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PROPAGATION OF SINGULARITIES AND
LOCAL SOLVABILITY IN GEVREY CLASSES

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The propagation of the Gevrey singularities has been investigated recently by many authors (see for example Cattabriga - Zanghirati [2] and the references there). Here we shall report on some results obtained in collaboration with Zanghirati [7] and Liess [5] concerning propagation of Gevrey singularities for pseudo differential operators with multiple characteristics; we shall also consider the strictly related problem of the Gevrey local solvability, already discussed in Rodino [6].

Let us denote by $G^s(\Omega)$ the Gevrey class of order s , $1 < s < \infty$, in the open subset Ω of \mathbb{R}^n . Let us write $G_0^s(\Omega) = G^s(\Omega) \cap C_0^\infty(\Omega)$; the space of the s -ultradistributions $G_0^{(s)' }(\Omega)$ and the space of the s -ultradistributions with compact support $G^{(s)' }(\Omega)$ are then defined as the duals of $G_0^s(\Omega)$, $G^s(\Omega)$, respectively. We shall also use the standard notion of Gevrey wave front set of order s of $v \in G^{(s)' }(\Omega)$, $WF_s v \subset \Omega \times (\mathbb{R}^n \setminus 0)$.

Our arguments will be microlocal in a small conic neighborhood Γ of a point $(x_0, \xi_0) \in \Omega \times (\mathbb{R}^n \setminus 0)$; we shall consequently refer to the factor-space of the s -microfunctions in Γ

$$M^s(\Gamma) = G_0^{(s)'}(\Omega) / \sim ,$$

where $f \sim g$ means that $\Gamma \cap WF_s(f-g) = \emptyset$.

Let us consider a classical analytic pseudo differential operator $P = p(x, D)$ with symbol

$$p(x, \xi) \sim \sum_{j=0}^{\infty} p_{m-j}(x, \xi)$$

defined in a conic neighborhood of (x_0, ξ_0) . We shall assume the principal part $p_m(x, \xi)$ satisfies for some integer $k \geq 2$ the following condition

- (1) we may write $p_m(x, \xi) = q_{m-k}(x, \xi) a_1(x, \xi)^k$, where $q_{m-k}(x, \xi)$ is an elliptic symbol homogeneous of order $m-k$, and the first order symbol $a_1(x, \xi)$ is real valued and of principal type, i.e. $d_{x, \xi} a_1(x, \xi)$ never vanishes and it is not parallel to $\sum_h \xi_h dx_h$ on $\Sigma = \{(x, \xi) \in \Gamma, a_1(x, \xi) = 0\} \neq \emptyset$.

This is equivalent to say that our operator P can be reduced, by conjugation with analytic Fourier integral

operators and multiplication by elliptic factors, to the form

$$P = D_{x_n}^k + \text{pseudo differential operators of order} \\ \leq k-1 .$$

The hypothesis (1) is sufficient to conclude non-analytic hypoellipticity of P and propagation of the analytic wave front set along the bicharacteristic strips associated to P (see Bony-Shapira [1]), whereas to obtain a similar result in the C^∞ category it is necessary to add the so-called Levi condition on the lower order terms (see Chazarain [3]). A natural interpolation of these results can be expressed in the frame of the Gevrey classes under the following ρ -Levi condition, $0 < \rho < 1$:

- (2) Let A be a classical analytic ps.diff.operator whose principal symbol is given by the function $a_1(x, \xi)$ in (1); then P can be written in the form $P = \sum_{j=0}^k Q_j A^{k-j}$, where Q_j , $j=0, \dots, k$, are classical analytic pseudo differential operators of order $\leq m-k + \rho j$.

If in (2) we set $\rho = 0$, we obtain the standard C^∞ Levi condition; in the other limit case $\rho = 1$, nothing is imposed on the lower order terms.

An operator P satisfying (1) and (2) is microlocal

ly equivalent to the model

$$(3) \quad P = D_{x_n}^k + \sum_{j=1}^k Q_j D_{x_n}^{k-j},$$

where the Q_j , $j=1, \dots, k$, are here classical analytic pseudo differential operators of order $\leq \rho j$.

THEOREM 1. (Rodino-Zanghirati [7]). Let (1), (2) be satisfied and let s be any real number with $1 < s < 1/\rho$. Write γ_0 for the bicharacteristic strip through $(x_0, \xi_0) \in \Sigma$ (we may define γ_0 to be integral curve of the Hamiltonian vector field H_{a_1} , with $a_1(x, \xi)$ as in (1), (2)).

Then, taking a sufficiently small neighborhood Γ of (x_0, ξ_0) :

- (i) There exists $v \in M^s(\Gamma)$ with $Pv = 0$ and $WF_s v = \gamma_0$.
- (ii) If v is in $M^s(\Gamma)$ with $Pv = 0$, then $(x_0, \xi_0) \in WF_s v$ implies $\gamma_0 \subset WF_s v$.
- (iii) For every $v \in M^s(\Gamma)$ there exists $w \in M^s(\Gamma)$ such that $Pw = v$.

For the proof we may refer to the model (3); its study can be further reduced to that of the first order operator:

$$(4) \quad D_{x_n} + \lambda(x, D_x),$$

where $\lambda(x, D_x)$ is a $k \times k$ -matrix of pseudo differential operators of order $\leq \rho$. We then construct two matrices B^+, B^- of linear maps from $M^s(\Gamma)$ to $M^s(\Gamma)$, one inverse of the other, which are s -micro-local and satisfy

$$(5) \quad B^-(D_{x_n} + \lambda(x, D_x)) B^+ = D_{x_n}.$$

In this way we are reduced to prove the theorem for $P = D_{x_n}$, and that is trivial. The formal construction of B^\pm as pseudo differential operators is easy by solving transport equations. However, the symbols which one obtains have an exponential growth and to give a precise meaning to B^\pm we have to refer to a suitable theory of Gevrey infinite order operators (cf. Cattabriga-Zanghirati [2]).

Under the assumptions (1), (2), the conclusions of Theorem 1 fail in general for $1/\rho \leq s < \infty$ and the study of the corresponding G^s regularity requires then a further analysis of the operators Q_j in (2), (3).

We shall illustrate the new phenomena which may occur by arguing on the model (4). For sake of simplicity, we shall suppose here $\lambda(x, D_x)$ is a scalar operator with symbol

$$\lambda(x, \xi) = \lambda_\rho(x, \xi') + \lambda_0(x, \xi),$$

where $\lambda_\rho(x, \xi')$ is homogeneous of order ρ , $0 < \rho < 1$, with respect to $\xi' = (\xi_1, \dots, \xi_{n-1})$ and $\lambda_0(x, \xi)$ is a classical analytic symbol of order zero. Our arguments will be microlocal in a neighborhood of a point (x_0, ξ_0) with $\xi_0 = (\xi'_0, 0)$. For the operator

$$(6) \quad P = D_{x_n} + \lambda_\rho(x, D_{x'}) + \lambda_0(x, D)$$

the conclusions of Theorem 1 (non-hypoellipticity, propagation, local solvability) apply when $1 < s < 1/\rho$, whereas for $1/\rho \leq s < \infty$ we have:

Theorem 2 . Assume $\text{Im } \lambda_\rho(x_0, \xi'_0) \neq 0$. Then for $1/\rho \leq s < \infty$ the operator P in (6) is G^s -hypoelliptic in a neighborhood Γ of (x_0, ξ_0) , i.e.

$$\text{WF}_s^+ Pv = \text{WF}_s^+ v \quad \text{for all } v \in M^s(\Gamma),$$

and the solvability property (iii) in Theorem 1 is still valid.

In fact, a parametrix P' of P can be easily constructed, $P'P = PP' = \text{identity}$ on $M^s(\Gamma)$, $1/\rho \leq s < \infty$, with symbol in a Gevrey version of the class $S_{\rho, 0}^0$ of Hörmander (see for example Liess-Rodino [4]).

Theorem 3. Assume $\lambda_\rho(x, \xi')$ is real valued in a conic neighborhood of (x_0, ξ'_0) . All the conclusions of Theorem 1 are valid for P in (6) also when $1/\rho \leq s < \infty$.

This is a consequence of a much more general result in Liess-Rodino [5], concerning Gevrey propagation for operators of non-homogeneous type. Precisely, under the assumption in Theorem 3, we may construct Fourier integral operators B^\pm , with non-homogeneous (real) phase function, for which (5) is satisfied on $M^s(\Gamma)$, $1/\rho \leq s < \infty$; in this way we are again reduced to the trivial study of the operator $BP = D_{x_n}$.

When $\lambda_\rho(x, \xi')$ takes values in the complex domain, but $\text{Im } \lambda_\rho(x, \xi')$ vanishes at (x_0, ξ'_0) , then the solvability property (iii) in Theorem 1 may fail for $1/\rho < s < \infty$.

A representative example in this connection is given by the model in \mathbb{R}^2

$$(7) \quad P_\rho = D_{x_2} + ix_2^h |D_{x_1}|^\rho,$$

where h is an odd integer and $0 < \rho < 1$; the symbol $\lambda_\rho = ix_2^h |\xi_1|^\rho$ is here considered in a neighborhood of $x_0 = (0, 0)$, $\xi_0 = (1, 0)$.

Theorem 4. Assume $1/\rho < s < \infty$. Then there exists

$v \in M^S(\mathbb{R}^2)$ such that $(x_0, \xi_0) \in WF_S(v - P_\rho v)$ for all
 $v \in M^S(\mathbb{R}^2)$.

The theorem is proved in Rodino [6] by considering the Fourier integral operator

$$\Pi_\rho f(x) = \iint_{\vartheta > 0} e^{i\omega(x, y, \vartheta)} \vartheta^{\rho/(h+1)} f(y) dy d\vartheta$$

with non-homogeneous complex phase function

$$\omega(x, y, \vartheta) = \vartheta(x_1 - y_1) + i\vartheta^\rho(x_2^{h+1} + y_2^{h+1})/(h+1) .$$

The operator Π_ρ maps $G_0^S(\mathbb{R}^2)$ into $G^S(\mathbb{R}^2)$, and $G^{(s)'}(\mathbb{R}^2)$ into $G_0^{(s)'}$ (\mathbb{R}^2), for $1/\rho < s < \infty$. For the same values of s , the operator Π_ρ is s-micro local, so it is well defined on the s-microfunctions in a neighborhood of the origin, and we also have:

$$\Pi_\rho P_\rho = 0 .$$

If we take $v \in M^S(\mathbb{R}^2)$ such that $(x_0, \xi_0) \in \Pi_\rho v$, then we obtain $(x_0, \xi_0) \in WF_S(v - P_\rho v)$ for all $v \in M^S(\mathbb{R}^2)$; in fact $P_\rho v = v$ in a conic neighborhood of (x_0, ξ_0) would imply

$$\Pi_\rho (P_\rho v - v) = \Pi_\rho P_\rho v - \Pi_\rho v = \Pi_\rho v = 0$$

in the same neighborhood.

If we limit ourselves to the local point of view, the proceeding shows that for $1/\rho < s < \infty$ the operator P_ρ is non-s-locally solvable at $x_0 = (0,0)$, i.e. there exists $f \in G_0^s(\mathbb{R}^2)$ such that the equation $P_\rho v = f$ has no solution $v \in G_0^{(s)'}(\mathbb{R}^2)$ in any neighborhood of the origin. In view of the obvious inclusions $G_0^s(\mathbb{R}^2) \subset C_0^\infty(\mathbb{R}^2)$, $D'(\mathbb{R}^2) \subset G_0^{(s)'}(\mathbb{R}^2)$, we have in particular that P_ρ is non-locally solvable in the standard C^∞ sense.

However, a solution v of the equation $P_\rho v = f \in C^\infty(\mathbb{R}^2)$ always exists if we allow v to be in $G_0^{(s)'}(\mathbb{R}^2)$ with $1 < s < 1/\rho$ (This follows from the local version of (iii) in Theorem 1). It is worth particularizing the computations of Rodino-Zanghirati [7] for the operator P_ρ in (7), to see explicitly how "unsolvable equations can be solved" in an ultra-distribution sense. We have to consider the pseudo differential operators

$$B_\rho^\pm f(x) = (2\pi)^{-2} \int e^{ix\xi} b_\rho^\pm(x, \xi) \hat{f}(\xi) d\xi$$

with infinite order symbols

$$b_\rho^\pm(x, \xi) = \exp. [\pm x_2^{h+1} |\xi_1|^\rho / (h+1)] .$$

They are one inverse of the other and satisfy the i dentity (5), i.e.:

$$B_{\rho}^{-} P_{\rho} B_{\rho}^{+} = D_{x_2} .$$

Therefore a solution of $P_{\rho} v = f \in C_0^{\infty}(\mathbb{R}^2)$ is obtained by considering

$$\tilde{f}(x) = i \int_0^{x_2} B_{\rho}^{-} f(x_1, y_2) dy_2 ,$$

which is still a C^{∞} function, and setting finally $v = B_{\rho}^{+} \tilde{f}$ (which is in general a true ultradistribution in $G_0^{(s)}$ (\mathbb{R}^2), $1 < s < 1/\rho$). For a more detailed discussion of the problem of the Gevrey-local solvability, we refer to Rodino [6].

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