To obtain interesting results we have to impose restrictions on the nature of the "boundary at infinity" of  $M^n$ . The main restriction that we impose is that  $\Gamma$  has a geometrically finite fundamental domain in  $B^n$  (i.e. a fundamental domain bounded by a finite number of totally geodesic hypersurfaces). We shall compactify  $M^n$  by adding to it its boundary at infinity. Roughly and informally this is done as follows. One represents  $M^n$  by a canonical fundamental domain  $\mathcal{F}$  in  $B^n$  with sides identified according to

the action of  $\Gamma$ . The boundary at infinity  $\partial M^n$  is then represented by the set  $\overline{\mathcal{F}} \cap \partial B^n$  ( $\overline{\mathcal{F}}$  denotes closure in  $B^n$ ) with the proper identifications.

The compactified manifold:  $\overline{M}^n = M^n \cup \partial M^n$  is a real analytic Riemannian manifold with a boundary having a finite number of singularities known as cusps. The cusps correspond to parabolic elements of  $\Gamma$ . We shall impose on  $\Gamma$  a last restriction that it does not contain parabolic elements. This last assumption implies that  $\partial M^n$  is a compact real analytic manifold without singularities of dimension  $n-1$  (possibly disconnected).

**Remark:** The theory we describe here can be extended to the case where  $M^n$  is allowed to have cusps which are all of maximal rank. In particular for  $n = 2$  the theory can be extended to all finitely generated discrete groups  $\Gamma$ .

Some historical remarks. The interest in the spectral theory of the Laplacian on hyperbolic manifolds of the form (2.1) arose originally in number theory, for the case  $n = 2$ , for special groups  $\Gamma$  having a finite area fundamental domain. This followed an observation made by Maass [11] that Eisenstein series are automorphic generalized eigenfunctions of the Laplace-Beltrami operator in the hyperbolic plane. The work of Selberg [19] on the trace theorem gave a special impetus to the study of the spectral properties of the Laplacian on automorphic functions in the hyperbolic plane. Among the many important contributions to this study we mention (a partial list): Roelcke [18], Faddeev [4], Faddeev and Pavlov [5], Elstrodt [3], Patterson [16], Fay [6], Lax and Phillips [9], Venkov [20].

Spectral properties of the Laplacian on non-compact  $n$ -dimensional hyperbolic manifolds were studied in recent years by various authors. We mention in particular the work of Müller [15] dealing with spectral properties of the Laplacian on general Riemannian manifolds with cusps, and the work of Lax and Phillips [10] dealing with spectral properties of  $\Delta_\Gamma$  on  $B^n/\Gamma$  for general geometrically finite discrete groups  $\Gamma$ .

One of the interesting problems in the theory (related to Selberg trace formula) is the problem of the meromorphic continuation of the Eisenstein series associated with  $\Gamma$  (= generalized eigenfunctions of  $\Delta_\Gamma$ ). In the two dimensional case the problem was solved by Patterson [16] and Fay [6] for finitely generated groups  $\Gamma$  having no parabolic elements (see also [1]). For  $n = 3$  Mandouvalos [12] has derived the meromorphic continuation for a certain class of groups  $\Gamma$ . Recently Mazzeo and Melrose [13] have shown that in any dimension  $n$ , for geometrically finite groups  $\Gamma$  with no parabolic elements, Eisenstein series admit meromorphic continuation to the entire complex plane. They derive the result as a corollary of a theorem on the meromorphic continuation of the resolvent kernel of  $\Delta_\Gamma$ . In the following we shall describe an alternative solution to the problem of meromorphic continuation of Eisenstein series which follows more closely Selberg's original ideas. This approach uses technics which proved successful in the study of Schrödinger operators. We mention that Peter Perry has informed us in a letter that he had obtained a proof of the Mazzeo - Melrose result using similar technics.

Returning to our main discussion we consider the operator  $\Delta_\Gamma$  on  $M^n = B^n/\Gamma$  where  $\Gamma$  verifies the restrictions imposed before. We set:

$$P_{\Gamma} = -\Delta_{\Gamma} - \left(\frac{n-1}{2}\right)^2.$$

$P_{\Gamma}$  has a unique self-adjoint realization in  $L^2(M^n; d\mu)$  also denoted by  $P_{\Gamma}$ . It can be shown that its spectrum  $\sigma(P_{\Gamma}) = \sigma_d(P_{\Gamma}) \cup [0, \infty]$  where  $\sigma_d(P_{\Gamma})$  consists of a finite set of negative eigenvalues. There are no eigenvalues imbedded in the positive spectrum (c.f. [10]).

Next, we consider the resolvent operator  $G_{\Gamma}(k) = (P_{\Gamma} - k^2)^{-1}$ . It follows from standard results in elliptic theory that  $G_{\Gamma}(k)$  is an integral operator with a smooth kernel  $G_{\Gamma}(x, x'; k)$  for  $x, x' \in M^n, x \neq x'$ , for any  $k$  in the half-plane  $\text{Im } k > 0, k^2 \notin \sigma_d(P_{\Gamma})$ . We refer to  $G_{\Gamma}(x, x'; k)$  as the resolvent kernel. It is a meromorphic function of  $k$  in the upper half-plane with simple poles at the points  $k_j$  verifying:  $k_j^2 \in \sigma_d(P_{\Gamma})$ .

There is an explicit series representation for  $G_{\Gamma}$  if  $\text{Im } k > (n-1)/2$ . Identifying  $M^n$  with  $\mathcal{F} \subset \mathbb{B}^n$  it can be shown that

$$G_{\Gamma}(x, x'; k) = \sum_{g \in \Gamma} G(gx, x'; k) \quad (2.2)$$

where  $G$  is the resolvent kernel of the modified Laplacian on  $\mathbb{B}^n$  given by (1.2).

Next we look for the functions which should play the role of the generalized eigenfunctions of the Laplacian on  $M^n$ . Natural candidates for such functions are the functions defined by the series:

$$e_{\Gamma}(x, \omega; k) = \sum_{g \in \Gamma} E(gx, \omega; k) \quad (2.3)$$

where  $E(x, \omega; k)$  denotes the generalized eigenfunctions of the Laplacian on  $\mathbb{B}^n$ , defined by (1.3). (Here,  $M^n$  is represented by a fundamental domain  $\mathcal{F} \subset \mathbb{B}^n$  and

$\omega \in \partial M^n$ , identified with  $\overline{\mathcal{F}} \cup \partial B^n$ ).

The series (2.3) are the Eisenstein series associated with  $\Gamma$ . They can be shown to converge in  $B^n$ , uniformly on compact subsets, for any  $k$  in the half-plane  $Im k > (n-1)/2$  and any  $\omega \in \partial B^n$ ,  $\omega$  not in the limit set of  $\Gamma$ , to a  $\Gamma$ -automorphic solution of the differential equation:  $Pu = k^2u$  in  $B^n$ .

The functions  $e_\Gamma$  are essentially the desired generalized eigenfunction for  $k$  in the half-plane  $Im k > (n-1)/2$ . However, to be useful in spectral problems one needs to renormalize the  $e_\Gamma$  for the following reason. When one considers  $e_\Gamma(x, \omega; k)$  as a function of  $(x, \omega)$  on  $M^n \times \partial M^n$ , one finds that  $e_\Gamma$  is smooth in  $x$  on  $M^n$  but is not smooth in  $\omega$  on  $\partial M^n$ . For this and other reasons we shall use another method to define the generalized eigenfunctions.

We start by defining for  $x, x' \in B^n$ , the function:

$$\sigma_\Gamma(x, x') = \left( \sum_{g \in \Gamma} \sigma(gx, x')^{-n} \right)^{-\frac{1}{n}} \tag{2.4}$$

where  $\sigma(x, x')$  is the pair invariant function (1.1). It can be shown that  $\sigma_\Gamma$  is a well defined positive, smooth,  $\Gamma$  automorphic function in  $x$  and  $x'$ . Next, considering  $\sigma_\Gamma$  as a function on  $M^n \times M^n$  we fix a point  $x_0 \in M^n$  and set:

$$\tau(x) = \sigma_\Gamma(x, x_0). \tag{2.5}$$

The function  $\tau(x)$  is a positive smooth function of  $x$  on  $M^n$ . It verifies the growth relation:

$$\rho_\Gamma(x, x_0) - C \leq \log \tau(x) \leq \rho_\Gamma(x, x_0) + C$$

for some constant  $C$ ,  $\rho_\Gamma(x, x_0)$  denotes the geodesic distance between  $x$  and  $x_0$  on

$M^n$ . For any point  $\omega \in \partial M^n$  and  $k$  in the half-plane  $\text{Im } k > (n-1)/2$ , we now define:

$$E_\Gamma(x, \omega; k) = \lim_{x' \rightarrow \omega} \tau(x')^s G_\Gamma(x, x'; k), \quad s = \frac{n-1}{2} - ik \quad (2.6)$$

It follows from (2.2) and (1.2), and properties of the function  $\tau$ , that the limit (2.6) exists for  $k$  in the half-plane  $\text{Im } k > (n-1)/2$ . It follows further that  $E_\Gamma(x, \omega; k)$  is a real analytic function in  $(x, \omega)$  on  $M^n \times \partial M^n$ , analytic in  $k$  in the half-plane  $\text{Im } k > (n-1)/2$ , satisfying the differential equation:

$$P_\Gamma u = k^2 u \quad \text{in } M^n. \quad (2.7)$$

It can be shown that the generalized eigenfunctions  $E_\Gamma$  are renormalized Eisenstein series:  $E_\Gamma(x, \omega; k) = m(\omega, k) e_\Gamma(x, \omega; k)$  where  $m$  is a non-vanishing entire analytic function of  $k$ .

The next step is to continue analytically the generalized eigenfunction  $E_\Gamma$  beyond the half-plane  $\text{Im } k > (n-1)/2$ . Now, using relatively simple P.D.E. estimates one can show that the limit on the r.h.s. of (2.6) exists (uniformly on compact subsets) for all  $k$  in the half-plane  $\text{Im } k > 0, k^2 \notin \sigma_d(P_\Gamma)$ . Hence formula (2.6) gives the meromorphic continuation of  $E_\Gamma(x, \omega; k)$  to the half-plane  $\text{Im } k > 0$ .

We come to a crucial intermediate step in the analytic continuation proof. It consists in showing that both  $G_\Gamma$  and  $E_\Gamma$  possess continuous boundary values on the real axis in the  $k$ -plane. The proof of this result requires some delicate estimates "on the continuous spectrum" of the type used to prove the limiting absorption principle for Schrödinger operators (c.f. [2], [8]). Such estimates in the hyperbolic case are dis-

cussed in [17].

Summing up, one finds at this stage that both  $G_\Gamma(x, x'; k)$  and  $E_\Gamma(x, \omega; k)$  exist and are analytic in the half-plane  $Im k > 0$ , continuous in  $Im k \geq 0$ , except for a finite number of poles at the points  $\{k_j\}$  such that  $k_j^2 \in \sigma_d(P_\Gamma)$ . The  $E_\Gamma$  satisfy (2.7) and are related to  $G_\Gamma$  by (2.6) (also for  $k$  real). An important point is that  $E_\Gamma$  is a smooth (real analytic) function in  $(x, \omega)$  on  $M^n \times \partial M^n$ .

We denote by  $\mathcal{A}'(\partial M^n)$  and  $\mathcal{D}'(\partial M^n)$  the classes the hyperfunctions and distributions on  $\partial M^n$ , respectively. We shall denote by  $\mathcal{E}_\lambda(M^n)$ ,  $\lambda \in \mathbb{C}$ , the class of solutions of the differential equation:  $P_\Gamma u = \lambda u$  in  $M^n$ . The generalized eigenfunctions in  $\mathcal{E}_\lambda$  is a distinguished subclass. We have the following representation theorem which generalizes for the class of manifolds  $M^n$  a theorem proved by Helgason [7] and Minemura [14] in the case  $M^n = \mathbb{B}^n$ .

**Theorem 1 :** *The functions  $u(x)$  in  $\mathcal{E}_{k^2}(M^n)$ , for any  $k$  in the half-plane  $Im k \geq 0$ ,  $k^2 \notin \sigma_d(P_\Gamma)$ , are precisely the functions given by*

$$u(x) = \langle f, E_\Gamma(x, \cdot; k) \rangle \tag{2.8}$$

where  $f \in \mathcal{A}'(\partial M^n)$  ( $f$  acts on  $E_\Gamma$  as a function of  $\omega$ ). Moreover, the mapping:  $f \rightarrow u$  is a bijection of  $\mathcal{A}'(\partial M^n)$  onto  $\mathcal{E}_{k^2}(M^n)$ . Also,  $f \in \mathcal{D}'(\partial M^n)$  if and only if  $u(x)$  grows at most exponentially, i.e. if  $|u(x)| \leq C \tau(x)^\alpha$  on  $M^n$  for some constants  $C, \alpha$ .

We shall give now some indications on the scattering matrix  $S_\Gamma(k)$  which is a more subtle mathematical object appearing in the spectral theory of the Laplacian on  $M^n$ .  $S_\Gamma(k)$  is an operator valued function defined initially for  $k$  in the half-plane  $Im k \geq 0$

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except for a discrete set of points  $\Sigma$  on the positive imaginary axis, consisting of points  $k$  with  $k^2 \in \sigma_d(P_\Gamma)$  and the set  $i(\frac{n-1}{2} + \mathbb{Z}_+)$ .  $(S_\Gamma(k))$  acts on functions, distributions and hyperfunctions on  $\partial M^n$ . One finds that for any  $k$   $S_\Gamma(k)$  is an elliptic pseudo-differential operator on  $\partial M^n$  of complex order  $-2ik$ .  $S_\Gamma(k)$  is an analytic function of  $k$  for  $Im k > 0$  (in a weak sense), and continuous for  $Im k \geq 0$ , except for poles in the set  $\Sigma$ . The scattering matrix makes its appearance in the following asymptotic expansion theorem for functions in  $\mathcal{E}_{k^2}(M^n)$ . The theorem can be used to give the *definition* of the scattering matrix.

**Theorem 2:** Let  $u(x) \in \mathcal{E}_{k^2}(M^n)$  where  $Im k \geq 0, k^2 \notin \sigma_d(P_\Gamma)$ . Suppose also that  $2ik \notin \mathbb{Z}$ . Then the following asymptotic formula holds:

$$\begin{aligned} u(x) \sim \tau(x)^{-\left(\frac{n-1}{2} + ik\right)} \sum_{m=0}^{\infty} \tau(x)^{-m} A_m(k) f + \\ + \tau(x)^{-\left(\frac{n-1}{2} - ik\right)} \sum_{m=0}^{\infty} \tau(x)^{-m} B_m(k) S_\Gamma(k) f \end{aligned} \quad (2.9)$$

as  $x \rightarrow \partial M^n$ , for some hyperfunction  $f$  on  $\partial M^n$ , where  $A_m(k)$  and  $B_m(k)$  are differential operators of order  $\leq m$  on  $\partial M^n$  with smooth (real analytic) coefficients. The operators  $A_m$  and  $B_m$  depend on  $\Gamma, \tau, k$  and  $m$ .  $A_0$  and  $B_0$  are constants given by

$$A_0(k) = \frac{i}{k}, \quad B_0(k) = \frac{\Gamma(ik)}{\Gamma(1-ik)}.$$

**Remark:** The asymptotic relation (2.9) holds in some generalized sense. If  $F \in C^\infty(\partial M^n)$  then (2.9) holds in the usual pointwise sense.

Note that for  $k$  real, taking the two dominant terms in (2.9), we have:

$$u(x) \sim \frac{i}{k} \tau(x)^{-\left(\frac{n-1}{2} + ik\right)} f + \frac{\Gamma(ik)}{\Gamma(1-ik)} \tau(x)^{-\left(\frac{n-1}{2} - ik\right)} S_{\Gamma}(k) f \quad (2.10)$$

as  $x \rightarrow \partial M^n$ .

The relation between Theorem 1 and Theorem 2 is given in

**Theorem 3:** *There exists on  $\partial M^n$  a measure  $d\omega'$  of the form:  $d\omega' = \psi(\omega)d\omega$ , where  $\psi(\omega)$  is a positive real analytic function on  $\partial M^n$  and  $d\omega$  is the induced Lebesgue measure on  $\partial M^n$  (considered as a subset of  $\partial B^n$ ), such that if  $u \in \mathcal{E}_{k^2}(M^n)$  has the asymptotic expansion (2.9) with  $f \in \mathcal{A}'(\partial M^n)$ , then  $u$  has the representation:*

$$u(x) = \int_{\partial M^n} f(\omega) E_{\Gamma}(x, \omega; k) d\omega' \quad (2.11)$$

where (2.11) is interpreted in an obvious way (by continuity) if  $f$  is not a function.

As easy applications of the preceding theorems we derive two basic formulas.

Let  $k$  be real,  $k \neq 0$ .

Consider the function:

$$u(x) = G_{\Gamma}(x, x'; k) - G_{\Gamma}(x, x'; -k),$$

$x'$  being fixed. Clearly  $u \in \mathcal{E}_{k^2}(M^n)$ . From (2.6) it follows that

$$u(x) \sim \tau(x)^{-\left(\frac{n-1}{2} - ik\right)} E_{\Gamma}(x', \omega; k) - \tau(x)^{-\left(\frac{n-1}{2} + ik\right)} E_{\Gamma}(x', \omega; -k) \quad (2.12)$$

as  $x \rightarrow \omega \in \partial M^n$ . Hence, applying Theorem 3, it follows from (2.11) that

$$G_{\Gamma}(x, x'; k) - G_{\Gamma}(x, x'; -k) = ki \int_{\partial M^n} E_{\Gamma}(x', \omega; -k) E_{\Gamma}(x, \omega; k) d\omega'. \quad (2.13)$$

Also, by comparing (2.12) and (2.10), we find that

$$E_{\Gamma}(x, \omega; k) = -\frac{\Gamma(ik)}{\Gamma(-ik)} S_{\Gamma}(k) E_{\Gamma}(x, \cdot; -k). \quad (2.14)$$

We note that an easy application of formula (2.13) is the spectral representation theorem for  $\Delta_{\Gamma}$ . Another immediate application of the formula is a proof that  $G_{\Gamma}(x, x'; k)$  admits a meromorphic continuation to the entire complex plane once it is shown that  $E_{\Gamma}(x, \omega; k)$  has this property.

Formula (2.14) is the functional equation for the generalized eigenfunctions. An easy application of the formula is that  $E_{\Gamma}(x, \omega; k)$ , and thus also the Eisenstein series  $e_{\Gamma}(x, \omega; k)$ , admit meromorphic continuations to the entire complex plane once it is shown that  $S_{\Gamma}(k)$  has this property.

We conclude by giving some very brief indications concerning the proof that  $S_{\Gamma}(k)$  admits a meromorphic continuation to the entire complex plane. As explained above this is a main (and last) step in the proof of the theorem that both the Eisenstein series and the resolvent kernel admit such meromorphic continuations.

To derive the meromorphic continuation of  $S_{\Gamma}(k)$  into the lower half-plane one starts by showing that for all *real*  $k$  the pseudo-differential operator  $S_{\Gamma}(k)$  has an inverse satisfying the functional relation:

$$S_{\Gamma}(k)^{-1} = S_{\Gamma}(-k). \quad (2.15)$$

(This follows essentially from the functional equation (2.14)).

Next one uses the fact that  $S_{\Gamma}(k)$  is an *elliptic* pseudo-differential operator of order  $-2ik$  for any  $k$  in the half-plane  $Im\ k \geq 0, k \notin \Sigma$ , and that  $S_{\Gamma}(k)$  is a (weakly) analytic operator valued function in  $k$  for  $Im\ k > 0$ , continuous for  $Im\ k \geq 0$ , except

for poles in  $\Sigma$ . Using the calculus of pseudo-differential operators and the analytic Fredholm theory, it then follows by standard arguments that  $S_{\Gamma}(k)^{-1}$  exists as a meromorphic operator valued function for  $Im k > 0$ ,  $S_{\Gamma}(k)^{-1}$  admitting also continuous boundary values on the real axis. These properties together with the functional equation (2.15) yield the meromorphic continuation of  $S_{\Gamma}$  to the entire complex plane.

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